

Series in Mathematical Analysis and Applications

Edited by Ravi P. Agarwal and Donal O'Regan

VOLUME 8

**NONSMOOTH CRITICAL
POINT THEORY AND
NONLINEAR BOUNDARY
VALUE PROBLEMS**

Leszek Gasiński and Nikolaos S. Papageorgiou



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SERIES IN MATHEMATICAL ANALYSIS AND APPLICATIONS

Series in Mathematical Analysis and Applications (SIMAA) is edited by Ravi P. Agarwal, Florida Institute of Technology, USA and Donal O'Regan, National University of Ireland, Galway, Ireland.

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Volume 8

Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems

Leszek Gasiński and Nikolaos S. Papageorgiou

To Krystyna, Halszka and Krystyna (LG)
To my brother A.S. Papageorgiou (NSP)

Preface

Variational methods have turned out to be a very effective analytical tool in the study of nonlinear problems. The idea behind them is to try to find solutions of a given boundary value problem by looking for critical (stationary) points of a suitable “energy” functional defined on an appropriate function space dictated by the data of the problem. Then the boundary value problem under consideration is the Euler-Lagrange equation satisfied by a critical point. In many cases of interest, the energy functional is unbounded (from both above and below; indefinite functional) and so we cannot hope for a global maximum or minimum. Therefore we must look for local extrema and for saddle points obtained by minimax arguments.

One useful technique in obtaining critical points is based on deformations along the paths of steepest descent of the energy functionals. Another approach can be based on the Ekeland variational principle. The classical critical point theory was developed in the sixties and seventies for C^1 -functionals. The needs of specific applications (such as nonsmooth mechanics, nonsmooth gradient systems, mathematical economics, etc.) and the impressive progress in nonsmooth analysis and multivalued analysis led to extensions of the critical point theory to nondifferentiable functions, in particular locally Lipschitz and even continuous functions. The resulting theory succeeded in extending a big part of the smooth (C^1) theory.

In this book, we present the existing nonsmooth critical point theories (Chapter 2) and use them to study nonlinear boundary value problems of ordinary and partial (elliptic) differential equations, which are in variational form. We also investigate nonlinear boundary value problems (BVPs) in nonvariational form, using a great variety of methods and techniques which involve upper-lower solutions, fixed point and degree theories, nonlinear operator theory, nonsmooth analysis, and multivalued analysis (Chapter 3 and Chapter 4). The necessary mathematical background to understand these methods is developed in Chapter 1 (see also the Appendix). This way we present a large part of the methods used today in the study of nonlinear boundary value problems with nonsmooth and multivalued terms.

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Contents

1	Mathematical Background	1
1.1	Sobolev Spaces	2
1.1.1	Basic Definitions and Properties	2
1.1.2	Embedding Theorems	6
1.1.3	Poincaré Inequality	7
1.1.4	Dual Space	8
1.1.5	Green Formula	9
1.1.6	One Dimensional Sobolev Spaces	10
1.2	Set-Valued Analysis	14
1.2.1	Upper and Lower Semicontinuity	15
1.2.2	h -Lower and h -Upper Semicontinuity	18
1.2.3	Measurability of Multifunctions	21
1.2.4	Measurable Selections	22
1.2.5	Continuous Selections	24
1.2.6	Convergence in the Kuratowski Sense	28
1.3	Nonsmooth Analysis	32
1.3.1	Convex Functions	32
1.3.2	Fenchel Transform	38
1.3.3	Subdifferential of Convex Functions	42
1.3.4	Clarke Subdifferential	48
1.3.5	Weak Slope	62
1.4	Nonlinear Operators	67
1.4.1	Compact Operators	68
1.4.2	Maximal Monotone Operators	70
1.4.3	Yosida Approximation	81
1.4.4	Pseudomonotone Operators	83
1.4.5	Nemytskii Operators	87
1.4.6	Ekeland Variational Principle	89
1.5	Elliptic Differential Equations	93
1.5.1	Ordinary Differential Equations	93
1.5.2	Partial Differential Equations	99
1.5.3	Regularity Results	112
1.6	Remarks	120

2	Critical Point Theory	123
2.1	Locally Lipschitz Functionals	123
2.1.1	Compactness-Type Conditions	123
2.1.2	Critical Points and Deformation Theorem	128
2.1.3	Linking Sets	136
2.1.4	Minimax Principles	138
2.1.5	Existence of Multiple Critical Points	145
2.2	Constrained Locally Lipschitz Functionals	149
2.2.1	Critical Points of Constrained Functions	149
2.2.2	Deformation Theorem	151
2.2.3	Minimax Principles	154
2.3	Perturbations of Locally Lipschitz Functionals	159
2.3.1	Critical Points of Perturbed Functions	159
2.3.2	Generalized Deformation Theorem	162
2.3.3	Minimax Principles	166
2.4	Local Linking and Extensions	171
2.4.1	Local Linking	171
2.4.2	Minimax Principles	181
2.4.3	Palais-Smale-Type Conditions	184
2.5	Continuous Functionals	187
2.5.1	Compactness-Type Conditions	187
2.5.2	Deformation Theorem	188
2.5.3	Minimax Principles	195
2.6	Multivalued Functionals	197
2.6.1	Compactness-Type Conditions	197
2.6.2	Multivalued Deformation Theorem	199
2.6.3	Minimax Principles	200
2.7	Remarks	203
3	Ordinary Differential Equations	207
3.1	Dirichlet Problems	208
3.1.1	Formulation of the Problem	208
3.1.2	Approximation of the Problem	212
3.1.3	Existence Results	225
3.1.4	Problems with Non-Convex Nonlinearities	231
3.2	Periodic Problems	233
3.2.1	Auxiliary Problems	233
3.2.2	Formulation of the Problem	239
3.2.3	Approximation of the Problem	240
3.2.4	Existence Results	249
3.2.5	Problems with Non-Convex Nonlinearities	258
3.2.6	Scalar Problems	259
3.3	Nonlinear Boundary Conditions	268

3.4	Variational Methods	283
3.4.1	Existence Theorems	284
3.4.2	Homoclinic Solutions	308
3.4.3	Scalar Problems	317
3.4.4	Multiple Periodic Solutions	343
3.4.5	Nonlinear Eigenvalue Problems	353
3.4.6	Problems with Nonlinear Boundary Conditions	356
3.4.7	Multiple Solutions for “Smooth” Problems	365
3.5	Method of Upper and Lower Solutions	372
3.6	Positive Solutions and Other Methods	389
3.6.1	Positive Solutions	389
3.6.2	Method Based on Monotone Operators	403
3.7	Hamiltonian Inclusions	414
3.8	Remarks	448
4	Elliptic Equations	453
4.1	Problems at Resonance	454
4.1.1	Semilinear Problems at Resonance	455
4.1.2	Nonlinear Problems at Resonance	474
4.1.3	Variational-Hemivariational Inequality at Resonance	485
4.1.4	Strongly Resonant Problems	495
4.2	Neumann Problems	500
4.2.1	Spectrum of $(-\Delta_p, W^{1,p}(\Omega))$	501
4.2.2	Homogeneous Neumann Problems	516
4.2.3	Nonhomogeneous Neumann Problem	526
4.3	Problems with an Area-Type Term	536
4.4	Strongly Nonlinear Problems	558
4.5	Method of Upper and Lower Solutions	586
4.5.1	Existence of Solutions	587
4.5.2	Existence of Extremal Solutions	598
4.6	Multiplicity Results	608
4.6.1	Semilinear Problems	608
4.6.2	Nonlinear Problems	640
4.7	Positive Solutions	661
4.8	Problems with Discontinuous Nonlinearities	683
4.9	Remarks	700
A	Appendix	707
A.1	Set Theory and Topology	707
A.2	Measure Theory	714
A.3	Functional Analysis	718
A.4	Nonlinear Analysis	724

List of Symbols	727
References	735
Index	763

Chapter 1

Mathematical Background

In this chapter, we review the basic mathematical material that we need in the development of the nonsmooth critical point theories and in the study of the nonlinear boundary value problems (ordinary and partial) that follow. So in the first section we outline the basic facts about Sobolev spaces. Sobolev spaces provide the appropriate functional framework for the analysis of the ordinary and partial differential equations problems that we consider in this volume. The subdifferential of a nonsmooth (nondifferentiable) function is a multivalued map. So the resulting nonsmooth critical point theories and the corresponding boundary value problems are of multivalued nature, since the potential function is nonsmooth. Moreover, in our formulation of the problems we allow the nonlinear perturbation term to be set-valued. Therefore, to handle such problems we need to know a few basic facts about Set-Valued Analysis. In Section 1.2 we review from the theory the main items that will be helpful in what follows. Since one of our goals in this volume is to present the main facts about the existing nonsmooth critical point theories, we need the notions and results of Nonsmooth Analysis. In Section 1.3, we review the main items of Nonsmooth Analysis, which are needed for what follows. Nonsmooth Analysis is closely related to Set-Valued Analysis and to the theory of nonlinear operators. Set-Valued Analysis has already been covered in Section 1.2. So in Section 1.4 we deal with nonlinear operators, with particular emphasis on operators of monotone type. We also discuss briefly the Nemytskii (superposition) operator and present various forms of the Ekeland Variational Principle. Finally in Section 1.5, we present some basic facts about semilinear and nonlinear elliptic equations. Our starting point is the derivation of the spectra of the ordinary and partial Laplacian and p -Laplacian differential operators under Dirichlet and periodic boundary conditions. We also consider certain weighted eigenvalue problems driven by a strongly elliptic linear partial differential operator. We establish the existence of eigenvalues, provide variational characterizations of them (via the Rayleigh quotient) and examine the corresponding eigenfunctions. This analysis is based on some regularity results and maximum principles that we also present.

1.1 Sobolev Spaces

For the reader's convenience, in this section we present a quick review of the theory of Sobolev spaces. The results that we present here are standard and their proofs as well as a more detailed and deeper analysis can be found in several classical textbooks on the subject such as Adams (1975), Brézis (1983) and Kufner, John & Fučík (1977).

1.1.1 Basic Definitions and Properties

Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open set. By $\partial\Omega$ we denote the **boundary** of Ω , i.e. $\partial\Omega \stackrel{df}{=} \overline{\Omega} \cap \Omega^c = \overline{\Omega} \setminus \Omega$. Also we say that another open set Ω' is **strongly included** in Ω , denoted by $\Omega' \subset\subset \Omega$, if Ω' is bounded and $\overline{\Omega'} \subseteq \Omega$. For a **multi-index** $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, by $|\alpha|$ we denote the **length of the multi-index**, defined by

$$|\alpha| \stackrel{df}{=} \sum_{k=1}^N \alpha_k$$

and by $D^\alpha u$ we denote the **weak derivative** of u of order α , i.e.

$$D^\alpha u \stackrel{df}{=} \frac{\partial^{|\alpha|} u}{\partial z_1^{\alpha_1} \dots \partial z_N^{\alpha_N}}.$$

By $C_c^\infty(\Omega)$ we denote the space of functions $\vartheta \in C^\infty(\Omega)$ for which their **support**, defined by

$$\text{supp } \vartheta \stackrel{df}{=} \overline{\{x \in \Omega : \vartheta(x) \neq 0\}},$$

is a compact set contained in Ω . We furnish $C_c^\infty(\Omega)$ with a convergence notion according to which $\{\vartheta_n\}_{n \geq 1} \subseteq C_c^\infty(\Omega)$ converges to 0 if and only if there exists a compact set $K \subseteq \Omega$, such that

$$\bigcup_{n \geq 1} \text{supp } \vartheta_n \subseteq K$$

and the sequence $\{D^\alpha \vartheta_n\}_{n \geq 1}$ converges uniformly to 0 for all $\alpha \in \mathbb{N}_0^N$. Usually $C_c^\infty(\Omega)$ equipped with this convergence notion is denoted by $\mathcal{D}(\Omega)$ and is known as the **space of test functions**. Recall that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, +\infty)$. By $\mathcal{D}'(\Omega)$ we denote the **space of distributions**, i.e. the space of all linear maps $L: \mathcal{D}(\Omega) \rightarrow \mathbb{R}$, such that $L(\vartheta_n) \rightarrow 0$ for all $\{\vartheta_n\}_{n \geq 1} \subseteq \mathcal{D}(\Omega)$, such that $\vartheta_n \rightarrow 0$. For a given distribution $L \in \mathcal{D}'(\Omega)$ and for all $\alpha \in \mathbb{N}_0^N$, we define the distribution $D^\alpha L$ by

$$D^\alpha L(\vartheta) \stackrel{df}{=} (-1)^{|\alpha|} L(D^\alpha \vartheta) \quad \forall \vartheta \in \mathcal{D}(\Omega).$$

For every $u \in L^1_{\text{loc}}(\Omega)$, we can introduce the so-called **regular distribution** L_u by

$$L_u(\vartheta) \stackrel{\text{df}}{=} \int_{\Omega} u(x)\vartheta(x)dx \quad \forall \vartheta \in \mathcal{D}(\Omega).$$

We have $L_u = L_v$ if and only if $u(x) = v(x)$ for almost all $x \in \Omega$. For given $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^N$ we write $v = D^\alpha u$ to express the equality $L_v = D^\alpha L_u$. So it is equivalent to saying that

$$\int_{\Omega} v(x)\vartheta(x)dx = (-1)^{|\alpha|} \int_{\Omega} u(x)D^\alpha \vartheta(x)dx \quad \forall \vartheta \in \mathcal{D}(\Omega).$$

We say that $D^\alpha u \in L^1_{\text{loc}}(\Omega)$, if we can find $v \in L^1_{\text{loc}}(\Omega)$, such that $D^\alpha u = v$. We say that $D^\alpha u \in L^p(\Omega)$ (with $1 \leq p \leq +\infty$), if we can find $v \in L^p(\Omega)$, such that $D^\alpha u = v$. Note that, if $u \in C^{|\alpha|}(\Omega)$, then this generalized derivative coincides with the usual (classical) partial derivative.

DEFINITION 1.1.1 For $m \in \mathbb{N}_0 \stackrel{\text{df}}{=} \mathbb{N} \cup \{0\}$ and $1 \leq p \leq +\infty$, we define the **Sobolev space**

$$W^{m,p}(\Omega) \stackrel{\text{df}}{=} \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \leq m\}.$$

For every $u \in W^{m,p}(\Omega)$, we define

$$\|u\|_{W^{m,p}(\Omega)} \stackrel{\text{df}}{=} \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < +\infty,$$

where $\|\cdot\|_p$ is the norm of $L^p(\Omega)$, and

$$\|u\|_{W^{m,\infty}(\Omega)} \stackrel{\text{df}}{=} \sum_{|\alpha| \leq m} \|D^\alpha u\|_{\infty},$$

where $\|\cdot\|_{\infty}$ is the norm of $L^{\infty}(\Omega)$. We also set

$$W_0^{m,p}(\Omega) \stackrel{\text{df}}{=} \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}.$$

REMARK 1.1.1 The space $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is a Banach space, which is reflexive and uniformly convex if $p \in (1, +\infty)$ and separable if $p \in [1, +\infty)$. $(W_0^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is a closed subspace of $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$. If $p = 2$, we write

$$H^m(\Omega) \stackrel{\text{df}}{=} W^{m,2}(\Omega) \quad \text{and} \quad H_0^m(\Omega) \stackrel{\text{df}}{=} W_0^{m,2}(\Omega).$$

These spaces are Hilbert spaces with inner product given by

$$(u, v)_{H^m(\Omega)} \stackrel{df}{=} \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_2 = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u(x) D^\alpha v(x) dx.$$

□

The next theorem is known as the Meyers-Serrin Theorem and it says that Sobolev functions can be approximated by smooth ones.

THEOREM 1.1.1 (Meyers-Serrin Theorem)

If $\Omega \subseteq \mathbb{R}^N$ is open, $m \in \mathbb{N}_0$ and $p \in [1, +\infty)$,
then $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.

REMARK 1.1.2 Note that in Theorem 1.1.1 we do not claim that the approximating sequence of smooth functions belongs in $C^\infty(\overline{\Omega})$. To be able to approximate Sobolev functions by functions which are smooth all the way up to the boundary, we need to strengthen our hypotheses about the geometry of Ω . □

DEFINITION 1.1.2 We say that the boundary $\partial\Omega$ of an open set $\Omega \subseteq \mathbb{R}^N$ is **Lipschitz**, if for each $x = (x_1, \dots, x_N) \in \partial\Omega$, there exist $r > 0$ and a Lipschitz continuous map $\gamma: \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ which, after rotation and relabelling of the coordinate axes if necessary, satisfies

$$\Omega \cap C_r(x) = \{ (y_1, \dots, y_N) \in \mathbb{R}^N : \gamma(y_1, \dots, y_{N-1}) < y_N \} \cap C_r(x),$$

where

$$C_r(x) \stackrel{df}{=} \{ (y_1, \dots, y_N) \in \mathbb{R}^N : |x_k - y_k| < r \text{ for } k \in \{1, \dots, N\} \}.$$

REMARK 1.1.3 So $\partial\Omega$ is Lipschitz, if locally it is the graph of a Lipschitz continuous function. By Rademacher's theorem (see Theorem A.2.4), the outer unit normal $n(z)$ to Ω exists for almost all $z \in \partial\Omega$ (on $\partial\Omega$ we consider the $(N-1)$ -dimensional Hausdorff (surface) measure; see Definition A.2.3). □

Using this notion we can have a stronger approximation result by smooth functions.

THEOREM 1.1.2

If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with Lipschitz boundary $\partial\Omega$ and $u \in W^{1,p}(\Omega)$ with $p \in [1, +\infty)$,

then we can find a sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$, such that $u_n \rightarrow u$ in $W^{1,p}(\Omega)$.

The next theorem (known as the **Trace Theorem**), for every $u \in W^{m,p}(\Omega)$ assigns a meaning to expressions like $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial n}$ (the normal derivative on $\partial\Omega$). Because in general the N -dimensional Lebesgue measure of $\partial\Omega$ is zero, it is not meaningful to talk *a priori* of $u|_{\partial\Omega}$ when $u \in W^{1,p}(\Omega)$, unless u is at least continuous. So we have to generalize the meaning of boundary values for Sobolev functions.

THEOREM 1.1.3 (Trace Theorem)

If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $p \in [1, +\infty)$, then there exists a unique continuous linear operator

$$\gamma_0: W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega),$$

such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega})$. We say that $\gamma_0(u)$ is the **trace** of $u \in W^{1,p}(\Omega)$ on $\partial\Omega$.

REMARK 1.1.4 For a bounded open set $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary, we have

$$\ker \gamma_0 = W_0^{1,p}(\Omega).$$

The range of γ_0 is less than $L^p(\partial\Omega)$. There are functions $v \in L^p(\partial\Omega)$ which are not the trace of an element $u \in W^{1,p}(\Omega)$. More precisely

$$\gamma_0(W^{1,p}(\Omega)) = W^{1-\frac{1}{p},p}(\partial\Omega),$$

where $v \in W^{1-\frac{1}{p},p}(\partial\Omega)$ if and only if $v \in L^p(\partial\Omega)$ and $\|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} < +\infty$, with

$$\|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \stackrel{df}{=} \left(\int_{\partial\Omega} |v(x)|^p d\sigma(x) + \int_{\partial\Omega \times \partial\Omega} \frac{|v(x) - v(x')|^p}{|x - x'|^{N+p-2}} d\sigma(x) d\sigma(x') \right)^{\frac{1}{p}}.$$

□

Clearly a function $u \in W_0^{1,p}(\Omega)$ can be extended by zero to a Sobolev function on all \mathbb{R}^N . Can we do this for any Sobolev function $u \in W^{1,p}(\Omega)$?

THEOREM 1.1.4 (Extension Theorem)

If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set and $\partial\Omega$ is Lipschitz, then there exists a bounded linear operator

$$E: W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N),$$