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Leszek Gasiński and Nikolaos S. Papageorgiou

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Volume 8
Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems
Leszek Gasiński and Nikolaos S. Papageorgiou

To Krystyna, Halszka and Krystyna (LG)
To my brother A.S. Papageorgiou (NSP)

Preface

Variational methods have turned out to be a very effective analytical tool in the study of nonlinear problems. The idea behind them is to try to find solutions of a given boundary value problem by looking for critical (stationary) points of a suitable "energy" functional defined on an appropriate function space dictated by the data of the problem. Then the boundary value problem under consideration is the Euler-Lagrange equation satisfied by a critical point. In many cases of interest, the energy functional is unbounded (from both above and below; indefinite functional) and so we cannot hope for a global maximum or minimum. Therefore we must look for local extrema and for saddle points obtained by minimax arguments.

One useful technique in obtaining critical points is based on deformations along the paths of steepest descent of the energy functionals. Another approach can be based on the Ekeland variational principle. The classical critical point theory was developed in the sixties and seventies for C^1-functionals. The needs of specific applications (such as nonsmooth mechanics, nonsmooth gradient systems, mathematical economics, etc.) and the impressive progress in nonsmooth analysis and multivalued analysis led to extensions of the critical point theory to nondifferentiable functions, in particular locally Lipschitz and even continuous functions. The resulting theory succeeded in extending a big part of the smooth (C^1) theory.

In this book, we present the existing nonsmooth critical point theories (Chapter 2) and use them to study nonlinear boundary value problems of ordinary and partial (elliptic) differential equations, which are in variational form. We also investigate nonlinear boundary value problems (BVPs) in nonvariational form, using a great variety of methods and techniques which involve upper-lower solutions, fixed point and degree theories, nonlinear operator theory, nonsmooth analysis, and multivalued analysis (Chapter 3 and Chapter 4). The necessary mathematical background to understand these methods is developed in Chapter 1 (see also the Appendix). This way we present a large part of the methods used today in the study of nonlinear boundary value problems with nonsmooth and multivalued terms.

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Contents

Chapter 1

Mathematical Background

In this chapter, we review the basic mathematical material that we need in the development of the nonsmooth critical point theories and in the study of the nonlinear boundary value problems (ordinary and partial) that follow. So in the first section we outline the basic facts about Sobolev spaces. Sobolev spaces provide the appropriate functional framework for the analysis of the ordinary and partial differential equations problems that we consider in this volume. The subdifferential of a nonsmooth (nondifferentiable) function is a multivalued map. So the resulting nonsmooth critical point theories and the corresponding boundary value problems are of multivalued nature, since the potential function is nonsmooth. Moreover, in our formulation of the problems we allow the nonlinear perturbation term to be set-valued. Therefore, to handle such problems we need to know a few basic facts about Set-Valued Analysis. In Section 1.2 we review from the theory the main items that will be helpful in what follows. Since one of our goals in this volume is to present the main facts about the existing nonsmooth critical point theories, we need the notions and results of Nonsmooth Analysis. In Section 1.3, we review the main items of Nonsmooth Analysis, which are needed for what follows. Nonsmooth Analysis is closely related to Set-Valued Analysis and to the theory of nonlinear operators. Set-Valued Analysis has already been covered in Section 1.2. So in Section 1.4 we deal with nonlinear operators, with particular emphasis on operators of monotone type. We also discuss briefly the Nemytskii (superposition) operator and present various forms of the Ekeland Variational Principle. Finally in Section 1.5, we present some basic facts about semilinear and nonlinear elliptic equations. Our starting point is the derivation of the spectra of the ordinary and partial Laplacian and p-Laplacian differential operators under Dirichlet and periodic boundary conditions. We also consider certain weighted eigenvalue problems driven by a strongly elliptic linear partial differential operator. We establish the existence of eigenvalues, provide variational characterizations of them (via the Rayleigh quotient) and examine the corresponding eigenfunctions. This analysis is based on some regularity results and maximum principles that we also present.

1.1 Sobolev Spaces

For the reader's convenience, in this section we present a quick review of the theory of Sobolev spaces. The results that we present here are standard and their proofs as well as a more detailed and deeper analysis can be found in several classical textbooks on the subject such as Adams (1975), Brézis (1983) and Kufner, John & Fučik (1977).

1.1.1 Basic Definitions and Properties

Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open set. By $\partial\Omega$ we denote the ***boundary*** of Ω, i.e. $\partial\Omega \stackrel{df}{=} \overline{\Omega} \cap \Omega^c = \overline{\Omega} \setminus \Omega$. Also we say that another open set Ω' is ***strongly included*** in Ω, denoted by $\Omega' \subset\subset \Omega$, if Ω' is bounded and $\overline{\Omega'} \subseteq \Omega$. For a ***multi-index*** $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, by $|\alpha|$ we denote the ***length of the multi-index***, defined by

$$|\alpha| \stackrel{df}{=} \sum_{k=1}^{N} \alpha_k$$

and by $D^\alpha u$ we denote the ***weak derivative*** of u of order α, i.e.

$$D^\alpha u \stackrel{df}{=} \frac{\partial^{|\alpha|} u}{\partial z_1^{\alpha_1} \ldots \partial z_N^{\alpha_N}}.$$

By $C_c^\infty(\Omega)$ we denote the space of functions $\vartheta \in C^\infty(\Omega)$ for which their ***support***, defined by

$$\operatorname{supp} \vartheta \stackrel{df}{=} \overline{\{x \in \Omega : \vartheta(x) \neq 0\}},$$

is a compact set contained in Ω. We furnish $C_c^\infty(\Omega)$ with a convergence notion according to which $\{\vartheta_n\}_{n\geq 1} \subseteq C_c^\infty(\Omega)$ converges to 0 if and only if there exists a compact set $K \subseteq \Omega$, such that

$$\bigcup_{n\geq 1} \operatorname{supp} \vartheta_n \subseteq K$$

and the sequence $\{D^\alpha \vartheta_n\}_{n\geq 1}$ converges uniformly to 0 for all $\alpha \in \mathbb{N}_0^N$. Usually $C_c^\infty(\Omega)$ equipped with this convergence notion is denoted by $\mathcal{D}(\Omega)$ and is known as the ***space of test functions***. Recall that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, +\infty)$. By $\mathcal{D}'(\Omega)$ we denote the ***space of distributions***, i.e. the space of all linear maps $L \colon \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$, such that $L(\vartheta_n) \longrightarrow 0$ for all $\{\vartheta_n\}_{n\geq 1} \subseteq \mathcal{D}(\Omega)$, such that $\vartheta_n \longrightarrow 0$. For a given distribution $L \in \mathcal{D}'(\Omega)$ and for all $\alpha \in \mathbb{N}_0^N$, we define the distribution $D^\alpha L$ by

$$D^\alpha L(\vartheta) \stackrel{df}{=} (-1)^{|\alpha|} L(D^\alpha \vartheta) \qquad \forall\, \vartheta \in \mathcal{D}(\Omega).$$

For every $u \in L^1_{\text{loc}}(\Omega)$, we can introduce the so-called ***regular distribution*** L_u by

$$L_u(\vartheta) \stackrel{df}{=} \int_\Omega u(x)\vartheta(x)dx \qquad \forall\, \vartheta \in \mathcal{D}(\Omega).$$

We have $L_u = L_v$ if and only if $u(x) = v(x)$ for almost all $x \in \Omega$. For given $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^N$ we write $v = D^\alpha u$ to express the equality $L_v = D^\alpha L_u$. So it is equivalent to saying that

$$\int_\Omega v(x)\vartheta(x)dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\vartheta(x)dx \qquad \forall\, \vartheta \in \mathcal{D}(\Omega).$$

We say that $D^\alpha u \in L^1_{\text{loc}}(\Omega)$, if we can find $v \in L^1_{\text{loc}}(\Omega)$, such that $D^\alpha u = v$. We say that $D^\alpha u \in L^p(\Omega)$ (with $1 \leq p \leq +\infty$), if we can find $v \in L^p(\Omega)$, such that $D^\alpha u = v$. Note that, if $u \in C^{|\alpha|}(\Omega)$, then this generalized derivative coincides with the usual (classical) partial derivative.

DEFINITION 1.1.1 *For $m \in \mathbb{N}_0 \stackrel{df}{=} \mathbb{N} \cup \{0\}$ and $1 \leq p \leq +\infty$, we define the* ***Sobolev space***

$$W^{m,p}(\Omega) \stackrel{df}{=} \left\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \leq m\right\}.$$

For every $u \in W^{m,p}(\Omega)$, we define

$$\|u\|_{W^{m,p}(\Omega)} \stackrel{df}{=} \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < +\infty,$$

where $\|\cdot\|_p$ is the norm of $L^p(\Omega)$, and

$$\|u\|_{W^{m,\infty}(\Omega)} \stackrel{df}{=} \sum_{|\alpha|\leq m} \|D^\alpha u\|_\infty,$$

where $\|\cdot\|_\infty$ is the norm of $L^\infty(\Omega)$. We also set

$$W_0^{m,p}(\Omega) \stackrel{df}{=} \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}.$$

REMARK 1.1.1 The space $\left(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)}\right)$ is a Banach space, which is reflexive and uniformly convex if $p \in (1, +\infty)$ and separable if $p \in [1, +\infty)$. $\left(W_0^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)}\right)$ is a closed subspace of $\left(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)}\right)$. If $p = 2$, we write

$$H^m(\Omega) \stackrel{df}{=} W^{m,2}(\Omega) \quad \text{and} \quad H_0^m(\Omega) \stackrel{df}{=} W_0^{m,2}(\Omega).$$

These spaces are Hilbert spaces with inner product given by

$$(u,v)_{H^m(\Omega)} \stackrel{df}{=} \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)_2 = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x) D^\alpha v(x)dx.$$

□

The next theorem is known as the Meyers-Serrin Theorem and it says that Sobolev functions can be approximated by smooth ones.

THEOREM 1.1.1 (Meyers-Serrin Theorem)
If $\Omega \subseteq \mathbb{R}^N$ *is open,* $m \in \mathbb{N}_0$ *and* $p \in [1,+\infty)$,
then $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ *is dense in* $W^{m,p}(\Omega)$.

REMARK 1.1.2 Note that in Theorem 1.1.1 we do not claim that the approximating sequence of smooth functions belongs in $C^\infty(\overline{\Omega})$. To be able to approximate Sobolev functions by functions which are smooth all the way up to the boundary, we need to strengthen our hypotheses about the geometry of Ω. □

DEFINITION 1.1.2 *We say that the boundary $\partial\Omega$ of an open set $\Omega \subseteq \mathbb{R}^N$ is* ***Lipschitz****, if for each $x = (x_1,\ldots,x_N) \in \partial\Omega$, there exist $r > 0$ and a Lipschitz continuous map $\gamma\colon \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$ which, after rotation and relabelling of the coordinate axes if necessary, satisfies*

$$\Omega \cap C_r(x) = \{ (y_1,\ldots,y_N) \in \mathbb{R}^N : \gamma(y_1,\ldots,y_{N-1}) < y_N \} \cap C_r(x),$$

where

$$C_r(x) \stackrel{df}{=} \{ (y_1,\ldots,y_N) \in \mathbb{R}^N : |x_k - y_k| < r \text{ for } k \in \{1,\ldots,N\} \}.$$

REMARK 1.1.3 So $\partial\Omega$ is Lipschitz, if locally it is the graph of a Lipschitz continuous function. By Rademacher's theorem (see Theorem A.2.4), the outer unit normal $n(z)$ to Ω exists for almost all $z \in \partial\Omega$ (on $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure; see Definition A.2.3). □

Using this notion we can have a stronger approximation result by smooth functions.

THEOREM 1.1.2
If $\Omega \subseteq \mathbb{R}^N$ *is a bounded open set with Lipschitz boundary* $\partial\Omega$ *and* $u \in W^{1,p}(\Omega)$ *with* $p \in [1,+\infty)$,

then we can find a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)\cap C^\infty(\overline{\Omega})$, such that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

The next theorem (known as the ***Trace Theorem***), for every $u \in W^{m,p}(\Omega)$ assigns a meaning to expressions like $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial n}$ (the normal derivative on $\partial\Omega$). Because in general the N-dimensional Lebesgue measure of $\partial\Omega$ is zero, it is not meaningful to talk *a priori* of $u|_{\partial\Omega}$ when $u \in W^{1,p}(\Omega)$, unless u is at least continuous. So we have to generalize the meaning of boundary values for Sobolev functions.

THEOREM 1.1.3 (Trace Theorem)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $p \in [1,+\infty)$, then there exists a unique continuous linear operator

$$\gamma_0\colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega),$$

*such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega})$. We say that $\gamma_0(u)$ is the **trace** of $u \in W^{1,p}(\Omega)$ on $\partial\Omega$.*

REMARK 1.1.4 For a bounded open set $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary, we have

$$\ker \gamma_0 = W_0^{1,p}(\Omega).$$

The range of γ_0 is less than $L^p(\partial\Omega)$. There are functions $v \in L^p(\partial\Omega)$ which are not the trace of an element $u \in W^{1,p}(\Omega)$. More precisely

$$\gamma_0\big(W^{1,p}(\Omega)\big) = W^{1-\frac{1}{p},p}(\partial\Omega),$$

where $v \in W^{1-\frac{1}{p},p}(\partial\Omega)$ if and only if $v \in L^p(\partial\Omega)$ and $\|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} < +\infty$, with

$$\|v\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \stackrel{df}{=} \left(\int\limits_{\partial\Omega} |v(x)|^p d\sigma(x) + \int\limits_{\partial\Omega\times\partial\Omega} \frac{|v(x)-v(x')|}{|x-x'|^{N+p-2}} d\sigma(x)d\sigma(x')\right)^{\frac{1}{p}}.$$

□

Clearly a function $u \in W_0^{1,p}(\Omega)$ can be extended by zero to a Sobolev function on all $\mathbb{R}^N$. Can we do this for any Sobolev function $u \in W^{1,p}(\Omega)$?

THEOREM 1.1.4 (Extension Theorem)
If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set and $\partial\Omega$ is Lipschitz, then there exists a bounded linear operator

$$E\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N),$$