

Linear Algebra | Serge Lang

Second Edition



LINEAR ALGEBRA

SECOND EDITION

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Foreword

The present book is meant as a text for a course in linear algebra, at the undergraduate level. Enough material has been included for a one-year course, but by suitable omissions, it will also be easy to use the book for one term.

During the past decade, the curriculum for algebra courses at the undergraduate level has shifted its emphasis towards linear algebra. The shift is partly due to the recognition that this part of algebra is easier to understand than some other parts (being less abstract, and in any case being directly motivated by spatial geometry), and partly because of the wide applications which exist for linear algebra. Consequently, I have started the book with the basic notion of vector in real Euclidean space, which sets the general pattern for much that follows. The chapters on groups and rings are included because of their important relation to the linear algebra, the group of invertible linear maps (or matrices) and the ring of linear maps of a vector space being perhaps the most striking examples of groups and rings. The fact that a vector space over a field can be viewed fruitfully as a module over its ring of endomorphisms is worth emphasizing as part of a linear algebra course. However, because of the general intent of the book, these chapters are not treated with quite the same degree of completeness which they might otherwise receive, and a short text on basic algebraic structures (groups, rings, fields, sets, etc.) will accompany this one to offer the opportunity of teaching a separate one-term course on these matters, principally intended for mathematics majors.

The tensor product, and especially the alternating product, are so important in courses in advanced calculus that it was imperative to insert a chapter on them, keeping the applications in mind. The limited purpose of the chapter here allows for concreteness and simplicity.

The appendix on convex sets pursues some of the geometric ideas of Chapter I, taking for granted some standard facts about continuous functions on compact sets, closures of sets, etc. It can essentially be read after Chapter I, and after knowing the definition of a linear map. Various odds and ends are given in a second appendix (including a proof of the algebraic closure of the complex numbers), which can be covered according to the judgement of the instructor.

The basic portion of this book, on vector spaces, matrices, linear maps, and determinants is now published separately as *Introduction to Linear Algebra*, with additional simplifications of language and text. For instance, we take vector spaces over the reals, we consider only the positive definite scalar product, we omit the dual space, etc., which are less worthy of emphasis for a first introduction, needed in immediate applications, e.g. in calculus. In the more complete text of a full course in linear algebra, these topics are of course included, as are many others, especially the structure theorems which form Part Two: spectral theorem, for symmetric, hermitian, unitary operators; triangulation theorems (including the Jordan normal form); primary decomposition; Schur's lemma; the Wedderburn-Rieffel theorem (with Rieffel's beautifully simple proof); etc. Of course, better students can handle the more complete book at once, but I hope that the separation will be pedagogically useful for others.

In this second edition, I have rewritten a few sections, and inserted a few new topics. I have also added many new exercises.

New York, 1970

SERGE LANG

Contents

Part One Basic Theory

CHAPTER I

Vectors

1. Definition of points in n -space	3
2. Located vectors	8
3. Scalar product	10
4. The norm of a vector	13
5. Lines and planes	24
6. The cross product	32
7. Complex numbers	34

CHAPTER II

Vector Spaces

1. Definitions	39
2. Bases	46
3. Dimension of a vector space	51
4. Sums and direct sums	55

CHAPTER III

Matrices

1. The space of matrices	59
2. Linear equations	64
3. Multiplication of matrices	69
Appendix. Elimination	79

CHAPTER IV

Linear Mappings

1. Mappings	83
2. Linear mappings	91
3. The kernel and image of a linear map	98
4. Composition and inverse of linear mappings	103
5. Geometric applications	109

CHAPTER V

Linear Maps and Matrices

1. The linear map associated with a matrix	117
2. The matrix associated with a linear map	118
3. Bases, matrices, and linear maps	122

CHAPTER VI

Scalar Products and Orthogonality

1. Scalar products	131
2. Orthogonal bases, positive definite case	137
3. Application to linear equations	147
4. Bilinear maps and matrices	152
5. General orthogonal bases	156
6. The dual space	159

CHAPTER VII

Determinants

1. Determinants of order 2	167
2. Existence of determinants	169
3. Additional properties of determinants	176
4. Cramer's rule	182
5. Permutations	185
6. Uniqueness	191
7. Determinant of a transpose	195
8. Determinant of a product	196
9. Inverse of a matrix	197
10. The rank of a matrix and subdeterminants	200
11. Determinants as area and volume	202

Part Two Structure Theorems

CHAPTER VIII

Bilinear Forms and the Standard Operators

1.	Bilinear forms	215
2.	Quadratic forms	220
3.	Symmetric operators	222
4.	Hermitian operators	227
5.	Unitary operators	231
6.	Sylvester's theorem	235

CHAPTER IX

Polynomials and Matrices

1.	Polynomials	241
2.	Polynomials of matrices and linear maps	244
3.	Eigenvectors and eigenvalues	246
4.	The characteristic polynomial	252

CHAPTER X

Triangulation of Matrices and Linear Maps

1.	Existence of triangulation	257
2.	Theorem of Hamilton-Cayley	260
3.	Diagonalization of unitary maps	262

CHAPTER XI

Spectral Theorem

1.	Eigenvectors of symmetric linear maps	265
2.	The spectral theorem	268
3.	The complex case	274
4.	Unitary operators	275

CHAPTER XII

Polynomials and Primary Decomposition

1.	The Euclidean algorithm	281
2.	Greatest common divisor	283

3. Unique factorization	286
4. The integers	290
5. Application to the decomposition of a vector space	292
6. Schur's lemma	295
7. The Jordan normal form	297

Part Three

Relations with Other Structures

CHAPTER XIII

Multilinear Products

1. The tensor product	305
2. Isomorphisms of tensor products	309
3. Alternating products: Special case	311
4. Alternating products: General case	315
Appendix. The vector space generated by a set	324

CHAPTER XIV

Groups

1. Groups and examples	327
2. Simple properties of groups	330
3. Cosets and normal subgroups	336
4. Cyclic groups	341
5. Free abelian groups	344

CHAPTER XV

Rings

1. Rings and ideals	349
2. Homomorphisms	354
3. Modules	357
4. Factor modules	361

APPENDIX 1

Convex Sets

1. Definitions	365
2. Separating hyperplanes	367

3. Extreme points and supporting hyperplanes	369
4. The Krein-Milman theorem	371

APPENDIX 2

Odds and Ends

1. Induction	373
2. Algebraic closure of the complex numbers	374
3. Equivalence relations	375

APPENDIX 3

Angles

379

Answers	387
--------------------------	-----

Index	397
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PART ONE

BASIC THEORY

CHAPTER I

Vectors

The concept of a vector is basic for the whole course. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full.

The cross product is included for the sake of completeness. It is almost never used in the rest of the book. It is the only aspect of the theory of vectors which is valid only in 3-dimensional space (not 2, nor 4, nor n -dimensional space). One significant feature of almost all the statements and proofs of this book (except for those concerning the cross product and determinants), is that they are neither easier nor harder to prove in 3- or n -space than they are in 2-space.

§1. Definition of points in n -space

We know that a number can be used to represent a point on a line, once a unit length is selected.

A pair of numbers (i.e. a couple of numbers) (x, y) can be used to represent a point in the plane.

These representations can be represented in a picture as follows.

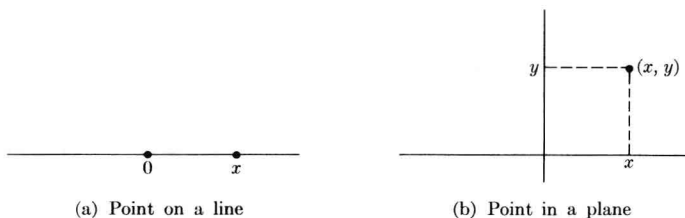


Figure 1

We now observe that a triple of numbers (x, y, z) can be used to represent a point in space, that is 3-dimensional space, or 3-space. We simply introduce one more axis. The following picture illustrates this.

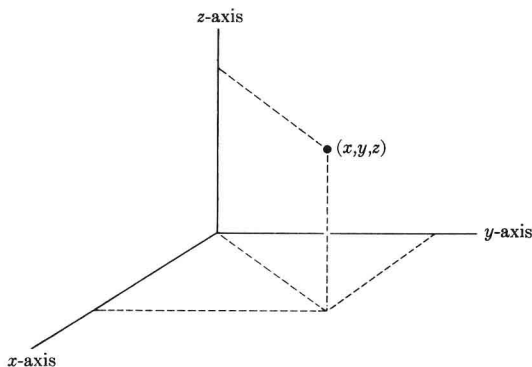


Figure 2

Instead of using x, y, z we could also use (x_1, x_2, x_3) . The line could be called 1-space, and the plane could be called 2-space.

Thus we can say that a single number represents a point in 1-space. A couple represents a point in 2-space. A triple represents a point in 3-space.

Although we cannot draw a picture to go further, there is nothing to prevent us from considering a quadruple of numbers

$$(x_1, x_2, x_3, x_4)$$

and decreeing that this is a point in 4-space. A quintuple would be a point in 5-space, then would come a sextuple, septuple, octuple, . . .

We let ourselves be carried away and define a **point in n -space** to be an n -tuple of numbers

$$(x_1, x_2, \dots, x_n),$$

if n is a positive integer. We shall denote such an n -tuple by a capital letter X , and try to keep small letters for numbers and capital letters for points. We call the numbers x_1, \dots, x_n the **coordinates** of the point X . For example, in 3-space, 2 is the first coordinate of the point $(2, 3, -4)$, and -4 is its third coordinate.

Most of our examples will take place when $n = 2$ or $n = 3$. Thus the reader may visualize either of these two cases throughout the book. However, two comments must be made: First, practically no formula or theorem is simpler by making such assumptions on n . Second, the case $n = 4$ does occur in physics, and the case $n = n$ occurs often enough in practice or theory to warrant its treatment here. Furthermore, part of our purpose is in fact to show that the general case is always similar to the case when $n = 2$ or $n = 3$.

Examples. One classical example of 3-space is of course the space we live in. After we have selected an origin and a coordinate system, we can

describe the position of a point (body, particle, etc.) by 3 coordinates. Furthermore, as was known long ago, it is convenient to extend this space to a 4-dimensional space, with the fourth coordinate as time, the time origin being selected, say, as the birth of Christ—although this is purely arbitrary (it might be more convenient to select the birth of the solar system, or the birth of the earth as the origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, and a point with positive time coordinate is an AD point.

Don't get the idea that "time is *the* fourth dimension", however. The above 4-dimensional space is only one possible example. In economics, for instance, one uses a very different space, taking for coordinates, say, the number of dollars expended in an industry. For instance, we could deal with a 7-dimensional space with coordinates corresponding to the following industries:

- | | | | |
|--------------|-------------|-------------------|---------|
| 1. Steel | 2. Auto | 3. Farm products | 4. Fish |
| 5. Chemicals | 6. Clothing | 7. Transportation | |

We agree that a megabuck per year is the unit of measurement. Then a point

$$(1,000, 800, 550, 300, 700, 200, 900)$$

in this 7-space would mean that the steel industry spent one billion dollars in the given year, and that the chemical industry spent 700 million dollars in that year.

We shall now define how to add points. If A, B are two points, say

$$A = (a_1, \dots, a_n), \quad B = (b_1, \dots, b_n),$$

then we define $A + B$ to be the point whose coordinates are

$$(a_1 + b_1, \dots, a_n + b_n).$$

For example, in the plane, if $A = (1, 2)$ and $B = (-3, 5)$, then

$$A + B = (-2, 7).$$

In 3-space, if $A = (-1, \pi, 3)$ and $B = (\sqrt{2}, 7, -2)$, then

$$A + B = (\sqrt{2} - 1, \pi + 7, 1).$$

Furthermore, if c is any number, we *define* cA to be the point whose coordinates are

$$(ca_1, \dots, ca_n).$$

If $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$.

We observe that the following rules are satisfied:

$$(1) (A + B) + C = A + (B + C).$$

$$(2) A + B = B + A.$$

$$(3) c(A + B) = cA + cB.$$

(4) If c_1, c_2 are numbers, then

$$(c_1 + c_2)A = c_1A + c_2A \quad \text{and} \quad (c_1c_2)A = c_1(c_2A).$$

(5) If we let $O = (0, \dots, 0)$ be the point all of whose coordinates are 0, then $O + A = A + O = A$ for all A .

(6) $1 \cdot A = A$, and if we denote by $-A$ the n -tuple $(-1)A$, then

$$A + (-A) = O.$$

[Instead of writing $A + (-B)$, we shall frequently write $A - B$.] All these properties are very simple to prove, and we suggest that you verify them on some examples. We shall give in detail the proof of property (3). Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$. Then

$$A + B = (a_1 + b_1, \dots, a_n + b_n)$$

and

$$\begin{aligned} c(A + B) &= (c(a_1 + b_1), \dots, c(a_n + b_n)) \\ &= (ca_1 + cb_1, \dots, ca_n + cb_n) \\ &= cA + cB, \end{aligned}$$

this last step being true by definition of addition of n -tuples.

The other proofs are left as exercises.

Note. Do not confuse the number 0 and the n -tuple $(0, \dots, 0)$. We usually denote this n -tuple by O , and also call it zero, because no difficulty can occur in practice.

We shall now interpret addition and multiplication by numbers geometrically in the plane (you can visualize simultaneously what happens in 3-space).

Take an example. Let $A = (2, 3)$ and $B = (-1, 1)$. Then

$$A + B = (1, 4).$$

The figure looks like a parallelogram (Fig. 3).

Take another example. Let $A = (3, 1)$ and $B = (1, 2)$. Then

$$A + B = (4, 3).$$

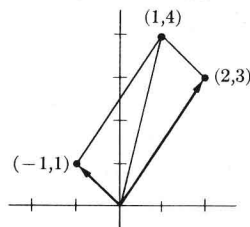


Figure 3

We see again that the geometric representation of our addition looks like a parallelogram (Fig. 4).

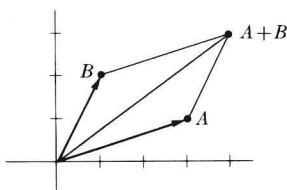


Figure 4

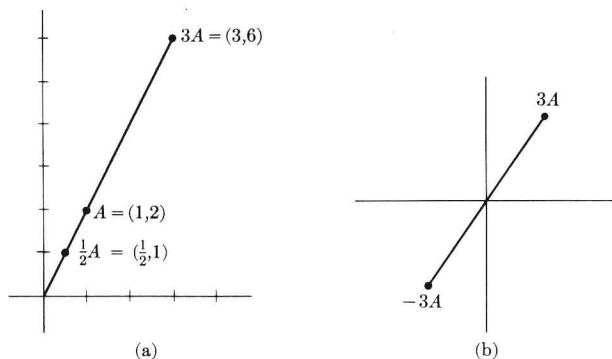


Figure 5

What is the representation of multiplication by a number? Let $A = (1, 2)$ and $c = 3$. Then $cA = (3, 6)$ as in Fig. 5(a).

Multiplication by 3 amounts to stretching A by 3. Similarly, $\frac{1}{2}A$ amounts to stretching A by $\frac{1}{2}$, i.e. shrinking A to half its size. In general, if t is a number, $t > 0$, we interpret tA as a point in the same direction as A from the origin, but t times the distance.

Multiplication by a negative number reverses the direction. Thus $-3A$ would be represented as in Fig. 5(b).

EXERCISES

Find $A + B$, $A - B$, $3A$, $-2B$ in each of the following cases.

- $A = (2, -1)$, $B = (-1, 1)$
- $A = (-1, 3)$, $B = (0, 4)$
- $A = (2, -1, 5)$, $B = (-1, 1, 1)$
- $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
- $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$
- $A = (15, -2, 4)$, $B = (\pi, 3, -1)$
- Draw the points of Exercises 1 through 4 on a sheet of graph paper.
- Let A, B be as in Exercise 1. Draw the points $A + 2B$, $A + 3B$, $A - 2B$, $A - 3B$, $A + \frac{1}{2}B$ on a sheet of graph paper.