# Electromagnetic Wave Theory

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# Preface

*Electromagnetic Wave Theory* is designed for an advanced course in electromagnetic waves at the senior undergraduate or first-year graduate level in an electrical engineering curriculum. It is assumed that an individual reading this book will have had at least one course in electromagnetics with knowledge of static fields, as well as an introduction to Maxwell's equations. Additionally, this book should appeal to practicing engineers as a useful reference. An attempt is made to present the subject in a self-contained format. The essential mathematics, such as vector analysis and special functions, are reviewed in a concise manner.

A central theme of the book is the impedance concept applied to the description of wave phenomena. This point of view has its origin in the profound ideas of Sergei Schelkunoff and Henry Booker in the 1940s. This approach is very fruitful in systematizing the solutions of a wide class of boundary value problems. We do not try to be comprehensive, but a representative selection of topics that yields to this approach is considered. In particular, problems of current relevance to radio wave propagation and antennas are emphasized. Some of these are drawn from the author's own research, but an attempt has been made to put the results in a general context.

Taking a somewhat traditional viewpoint, the book begins with a review of stationary fields. The concept of potential is important if one is to really appreciate the meaning of static-like fields. It is not sufficient just to let the frequency tend to zero in dynamic formulations. Also, the boundary value problems for stationary fields can be best illustrated when the fields can be derived from a scalar potential. This sets the scene for later vector formulations under truly dynamic conditions.

A somewhat novel departure from many texts on EM theory is that we develop many of the essential ideas in wave propagation by treating purely scalar problems in one, two, and three dimensions. The loss in generality is later recovered when we address some of the same problems when vector solutions of Maxwell's equations are required.

Not only is the impedance concept expounded, but transmission-line theory is used consistently to formulate boundary value problems even for fairly complicated geometries. Also, many of the wave theory results are interpreted by equivalent transmission line circuits. We feel this helps the reader see the essential unity of the subject.

Exercises for the reader are located at various spots in the text. These are phrased in such a manner that the desired results of the derivation are given. In some cases these exercises allow the reader the opportunity to test his understanding by working out a closely related problem.

A few bits of information should stand as guides for easier reading. The rationalized meter-kilogram-second (MKS) system of units is used consistently. This is now referred to as the System Internationale (S.I.). Equation numbers are in parentheses and cited as such in each chapter. Thus, for example, a reference to (9) in Chapter II refers to equation number 9 in this chapter. In a few cases where references are made in Chapter II to equations in other chapters, an explicit statement is made accordingly. References to the literature are listed at the end of each chapter. They are cited in the text by a number in square brackets. Thus, for example, [10] in Chapter II is the tenth reference cited. This is essentially the IEEE system. Appendixes to individual chapters are identified by lower case letters and cited as such in the foregoing chapter.

I am extremely grateful to Professor E. Bahar, Professor C. Balanis, and the other reviewers for their useful comments and critical remarks concerning the presentation of the derivations. Also, I wish to thank my colleagues at the University of Arizona and the University of Colorado for their support and encouragement over the years. In particular, I would like to mention Professors R. H. Mattson, D. G. Dudley, T. Triffet, and A. Q. Howard in Tucson and Professors D. C. Chang and S. W. Maley in Boulder. Some of the chapters were typed by Joann Main and Robin Voustas in Tucson, for which I am very appreciative.

JAMES R. WAIT

# Contents

# Chapter 1 Electrostatics and Magnetostatics

## 1.1 COULOMB'S LAW

We begin with a review of electrostatic field theory. This review serves as a useful background for treating more realistic and useful time-varying electromagnetic fields.

Electrostatics is founded on the basis that certain fundamental relations exist or characterize the field behavior. These relations have their origin primarily in experimental observations. For example, *Coulomb's law* is a statement that a force $f$ exists between two point charges $q_1$ and $q_2$ that are separated by a distance $r$. This force is found experimentally to obey an inverse square law according to

$$f = \text{constant} \times \frac{q_1 q_2}{r^2} \tag{1.1}$$

In the rationalized MKS (meter-kilogram-second) system of units this constant is $(4\pi\epsilon)^{-1}$, where $\epsilon$ is the permittivity of the dielectric that contains these charges. In this same system of units, which has been accepted internationally, $q_1$ and $q_2$ are expressed in coulombs, $f$ is in newtons, and $\epsilon$ is in farads per meter. Then for free space $\epsilon = \epsilon_0 \simeq 8.86 \times 10^{-12} \simeq 1/36\pi \times 10^{-9}$ F/m.

To be explicit, Coulomb's law in MKS units is

$$F = \frac{q_1 q_2}{4\pi\epsilon r^2} \tag{1.2}$$

where the direction of the force is along the line joining the two charges. As indicated, $f$ is an attractive force when $q_1$ and $q_2$ are of opposite sign (for example, an electron and a proton), but the force is repulsive when $q_1$ and $q_2$ have the same sign.

If we think of $q_1$ as being a source point charge, then the force acting on test charge $q_2$ is

$$f = Eq_2 \tag{1.3}$$

where

$$E = \frac{q_1}{4\pi\epsilon r^2} \tag{1.4}$$

is, by definition, the electric field strength at the point charge $q_2$. This value of $E$ does not depend on the test charge, and, in fact, we may allow $q_2$ to approach zero in the limit.

To indicate the vector nature of the electric field strength **E**, we are led to write

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon r^2}\,\mathbf{r} \tag{1.5}$$

where **r** is the unit vector in the $r$ or radial direction from the charge $q_1$ to the point where the field is to be observed. In what follows we drop the subscript 1 on the charge $q_1$; this procedure should not cause any confusion.

It is evident that **E** associated with a charge $q$ depends on the permittivity $\epsilon$ of the medium. Clearly, in the present context we can define a vector flux density **D** associated with a charge $q$ according to

$$\mathbf{D} = \frac{q}{4\pi r^2}\,\mathbf{r} \tag{1.6}$$

where we may note that $4\pi r^2$ is the area of the enclosed spherical surface of radius $r$. Here the vector flux density or vector displacement has units of coulombs per square meter. Furthermore, **D** does not depend on the permittivity $\epsilon$ of the surrounding medium.

By comparing Equations (1.5) and (1.6), we observe that

$$\mathbf{D} = \epsilon\mathbf{E} \tag{1.7}$$

Actually, this result holds for all isotropic media even when the permittivity $\epsilon$ is dependent on the coordinates (that is, the medium is inhomogeneous).

When the medium is anisotropic, we generalize (1.7) to the form

$$\mathbf{D} = [\epsilon]\mathbf{E} \tag{1.8}$$

which, in fact, is a statement that

$$D_x = \epsilon_{11}E_x + \epsilon_{12}E_y + \epsilon_{13}E_z \tag{1.9}$$
$$D_y = \epsilon_{21}E_x + \epsilon_{22}E_y + \epsilon_{23}E_z \tag{1.10}$$
$$D_z = \epsilon_{31}E_x + \epsilon_{32}E_y + \epsilon_{33}E_z \tag{1.11}$$

In the limiting case where

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon$$

and

$$\epsilon_{ij} = 0 \qquad \text{for } i \neq j$$

the medium becomes isotropic. We defer here any further reference to anisotropic media.

## 1.2 GAUSS' LAW

We now deal with *Gauss' law* by referring to the situation shown in Figure 1.1. As we see, the displacement or flux density vector **D** emanates radially from the charge $q$, and it subtends a local angle $\theta$ with the normal unit vector **n** to a closed surface $S$. The flux through the element $da$ in the surface is

$$d\Psi = D\, da \cos\theta$$

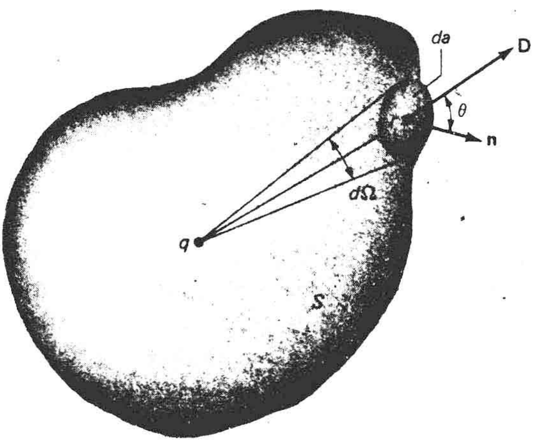


Figure 1.1 Point charge $q$ enclosed by closed surface $S$ and elemental area $da$ of surface.

But by definition, the element of solid angle subtended by $da$ at $q$ is

$$d\Omega = \frac{da \cos \theta}{r^2} \tag{1.12}$$

Thus

$$d\Psi = Dr^2 \, d\Omega \tag{1.13}$$

But

$$D = \frac{q}{4\pi r^2} \tag{1.14}$$

so that we have the incredibly simple result that

$$d\Psi = q \frac{d\Omega}{4\pi} \tag{1.15}$$

If we now integrate over all solid angles, noting that

$$\oint d\Omega = 4\pi \tag{1.16}$$

we deduce that

$$\Psi = q \tag{1.17}$$

This equation is a statement of Gauss' law, which tells us that the total displacement or flux through any closed surface is equal to the amount of charge enclosed.

A simple extension of Equation (1.17), written in vector form, is

$$\oint_S \mathbf{D} \cdot \mathbf{da} = \int_V \rho \, dV \tag{1.18}$$

where $\mathbf{da}$ is a vector aligned with the normal to the surface with area $da$ (note that $\mathbf{D} \cdot \mathbf{da} = D \, da \cos \theta$); on the right-hand side $\rho$ is the charge density in coulombs per cubic meter. Equation (1.18) is a statement of Gauss' law. An alternative form is simply $\nabla \cdot \mathbf{D} = \rho$. Or in the case of a homogeneous medium, $\nabla \cdot \mathbf{E} = \rho/\epsilon$.

## 1.3 POTENTIAL CONCEPT

We may now introduce the concept of potential $V$ at a point resulting from a charge $q$ in coulombs. The potential $V$ can be defined as the work done to move a unit test charge from infinity up to a point a distance $r$ from the source charge $q$. Clearly, the work is obtained from the relation

$$\text{Work} = -\int_{\infty}^{r} E_r \, dr \tag{1.19}$$

where $E_r$ is the force in the radial direction, given by

$$E_r = \frac{q}{4\pi\epsilon r^2} \tag{1.20}$$

Thus on integrating (1.20), we see that

$$V = \frac{q}{4\pi\epsilon r} \tag{1.21}$$

It is the so-called conservative property of static fields that the scalar quantity $V$ is the same for any path drawn from infinity up to the field point at a radial distance $r$ from $q$.

The relationship of electric field and potential follows quite easily. Here we might consider two points separated by a vector distance $\mathbf{ds}$ where the electric field $\mathbf{E}$ is measured. Now the work done $dV$ in moving a unit charge through this infinitesimal distance is clearly

$$dV = -\mathbf{E} \cdot \mathbf{ds} \tag{1.22}$$

But we can also write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = \nabla V \cdot \mathbf{ds} \tag{1.23}$$

where

$$\nabla V = \mathbf{i}_x \frac{\partial V}{\partial x} + \mathbf{i}_y \frac{\partial V}{\partial y} + \mathbf{i}_z \frac{\partial V}{\partial z} \tag{1.24}$$

and

$$\mathbf{ds} = \mathbf{i}_x\, dx + \mathbf{i}_y\, dy + \mathbf{i}_z\, dz \tag{1.25}$$

Here, of course, $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are unit vectors in the $x$, $y$, and $z$ directions, respectively. Now Equations (1.22)–(1.25) tell us that for conservative or static fields

$$\mathbf{E} = -\nabla V \tag{1.26}$$

where $\nabla$ is the gradient operator defined by (1.24). An equivalent statement, preferred by this writer, is

$$\mathbf{E} = -\text{grad } V \tag{1.27}$$

where grad is the abbreviation for gradient. Of course, (1.26) and (1.27) hold in any orthogonal coordinate system.

## 1.4 DIPOLE CONCEPT

Another important concept is the dipole. To illustrate, we first consider two charges $+q$ and $-q$ separated by a distance $\ell$, as indicated in Figure

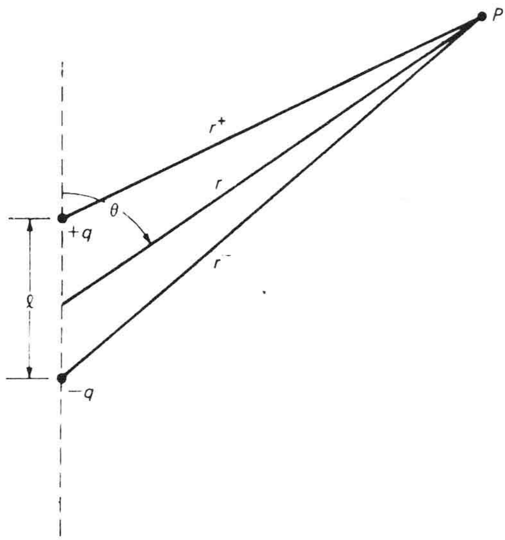


Figure 1.2 Geometry for calculating potential of two charges of equal and opposite sign.

1.2. The distance from $+q$ to the field point $P$ is $r^+$, while the distance from $-q$ to the field point is $r^-$. The *resultant* potential at $P$ is given by

$$V = \frac{1}{4\pi\epsilon}\left(\frac{q}{r^+} - \frac{q}{r^-}\right) \tag{1.28}$$

where $\epsilon$ is the permittivity of the homogeneous host medium. Now we will consider the case where $r \gg \ell$, whence

$$r^+ \simeq r - \left(\frac{\ell}{2}\right)\cos\theta \tag{1.29}$$

and

$$r^- \simeq r + \left(\frac{\ell}{2}\right)\cos\theta \tag{1.30}$$

where $\theta$ is the angle subtended by the line drawn from the center point of the charges to the field point and the vertical axis through $+q$ and $-q$.

**Exercise:** Derive an explicit expression for $V$, valid if $r$ and $\ell$ are arbitrary, in terms of $r$ and $\theta$.

From (1.29) and (1.30) it follows that (1.28) simplifies in the manner

$$V \simeq \frac{q}{4\pi\epsilon}\left[\frac{1}{r - (\ell/2)\cos\theta} - \frac{1}{r + (\ell/2)\cos\theta}\right]$$
$$\simeq \frac{q\ell}{4\pi\epsilon r^2}\cos\theta \tag{1.31}$$

Now bearing in mind that symmetry about the polar axis prevails, it follows that in spherical coordinates $(r, \theta, \phi)$

$$E_r = -\frac{\partial V}{\partial r} = \frac{q\ell}{2\pi\epsilon r^3}\cos\theta \tag{1.32}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{q\ell}{4\pi\epsilon r^3}\sin\theta \tag{1.33}$$

$$E_\phi = 0$$

As perceptive readers will note, the potential expression in (1.31) and the field expressions in (1.32) and (1.33) are only valid if $\ell \ll r$. But they might also assert that these expressions are valid for all nonzero values of $r$ if $\ell$ is infinitesimal or effectively so. In this case we should replace $\ell$ by, say, the differential $ds$; this replacement will be done in any further discussion of dipole fields. Incidentally, the term *dipole* should be reserved to describe the infinitesimal element and not be used in the context of linear antennas of finite length.

## 1.5 CHARGED LINE SOURCE

Another useful model is a uniform line charge. Such a configuration has an obvious cylindrical symmetry. Thus with reference to the cylindrical coordinates $(\rho, \phi, z)$, we locate the line charge along the $z$ axis, and it extends from $z = -\infty$ to $+\infty$ with a uniform charge $\hat{q}$ C/m. Now it is not difficult to see that the flux density vector has only a radial $\rho$ component, and it is given by

$$D_\rho = \frac{\hat{q}}{2\pi\rho} \quad \text{C/m}^2 \tag{1.34}$$

The corresponding electric field component is then

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{\hat{q}}{2\pi\epsilon\rho} \quad \text{V/m} \tag{1.35}$$

Now the potential $V$ at the radial distance $\rho$ is related to $E_\rho$ by

$$-\frac{\partial V}{\partial \rho} = E_\rho \tag{1.36}$$

This equation tells us that

$$V = -\int^\rho E_\rho \, d\rho \tag{1.37}$$

where we have an indefinite integral on the right. On using (1.35), we see that

$$V = -\frac{\hat{q}}{2\pi\epsilon}\ln\rho + \text{constant} \tag{1.38}$$

where, again, it is understood that the surrounding medium of permittivity $\epsilon$ is homogeneous and isotropic.

A simple application of the preceding results tells us that the difference of potential $\Delta V$, between the surfaces $\rho = a_1$ and $\rho = a_2$ for the line charge $\hat{q}$ at $\rho = 0$, is given by

$$\Delta V = \frac{\hat{q}}{2\pi\epsilon} \ln \frac{a_2}{a_1} \tag{1.39}$$

A slightly different formulation of the above problem would be to regard the surface $\rho = a$ as a perfectly conducting equipotential surface. Again we can let $\hat{q}$ be the total charge per unit length. Then (1.34) and (1.35) still apply for the region $\rho > a_1$. Now we also regard $\rho = a_2$ as a perfectly conducting surface. The difference of potential between these two concentric equipotential surfaces is given precisely by (1.39). The latter can be written in the equivalent form

$$\Delta V = \frac{\hat{q}}{C} \tag{1.40}$$

where

$$C = \frac{2\pi\epsilon}{\ln(a_2/a_1)} \tag{1.41}$$

by definition, is the capacitance per unit length. In other words, if we "apply" a voltage $\Delta V$ between these surfaces that are initially uncharged, we find that the inner surface has a total charge $\hat{q}$ per unit length given by

$$\hat{q} = C\,\Delta V$$

Then because total charge must be conserved, the charge on the outer concentric cylindrical is $-\hat{q}$ C/m.

## 1.6 PARALLEL CYLINDRICAL CONDUCTORS

Another cylindrical problem is to determine the capacity between two cylindrical conductors that are not concentric. One simple way to deal with this situation is to begin with the potential expression for two parallel line charges of strengths $\hat{q}$ and $-\hat{q}$, respectively. These charges are illustrated in Figure 1.3, where the line charges are located at points (+) and (−) separated by a distance $d$. The resultant potential at $P$ with rectangular coordinates $(x, y, z)$ is clearly

$$V = \frac{\hat{q}}{2\pi\epsilon} \ln \frac{r^-}{r^+} \tag{1.42}$$