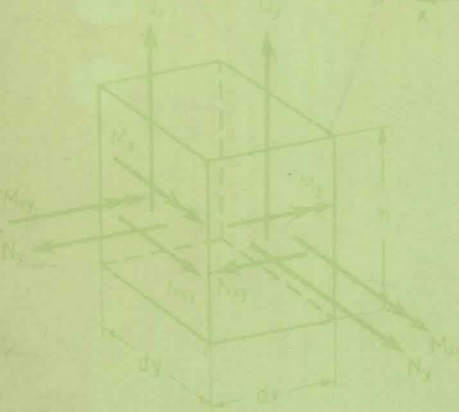
Variational Methods in Elasticity & Plasticity

THIRD EDITION

KYUICHIRO WASHIZU

University of Tokyo, Japan



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PREFACE TO THE THIRD EDITION

THE third edition divides roughly into two parts, namely Part A and Part B. The content of Part A, entitled *Formulations of Variational Principles in Elasticity and Plasticity*, is almost the same as that of the first edition, except for some topics newly introduced, e.g. in Chapters 5 and 7. The content of Part B, entitled *Variational Principles as a Basis of the Finite Element Method*, is intended to be an improved presentation of Appendix I of the second edition. There, conventional variational principles, together with modified variational principles with relaxed continuity requirements, are presented in a systematic manner for the small displacement theory of elastostatics, the finite displacement theory of elastostatics, the theory of elastodynamics, two incremental theories with geometrical and material nonlinearities, and bending of elastic plates. The last chapter is devoted to discrete analysis, where an introduction to the boundary element method is newly appended.

The repletion of Appendices is one of the main objectives in writing this new edition. It includes fourteen topics spanning from Appendix A to Appendix N. Among topics newly added are Variational Principles in Dynamics of a System of Particles (Appendix B), Notes on the Strain Energy Function and Complementary Energy Function (Appendix D), Notes on Several Kinds of Stress Tensors in Finte Displacement Theory (Appendix E), and Notes on the Boundary Element Method (Appendix N).

As Dr. Zienkiewicz states in the Preface of his momumental work entitled *The Finite Element Method*, Third Edition published in 1977, "the number of research publications on the finite element method has been increasing almost exponentially". The present author has found that a complete updating of the second edition of this book means that he should write a new book which needs an endless effort beyond his capacity and available time. Consequently, he is rather interested in partial updating and satisfied with improving the content of the book as much as possible so that it can be accepted as a good textbook for advanced engineering students. Again, the bibliography of this new edition is not intended to be complete.

The author would like to express his deep appreciation to Professor T. H. H. Pian of the Massachusetts Institute of Technology, Dr. Pin Tong of Transportation Systems Center, Department of Transportation,

Cambridge, Massachusetts, Professor Satya N. Atluri of the Georgia Institute of Technology, and Professor Herbert A. Mang of the Technische Universität Wien, for their advice, criticism and encouragement given to the manuscript of this third edition. Dr. Oscar Orringer of the Transportation Systems Center, Department of Transportation, Cambridge, Massachusetts, collaborated again with the author in correcting the writing of the manuscript. Dr. Tsukasa Nakayama and Mrs. Keiko Nakamichi of the University of Tokyo helped the author in preparing and typing the manuscript. The author would like to express his sincere appreciation to all of these people.

June 1980 K. WASHIZU

PREFACE TO THE FIRST EDITION

THE object of the author in writing this book is to provide a textbook on a treatment of variational formulations in elasticity and plasticity for advanced engineering students. Emphasis is placed on demonstrating the power of the principle of virtual work and related variational principles in deriving the governing equations and corresponding boundary conditions in a systematic way.

The book divides roughly into three parts. The first part, from Chapter 1 to Chapter 5, concerns fundamentals of the theory of elasticity. The first and second chapters deal with the small displacement theory of elasticity and the third with finite displacement theory in a rectangular Cartesian coordinate system. The finite displacement theory of elasticity is formulated in a curvilinear coordinate system in Chapter 4. The principle of virtual work and related variational principles are extended to initial stress, initial strain and dynamical problems in Chapter 5.

The second part, from Chapter 6 to Chapter 10, deals with applications of the principle of virtual work and variational principles to particular problems in elasticity. These include torsion of bars, beams, plates, shells and structures, and show the power of these principles in obtaining approximate governing equations and corresponding boundary conditions.

The third part, Chapters 11 and 12, deals with variational principles in the theories of plasticity. The deformation theory of plasticity is treated in Chapter 11. Variational principles and limit analysis in the flow theory of plasticity are treated in Chapter 12.

The reader is assumed to have some knowledge of elasticity, plasticity and variational calculus. Due to the limitation of space available, a description of the fundamentals of variational calculus is omitted in this book. The author is satisfied with listing several books on the calculus at the end of the Introduction for the reader's reference.

The author wishes to express his deepest appreciation to Professor R. L. Bisplinghoff of the Massachusetts Institute of Technology for the invitation to write this book and for his constant advice, criticism and encouragement given throughout the preparation of the manuscript. He wishes to thank Professor E. Reissner of the Massachusetts Institute of Technology and Professors A. Ono, T. Hayashi and S. Moriguti of the University of Tokyo for stirring up the author's interest in variational

methods. He also wishes to thank the authors of books and papers to which references have been made in the writing of this book. The author would like to mention several persons who offered assistance and gave criticism while the manuscript was being written: Professor T. F. O'Brien of the University of Washington, Professor M. Kuranishi of Nihon University, Professors Y. Yamamoto, T. Kawai, S. Kobayashi, H. Fujita and M. Iri of the University of Tokyo, and Professor T. Mura of Northwestern University. Mr. Oscar Orringer of the Massachusetts Institute of Technology collaborated with the author in correcting the writing of the manuscript. Mrs. Y. Yoshihara, Mr. H. Tomizawa and Miss M. Yoshida of the University of Tokyo helped the author in typing and preparing the manuscript. The author would like to express his sincere appreciation to all of these people and also to Messrs. D. J. Raymond and M. S. Gale of Pergamon Press who helped the author with thoughful arrangements and patience.

September 1965 K. WASHIZU

FOREWORD TO THE SECOND EDITION

THE variational principle and its application to many branches of mechanics including elasticity and plasticity has had a long history of development. However, the importance of this principle has been highlighted in recent years by developments in the use of finite element methods which have been widely employed in structural analysis since the pioneering work by M. J. Turner *et al.* appeared in Vol. 23, No. 9 issue of the *Journal of Aeronautical Sciences* in 1956. It has been shown repeatedly since that time that the variational principle provides a powerful tool in the mathematical formulation of the finite element approach. Conversely, the rapid development of the finite element method has given much stimulus to the advancement of the variational principle and new forms of the principle have been developed during the past decade as outlined in Section 1 of Appendix I of the present book.

The first edition of Professor Washizu's book, entitled *Variational Methods in Elasticity and Plasticity* and published in 1968, was well received by engineers, teachers and students working in solid and structural mechanics. Its publication was timely, because it coincided with a period of rapid growth of application of the finite element method. The principle features of the first edition was that of providing a systematic way of deriving variational principles in elasticity and plasticity, of transforming one variational principle to another and of providing a systematic basis for the mathematical formulation of the finite element method. The book was widely used and referenced frequently in literature related to the finite element method.

Now, Professor Washizu has prepared a revised edition which adds a new Appendix I. The new appendix introduces an outline of variational principles which are used frequently as a basis for mathematical formulations in elasticity and plasticity including those new variational principles developed in connection with the finite element method. As in the case of the first edition, Appendix I is written in the clear, concise and elegant style for which Professor Washizu is so widely known. The revised edition should form an extremely valuable addition to the libraries and reference shelves of all who are interested in solid and structural mechanics.

National Science Foundation, Washington D.C. R. L. BISPLINGHOFF

ACKNOWLEDGEMENTS TO THE SECOND EDITION

THE author feels extremely honored and wishes to express his deepest gratitude to Dr. R. L. Bisplinghoff, Deputy Director of National Science Foundation, for having given the Foreword to the revised edition of this book. The author would like to express his deepest appreciation to Professor T. H. H. Pian of the Massachusetts Institute of Technology and Professor R. H. Gallagher of Cornell University for having given valuable comments to the manuscript for the new appendix. Dr. Oscar Orringer of the Massachusetts Institute of Technology collaborated again with the author in correcting the writing of the manuscript of the new appendix. Moreover, the author should remember that he has been given numerous comments, criticisms and encouragements from the reader since the publication of the first edition of this book. The author would like to express his sincere appreciation to all of these people, without whose encouragement and collaboration, this revised edition could not be realized.

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Part A—Formulation of Variational Principles in Elasticity and Plasticity

PART A

Formulations of Variational Principles in Elasticity and Plasticity