volume 69

lecture notes in pure and applied mathematics

tables of dimensions, indices, and branching rules for representations of simple lie algebras

W. G. McKay
J. Patera

Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras

W. G. McKay and J. Patera

Centre de recherche de mathématiques appliquées
Université de Montréal
Montréal, Québec, Canada

MARCEL DEKKER, INC. New York and Basel

Library of Congress Cataloging in Publication Data

McKay, W G
Tables of dimensions, indices, and branching rules for representations of simple Lie algebras

(Lecture notes in pure and applied mathematics ; 69)
Includes bibliographical references.
1. Lie groups--Tables. 2. Representations of algebras--Tables. I. Patera, J., joint author.
II. Title.
QC174.17.G7M32 512'.55 80-27663
ISBN 0-8247-1227-7

MARCEL DEKKER, INC.

270 Madison Avenue, New York, New York 10016

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras

PURE AND APPLIED MATHEMATICS

A Program of Monographs, Textbooks, and Lecture Notes

Contributions to *Lecture Notes in Pure and Applied Mathematics* are reproduced by direct photography of the author's typewritten manuscript. Potential authors are advised to submit preliminary manuscripts for review purposes. After acceptance, the author is responsible for preparing the final manuscript in camera-ready form, suitable for direct reproduction. Marcel Dekker, Inc. will furnish instructions to authors and special typing paper. Sample pages are reviewed and returned with our suggestions to assure quality control and the most attractive rendering of your manuscript. The publisher will also be happy to supervise and assist in all stages of the preparation of your camera-ready manuscript.

LECTURE NOTES
IN PURE AND APPLIED MATHEMATICS

1. *N. Jacobson,* Exceptional Lie Algebras
2. *L.-Å. Lindahl and F. Poulsen,* Thin Sets in Harmonic Analysis
3. *I. Satake,* Classification Theory of Semi-Simple Algebraic Groups
4. *F. Hirzebruch, W. D. Newmann, and S. S. Koh,* Differentiable Manifolds and Quadratic Forms
5. *I. Chavel,* Riemannian Symmetric Spaces of Rank One
6. *R. B. Burckel,* Characterization of C(X) Among Its Subalgebras
7. *B. R. McDonald, A. R. Magid, and K. C. Smith,* Ring Theory: Proceedings of the Oklahoma Conference
8. *Y.-T. Siu,* Techniques of Extension of Analytic Objects
9. *S. R. Caradus, W. E. Pfaffenberger, and B. Yood,* Calkin Algebras and Algebras of Operators on Banach Spaces
10. *E. O. Roxin, P.-T. Liu, and R. L. Sternberg,* Differential Games and Control Theory
11. *M. Orzech and C. Small,* The Brauer Group of Commutative Rings
12. *S. Thomeier,* Topology and Its Applications
13. *J. M. López and K. A. Ross,* Sidon Sets
14. *W. W. Comfort and S. Negrepontis,* Continuous Pseudometrics
15. *K. McKennon and J. M. Robertson,* Locally Convex Spaces
16. *M. Carmeli and S. Malin,* Representations of the Rotation and Lorentz Groups: An Introduction
17. *G. B. Seligman,* Rational Methods in Lie Algebras
18. *D. G. de Figueiredo,* Functional Analysis: Proceedings of the Brazilian Mathematical Society Symposium
19. *L. Cesari, R. Kannan, and J. D. Schuur,* Nonlinear Functional Analysis and Differential Equations: Proceedings of the Michigan State University Conference
20. *J. J. Schäffer,* Geometry of Spheres in Normed Spaces
21. *K. Yano and M. Kon,* Anti-Invariant Submanifolds
22. *W. V. Vasconcelos,* The Rings of Dimension Two
23. *R. E. Chandler,* Hausdorff Compactifications
24. *S. P. Franklin and B. V. S. Thomas,* Topology: Proceedings of the Memphis State University Conference
25. *S. K. Jain,* Ring Theory: Proceedings of the Ohio University Conference
26. *B. R. McDonald and R. A. Morris,* Ring Theory II: Proceedings of the Second Oklahoma Conference
27. *R. B. Mura and A. Rhemtulla,* Orderable Groups
28. *J. R. Graef,* Stability of Dynamical Systems: Theory and Applications
29. *H.-C. Wang,* Homogeneous Banach Algebras
30. *E. O. Roxin, P.-T. Liu, and R. L. Sternberg,* Differential Games and Control Theory II
31. *R. D. Porter,* Introduction to Fibre Bundles
32. *M. Altman,* Contractors and Contractor Directions Theory and Applications
33. *J. S. Golan,* Decomposition and Dimension in Module Categories
34. *G. Fairweather,* Finite Element Galerkin Methods for Differential Equations
35. *J. D. Sally,* Numbers of Generators of Ideals in Local Rings
36. *S. S. Miller,* Complex Analysis: Proceedings of the S.U.N.Y. Brockport Conference
37. *R. Gordon,* Representation Theory of Algebras: Proceedings of the Philadelphia Conference
38. *M. Goto and F. D. Grosshans,* Semisimple Lie Algebras
39. *A. I. Arruda, N. C. A. da Costa, and R. Chuaqui,* Mathematical Logic: Proceedings of the First Brazilian Conference
40. *F. Van Oystaeyen,* Ring Theory: Proceedings of the 1977 Antwerp Conference

41. *F. Van Oystaeyen and A. Verschoren,* Reflectors and Localization: Application to Sheaf Theory
42. *M. Satyanarayana,* Positively Ordered Semigroups
43. *D. L. Russell,* Mathematics of Finite-Dimensional Control Systems
44. *P.-T. Liu and E. Roxin,* Differential Games and Control Theory III: Proceedings of the Third Kingston Conference, Part A
45. *A. Geramita and J. Seberry,* Orthogonal Designs: Quadratic Forms and Hadamard Matrices
46. *J. Cigler, V. Losert, and P. Michor,* Banach Modules and Functors on Categories of Banach Spaces
47. *P.-T. Liu and J. G. Sutinen,* Control Theory in Mathematical Economics: Proceedings of the Third Kingston Conference, Part B
48. *C. Byrnes,* Partial Differential Equations and Geometry
49. *G. Klambauer,* Problems and Propositions in Analysis
50. *J. Knopfmacher,* Analytic Arithmetic of Algebraic Function Fields
51. *F. Van Oystaeyen,* Ring Theory: Proceedings of the 1978 Antwerp Conference
52. *B. Kedem,* Binary Time Series
53. *J. Barros-Neto and R. A. Artino,* Hypoelliptic Boundary-Value Problems
54. *R. L. Sternberg, A. J. Kalinowski, and J. S. Papadakis,* Nonlinear Partial Differential Equations in Engineering and Applied Science
55. *B. R. McDonald,* Ring Theory and Algebra III: Proceedings of The Third Oklahoma Conference
56. *J. S. Golan,* Structure Sheaves over a Noncommutative Ring
57. *T. V. Narayana, J. G. Williams, and R. M. Mathsen,* Combinatorics, Representation Theory and Statistical Methods in Groups: YOUNG DAY Proceedings
58. *T. A. Burton,* Modeling and Differential Equations in Biology
59. *K. H. Kim and F. W. Roush,* Introduction to Mathematical Consensus Theory
60. *J. Banas and K. Goebel,* Measures of Noncompactness in Banach Spaces
61. *O. A. Nielsen,* Direct Integral Theory
62. *J. E. Smith, G. O. Kenny, and R. N. Ball,* Ordered Groups: Proceedings of the Boise State Conference
63. *J. Cronin,* Mathematics of Cell Electrophysiology
64. *J. W. Brewer,* Power Series Over Commutative Rings (in press)
65. *P. K. Kamthan and M. Gupta,* Sequence Spaces and Series (in press)
66. *T. G. McLaughlin,* Regressive Sets and the Theory of Isols (in press)
67. *H. W. Stech, S. M. Rankin, and T. L. Herdman,* Integral and Functional Differential Equations (in press)
68. *R. Draper,* Commutative Algebra: Analytic Methods (in press)
69. *W. G. McKay and J. Patera,* Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras

Other Volumes in Preparation

Preface

The nature of applications of semisimple Lie algebras and groups frequently requires the knowledge of how representations of an algebra reduce to representations of its subalgebras. The problem, although fairly standard from the point of view of mathematics, has no known easy general solution. There arises therefore a need to compile the results, within certain practical limits, in the form of tables. This volume contains the reduction (branching rules) of representations of complex simple Lie algebras of ranks not exceeding 8 to representations of their maximal semisimple subalgebras; the dimensions of representations given here are limited by 5000 for the classical Lie algebras, and 10000 for the five exceptional algebras. Also, there are tables of auxiliary quantities (dimensions, second and fourth indices of representations) useful in applications.

In 1971 a book of similar nature but much more limited content (dimensions < 1000) was prepared [1]. Since then the computation procedure [2] has been considerably speeded up. This volume contains results of an independent computation of branching rules; some conventions were modified in comparison with Ref. 1 and a few corrections were made. The text is completely rewritten.

We make no attempt to collect a complete bibliography on branching rules, which have been the subject of a number of publications during the last fifteen years. Efficient methods are known for particular algebra-subalgebra pairs [3] and extensive tables have been computed for some others often used in spectroscopy [4].

The present book is intended for quick reference by people knowing basic facts about semisimple Lie algebras and their finite dimensional representations [5] who use representations of semisimple Lie algebras as tools in their own work.

Since the publication of Ref. 1 we have received many comments and helpful suggestions. We are particularly grateful to Dr. F.W. Lemire, Dr. R.T. Sharp, and Dr. R. Slansky.

This work was supported in part by the Natural Science and Engineering Council of Canada and by the Ministère de l'Education du Québec.

W. G. McKay

J. Patera

Contents

Preface iii

1/Notations, Conventions, and Some Properties of Representations 1
Semisimple Lie Algebras 1
Contragredient, Orthogonal, and Symplectic Representations 3
Congruence Classes of Representations 6

2/Indices of Representations 9
Definition and Properties of the Indices 10
The Table of Indices 12
Examples of Applications 13

3/Branching Rules 17
Semisimple Subalgebras of Semisimple Lie Algebras and Branching Rules 17
The Table of Branching Rules 18
Branching Rules for A_2 19
Generating Functions for Branching Rules 21

References 23

Table 1 Table of Dimensions, Second and Fourth Indices of Irreducible Representations 25

Table 2 Table of Branching Rules 85

1/Notations, Conventions, and Some Properties of Representations

Semisimple Lie Algebras

There are nine types of complex simple Lie algebras: four classical series A_n, B_n, C_n, D_n of ranks $n = 1,2,\ldots,$ and five types of isolated exceptional algebras E_6, E_7, E_8, F_4 and G_2. A semisimple Lie algebra L decomposes into simple algebras (ideals); we write

$$L = L_1 + L_2 + \ldots + L_k \qquad (1.1)$$

There exist only the following isomorphisms among semisimple Lie algebras:

$$A_1 \sim B_1 \sim C_1 \sim D_1, \quad B_2 \sim C_2, \quad D_2 \sim A_1 + A_2, \quad A_3 \sim D_3 \qquad (1.2)$$

The symbols B_1, C_1, D_1, B_2, D_2, and D_3 do not therefore appear further in this book.

With every semisimple Lie algebra of rank n one associates a system of n real vectors α_i, $i = 1,2,\ldots,n$, called simple roots. The system is conveniently presented in the form of a Dynkin diagram (Fig. 1). A simple root is denoted by a circle. Two circles which are connected by a single, double or triple line correspond to two roots spanning an angle of 120°, 135°, and 150° respectively; two circles which are not connected directly correspond to mutually orthogonal simple roots. The Dynkin diagram of a

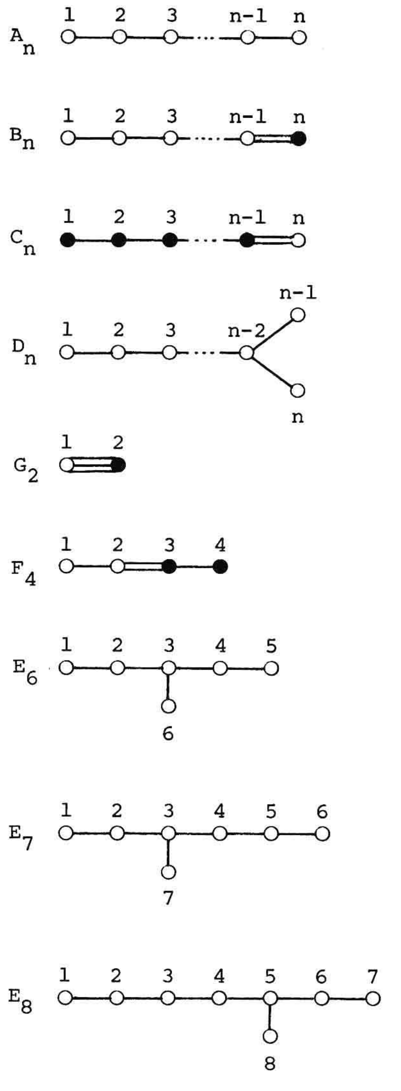


FIG. 1. Numbering of simple roots of simple Lie algebras

simple algebra is connected, the diagram of a nonsimple algebra (1.1) consists of k disconnected subdiagrams. Each simple algebra has simple roots of no more than two different lengths, the black circles denoting the shorter roots. The length of the longer roots is fixed by the convention that the scalar product (α,α), has value 2,

$$(\alpha,\alpha) = 2 \tag{1.3}$$

Furthermore, one has $(\alpha,\alpha) = 1$ for the shorter roots in the case of algebras B_n, C_n, and F_4, and $(\alpha,\alpha) = 2/3$ for G_2.

Throughout the book the simple roots are numbered as on Fig. 1.

Contragredient, Orthogonal, and Symplectic Representations

An irreducible representation $\phi(L)$ of a simple Lie algebra L is often denoted by its highest weight Λ. The highest weight Λ of a particular irreducible representation is specified by the coordinates

$$a_i = \frac{2(\Lambda,\alpha_i)}{(\alpha_i,\alpha_i)}, \qquad i = 1,2,\ldots,n \tag{1.4}$$

Thus one writes $\Lambda = (a_1 a_2 \ldots a_n)$. In particular, the convention (1.3) implies that an irreducible representation of A_1 (the algebra of "angular momentum, spin, isospin", etc. of quantum mechanics) is given by (a_1), where a_1 is an integer equal to *twice* the corresponding angular momentum.

An irreducible representation of a nonsimple algebra, say $L_1 + L_2$, is specified by

$$\phi(L_1 + L_2) = (a_1 a_2 \ldots a_n)(b_1 b_2 \ldots b_m) \tag{1.5}$$

where $(a_1 a_2 \ldots a_n)$ and $(b_1 b_2 \ldots b_m)$ are representations of L_1 and L_2, respectively. In order to improve the readability of the Table 2 below, we use there $(a_1 a_2 \ldots a_n - b_1 b_2 \ldots b_m)$ instead of (1.5). Sometimes it is advantageous to give a representation by the Dynkin diagram of the corresponding Lie algebra with the coordinates (1.4) shown at each simple root.

Some representations are of special interest. We list them here. The nontrivial representations of the lowest dimension are as follows:

$$
\begin{array}{lll}
(10\ldots0) \text{ and } (0\ldots01) & \text{for } A_n, & n \geq 1 \\
(10\ldots0) & \text{for } B_n, & n \geq 3 \\
(10\ldots0) & \text{for } C_n, & n \geq 2 \\
(1000), (0010), \text{ and } (0001) & \text{for } D_4 & \\
(10\ldots0) & \text{for } D_n, & n \geq 5 \\
(100000) \text{ and } (000010) & \text{for } E_6 & \\
(0000010) & \text{for } E_7 & \\
(1000000) & \text{for } E_8 & \\
(0001) & \text{for } F_4 & \\
(01) & \text{for } G_2 &
\end{array}
\tag{1.6}
$$

The adjoint representations are

$$
\begin{array}{lll}
(2) & \text{for } A_1 & \\
(10\ldots01) & \text{for } A_n, & n \geq 2 \\
(010\ldots0) & \text{for } B_n, & n \geq 3 \\
(20\ldots0) & \text{for } C_n, & n \geq 2 \\
(010\ldots0) & \text{for } D_n, & n \geq 4 \\
(000001) & \text{for } E_6 & \\
(1000000) & \text{for } E_7 & \\
(10000000) & \text{for } E_8 & \\
(1000) & \text{for } F_4 & \\
(10) & \text{for } G_2 &
\end{array}
\tag{1.7}
$$

A representation $\phi(L)$ acting in a space S_ϕ is selfcontragredient if there exists an invariant bilinear form (x,y), where $x,y \in S_\phi$, such that

$$(Xx,y) + (x,Xy) = 0, \qquad X \in \phi(L) \tag{1.8}$$

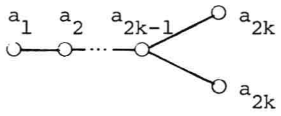
A selfcontragredient representation is orthogonal or symplectic if the form is symmetric, $(x,y) = (y,x)$, or skew symmetric, $(x,y) = -(y,x)$. The simple Lie algebras of types

$$A_1,\ B_n,\ C_n,\ D_{2k},\ E_7,\ E_8,\ F_4, \text{ and } G_2 \tag{1.9}$$

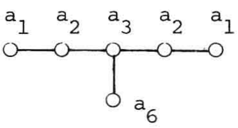
have only selfcontragredient representations. Among the irreducible representations of the remaining simple Lie algebras only the following are selfcontragredient

$$(a_1a_2\dots a_2a_1) \equiv$$

$$(a_1a_2\dots a_{2k-1}a_{2k}a_{2k}) \equiv \qquad (1.10)$$



$$(a_1a_2a_3a_2a_1a_6) \equiv$$



Here on the right hand side the representations are denoted using the Dynkin diagram which specifies the type of the algebra.

The irreducible selfcontragredient representations of

$$A_{2k},\ A_{4k+3},\ B_{4k},\ B_{4k+3},\ D_{2k+1},\ D_{4k},\ E_6,\ E_8,\ F_4,\ \text{and } G_2 \qquad (1.11)$$

are all orthogonal. An irreducible selfcontragredient representation of any of the remaining simple Lie algebras is orthogonal (symplectic) if a certain linear combination of the coordinates (1.4) of the highest weight is even (odd). These linear combinations are

a_{2k+1}	for A_{4k+1},	$k \geq 1$	
a_{4k+1}	for B_{4k+1},	$k \geq 1$	
a_{4k+2}	for B_{4k+2},	$k \geq 1$	
$a_1 + a_3 + a_5 + \dots$	for C_n,	$n \geq 2$	(1.12)
$a_{4k+1} + a_{4k+2}$	for D_{4k+2},	$k \geq 1$	
$a_4 + a_6 + a_7$	for E_7		

Two irreducible representations $\phi(L)$ and $\bar{\phi}(L)$ of a simple Lie algebra

L are mutually contragredient if they are one of the pairs

$$\phi(A_n) = (a_1 a_2 \ldots a_n) \quad \text{and} \quad \bar{\phi}(A_n) = (a_n a_{n-1} \ldots a_2 a_1)$$

$$\phi(D_{2k+1}) = (a_1 a_2 \ldots a_{2k-1} a_{2k} a_{2k+1}) \quad \text{and}$$
$$\bar{\phi}(D_{2k+1}) = (a_1 a_2 \ldots a_{2k-1} a_{2k+1} a_{2k}) \tag{1.13}$$

$$\phi(E_6) = (a_1 a_2 a_3 a_4 a_5 a_6) \quad \text{and} \quad \bar{\phi}(E_6) = (a_5 a_4 a_3 a_2 a_1 a_6)$$

A direct sum of selfcontragredient, orthogonal, symplectic representations is respectively selfcontragredient, orthogonal, or symplectic.

The direct sum $\phi(L) \oplus \bar{\phi}(L)$ of two mutually contragredient representations is both orthogonal and symplectic, i.e., there exists a symmetric as well as skew symmetric invariant bilinear form in the corresponding representation space $S_{\phi\oplus\bar{\phi}}$.

An irreducible representation

$$(a_1^1 a_2^1 \ldots a_{n_1}^1)(a_1^2 a_a^2 \ldots a_{n_2}^2) \ldots (a_1^k a_2^k \ldots a_{n_k}^k) \tag{1.14}$$

of a nonsimple algebra (1.1) is selfcontragredient if all the representations

$$(a_1^i a_2^i \ldots a_{n_i}^i), \qquad i = 1,2,\ldots,k \tag{1.15}$$

are selfcontragredient. A selfcontragredient representation (1.14) is orthogonal (symplectic) if it contains an even (odd) number of symplectic components (1.15).

Congruence Classes of Representations

Another convenient characterization of an irreducible representation of a semisimple algebra L is by its congruence class [6]. A particular case of that property well known in particle physics is the triality of SU(3) representations.

We say that two irreducible representations Λ and Λ' of a semisimple Lie algebra L belong to the same congruence class or are mutually congruent if and only if the difference $\Lambda - \Lambda'$ of their highest weights is an integer linear combination of simple roots of L. Each congruence class is characterized by a value of the congruence number c which, for an irreducible representation $(a_1 a_2 \ldots a_n)$ of a simple Lie algebra, is given by

$$c = \sum_{k=1}^{n} a_n \pmod{n+1} \qquad \text{for } A_n \tag{1.16}$$

$$c = a_n \pmod{2} \qquad \text{for } B_n \tag{1.17}$$

$$c = a_1 + a_3 + a_5 + \ldots \pmod{2} \qquad \text{for } C_n \tag{1.18}$$

$$c = a_1 - a_2 + a_4 - a_5 \pmod{3} \qquad \text{for } E_6 \tag{1.19}$$

$$c = a_4 + a_6 + a_7 \pmod{2} \qquad \text{for } E_7 \tag{1.20}$$

$$c = 0 \qquad \text{for all representations of } E_8,\ F_4, \text{ and } G_2 \tag{1.21}$$

Finally, when L is of the type D_n the congruence number c is a two component vector

$$c = (a_{n-1} + a_n,\ 2a_1 + 2a_3 + \ldots + 2a_{n-2} + (n-2)a_{n-1} + na_n) \quad \text{for } n \text{ odd} \tag{1.22}$$

$$c = (a_{n-1} + a_n,\ 2a_1 + 2a_3 + \ldots + 2a_{n-3} + (n-2)a_{n-1} + na_n) \quad \text{for } n \text{ even} \tag{1.23}$$

In (1.22) and (1.23) the first component is calculated mod 2 and the second one mod 4. One easily verifies that there are four congruence