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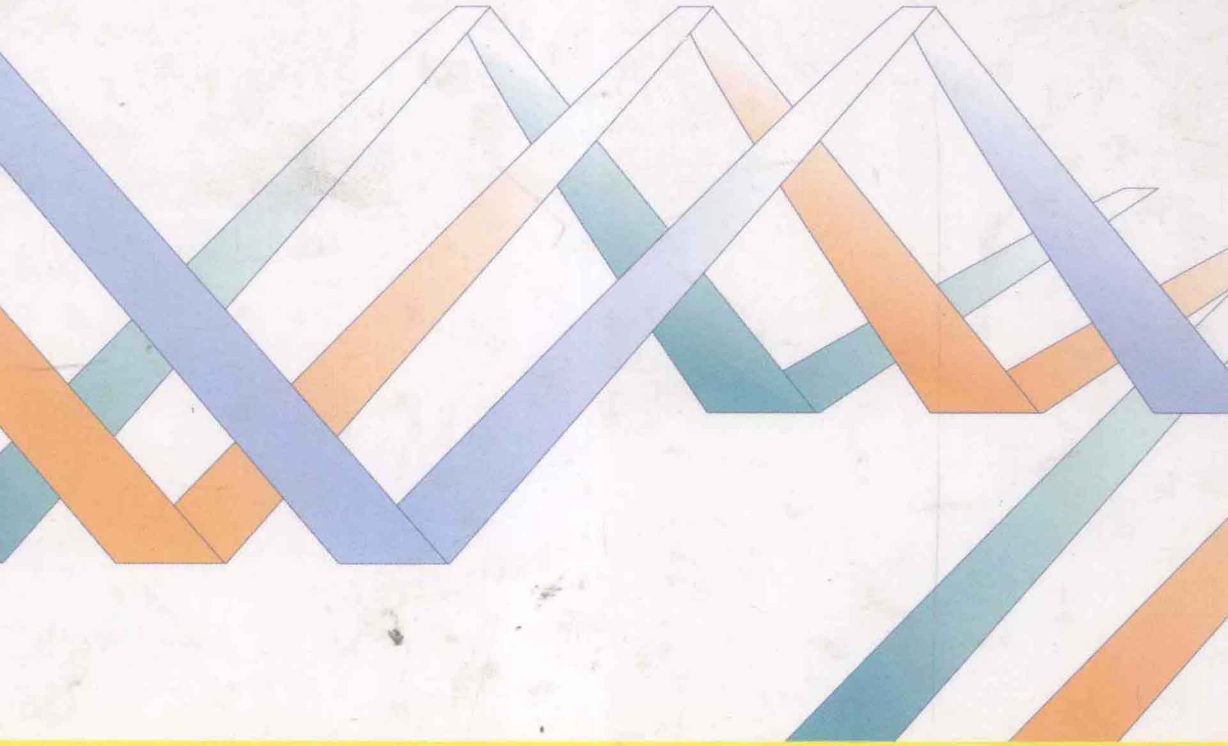

Lie Theory and Representation Theory

李理论与表示论

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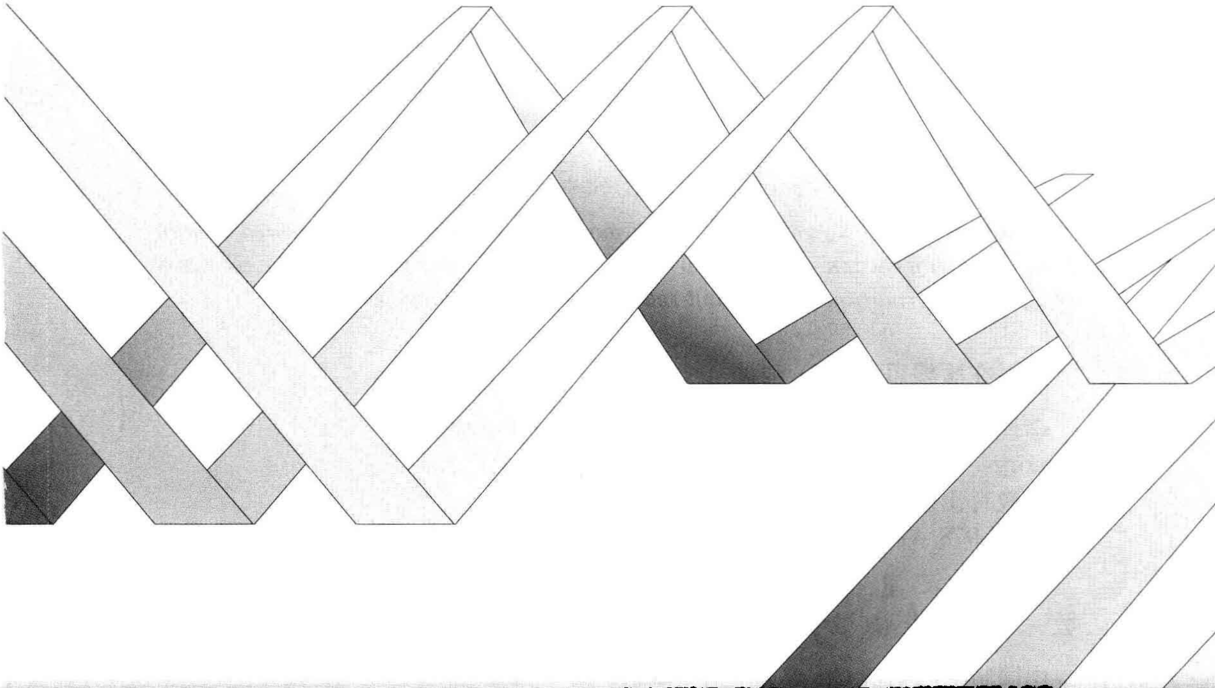

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LI LILUN YU BIAOSHILUN

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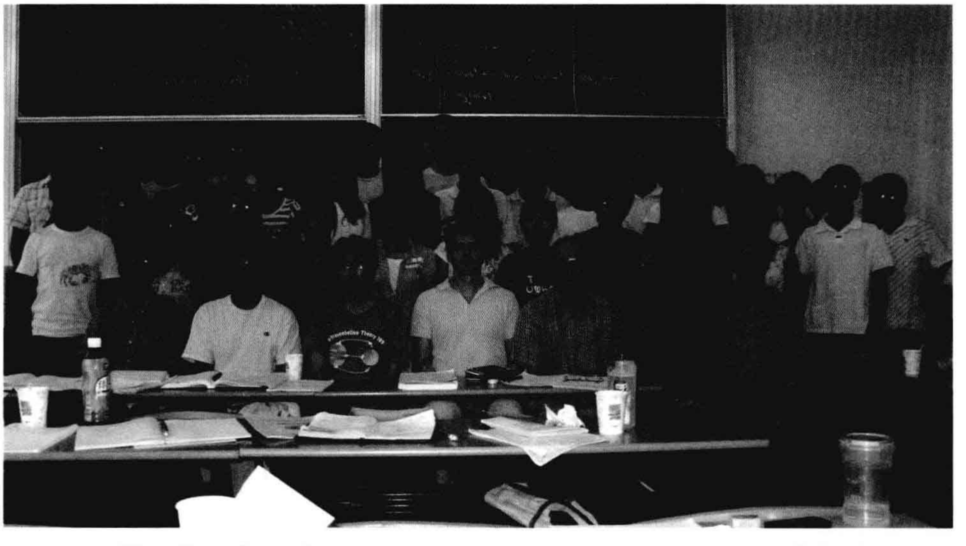
The Open Day of the East China Normal University Summer School
on Lie Theory and Representation Theory (II), 13, July, 2009



The Close Day of the East China Normal University Summer School
on Lie Theory and Representation Theory (II), 31, July, 2009

Workshop on Lie Theory and Representation Theory II

Preface

During the period from July 13 to July 31, 2009, East China Normal University hosted the second workshop and summer school on Lie Theory and Representation Theory. This volume contains the lecture notes of three courses in that summer school, together with the lecture notes of one course given in the first summer school which was held in 2006.

This volume consists of articles by Shun-Jen Cheng and Weiqiang Wang, Rolf Farnsteiner, Daniel K. Nakano, and Toshiyuki Tanisaki. These articles focus on different areas in Lie theory and representation theory. The article jointly by Cheng and Wang introduces some recent developments of representations of Lie superalgebras, explaining how Lie superalgebras of types $\mathfrak{gl}$ and $\mathfrak{osp}$ provide a natural framework for generalized Schur and Howe dualities, and how a super duality gives a conceptual solution to the irreducible character problem for these Lie superalgebras in terms of the classical Kazhdan-Lusztig polynomials.

Farnsteiner's article discusses combinatorial and geometric aspects of representation theory of finite group schemes, and focuses on the "classical" theory of co-commutative Hopf algebras, the defining algebras of affine algebraic group schemes.

Nakano's article gives a survey of recent developments in the representation theory and cohomology theory of reductive algebraic groups, their Frobenius kernels and their associated finite groups of Lie type.

Tanisaki's article presents an overview of the theory of D-modules and its application to representations of Lie algebras.

This volume is well suited for graduate students in the fields of Lie theory and representation theory and related topics, and also for researchers who wish to learn about some current core areas in Lie theory and representation theory and their applications.

At last, we sincerely express our thanks to the Department of Mathematics, the International Exchange Division and the Graduate School of East China Normal University for their financial support to the summer schools and workshops in 2006 and 2009. We are grateful to National Natural Science Foundation of China for financial support (Grant:10926022) in 2009. Our deep appreciation also goes to our colleagues Pei Gu, Youyi Wu and Hongyan Zhang for their assistance in organizing these activities.

Jianpan Wang
Bin Shu
Naihong Hu
In Shanghai
31 October, 2010

Contents

Lie Theory and
Representation Theory
SMM 2, pp. 1–46

Dualities for Lie Superalgebras

Shun-Jen Cheng* and Weiqiang Wang†

Abstract

We explain how Lie superalgebras of types $\mathfrak{gl}$ and $\mathfrak{osp}$ provide a natural framework generalizing the classical Schur and Howe dualities. This exposition includes a discussion of super duality, which connects the parabolic categories O between classical Lie superalgebras and Lie algebras. Super duality provides a conceptual solution to the irreducible character problem for these Lie superalgebras in terms of the classical Kazhdan-Lusztig polynomials.

2000 Mathematics Subject Classification: 17B10.

Keywords and Phrases: Lie superalgebras, Schur duality, Howe duality, super duality, irreducible characters.

0 Introduction

The study of Lie superalgebras, supergroups, and their representations was largely motivated by supersymmetry in mathematical physics, which puts bosons and fermions on the same footing. An earlier achievement is the Cartan-Killing type classification of finite-dimensional simple complex Lie superalgebras by Kac [K1] (also cf. [SNR] for an independent classification of the so-called classical Lie superalgebras). The most important basic classical Lie superalgebras consist of infinite series of types $\mathfrak{sl}, \mathfrak{osp}$. The basic classical Lie superalgebras afford root systems, Dynkin diagrams, Cartan subalgebras, triangular decomposition, Verma modules, category O, and so on. There has been much work on representation theory of Lie superalgebras (in particular, basic classical) in the last three decades, but conceptual approaches have been lacking until recently.

The aim of these lecture notes is to explain three different kinds of dualities for Lie superalgebras:

Schur duality, Howe duality, and Super duality.

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In the superalgebra setting, the first (i.e. Schur) duality was formulated by Sergeev, and the latter two dualities have been largely developed by the authors and their collaborators. These lecture notes are also intended to serve as a road map for a forthcoming book by the authors.

The Schur-Sergeev duality is an interplay between Lie superalgebras and the symmetric groups which incorporates the trivial and sign modules in a unified framework. On the algebraic combinatorial level, there is a natural super generalization of the notion of semistandard tableaux which is a hybrid of the traditional version and its conjugate counterpart.

It has been observed that much of the study of the classical invariant theory on polynomial algebras has parallels for exterior algebras, and both admit reformulation and extension in the theory of Howe's reductive dual pairs. Lie superalgebras allow a uniform treatment of Howe duality on the polynomial and exterior algebras.

Super duality has a different flavor. It views representation theories of Lie superalgebras and Lie algebras as two sides of the same coin, and it is an unexpected yet powerful approach developed in the past few years which allow us to overcome various superalgebra difficulties. We give an exposition on the new development on super duality which is an equivalence between parabolic categories $\mathcal{O}$ of Lie superalgebras and Lie algebras. Super duality provides a conceptual solution to the long-standing irreducible character problem for a wide class of modules over (a wide class of) Lie superalgebras in terms of Kazhdan-Lusztig polynomials. This is achieved despite the fact that there are no obvious Weyl groups controlling the linkage for super representation theory.

In Section 1, we give some basic constructions and structures of the general linear and the ortho-symplectic Lie superalgebras. We emphasize the super phenomena that are not observed in the ordinary Lie algebra setting, such as odd roots, non-conjugate Borel subalgebras, and so on. In Section 2, we present Kac's classification of finite-dimensional simple $\mathfrak{g}$-modules [K2]. The classification is very easy for type A, but nontrivial for $\mathfrak{osp}$. In the latter case we explain a new odd reflection approach by Shu and the second author [SW], using a more natural labeling of these modules by hook partitions. We note that odd reflection is also one of the main technical tools in super duality. In addition, we present the *typical* finite-dimensional irreducible character formula, following [K2].

The classical Schur duality relates the representation theory of the general linear Lie algebras and that of the symmetric groups. In Section 3, we explain Sergeev's generalization [Sv1] of Schur duality for the general linear Lie superalgebras $\mathfrak{gl}(m|n)$ (also see Berele and Regev [BeR] for additional insight and detail). More precisely, we establish a double centralizer theorem for the actions of $\mathfrak{gl}(m|n)$ and the symmetric group $\mathfrak{S}_d$ in d letters on the tensor space $(\mathbb{C}^{m|n})^{\otimes d}$. We then provide an explicit multiplicity-free decomposition of the tensor space into a $U(\mathfrak{gl}(m|n)) \otimes \mathbb{C}\mathfrak{S}_d$-modules. We further present a simple formula obtained in our latest work with Lam [CLW] for extremal weights in a simple polynomial $\mathfrak{gl}(m|n)$-module with respect to all Borel subalgebras, which has an explicit diagramatic interpretation from a Young diagram.

Howe's theory of reductive dual pairs [H1, H2] can be viewed as a representation theoretic reformulation and extension of the classical invariant theory (see Weyl [We]). For example, the first fundamental theorem on invariants for classical groups are reformulated in terms of double centralizer properties of two classical Lie groups/algebras. One advantage of Howe duality is that it allows natural generalizations to classical Lie groups/algebras (and superalgebras) other than type A.

We mainly use two examples of dual pairs to illustrate the main ideas of Howe duality and the new phenomena of superalgebra generalizations. For more detailed case study of Howe duality for Lie superalgebras, we refer to the original papers [BPT, CW1, CW2, CW3, CL1, CLZ, CZ2, CKW, LZ, Sv2]. In Section 4, we formulate the $(\mathfrak{gl}(m|n), \mathfrak{gl}(d))$-Howe duality and find the highest weight vectors for each isotypical component in the corresponding multiplicity-free decomposition. In Section 5 we present the $(\mathrm{Sp}(d), \mathfrak{osp}(2m|2n))$-Howe duality and its multiplicity-free decomposition. The application of Howe duality to irreducible characters over Lie superalgebras follows the simpler approach in our work with Kwon [CKW] (which uses Howe duality for infinite-dimensional Lie algebras [Wa]).

We recall some truly super phenomena that have been the main obstacles towards a better understanding of super representation theory:

1. There exist odd roots as well as non-conjugate Borel subalgebras for a Lie superalgebra. A homomorphism between Verma modules may not be injective.
2. The linkage in category $\mathcal{O}$ of modules for a Lie superalgebra is NOT controlled by the Weyl group of $\mathfrak{g}_{\bar{0}}$; see e.g. $\mathfrak{gl}(1|1)$.
3. There is no uniform Weyl-type irreducible finite-dimensional character formula for Lie superalgebras.
4. The super geometry behind super representation theory is still inadequately developed.

In light of these super phenomena, it was a rather unexpected discovery [CWZ, CW4], which was partly inspired by Brundan [Br1], that there exists a (conjectural) equivalence of categories between Lie algebras and Lie superalgebras of type A (at a certain suitable limit at infinity), which was termed *Super Duality*. This conjecture in the full generality of [CW4] has been proved in [CL2], which in particular offers an elementary and conceptual solution to the character problem for all finite-dimensional simple modules and for a large class of infinite-dimensional simple highest weight modules over Lie superalgebras of type A.

Super duality has been subsequently formulated and established between various Lie superalgebras of type $\mathfrak{osp}$ and the corresponding classical Lie algebras in our very recent work with Lam [CLW]. This in particular offers a conceptual solution of the irreducible character problem for a wide class of modules, which include all finite-dimensional irreducibles, of Lie superalgebras of type $\mathfrak{osp}$ in terms of Kazhdan-Lusztig polynomials for classical Lie algebras [KL, BB, BK] (for more on Kazhdan-Lusztig theory see Tanisaki's lectures [Ta]). In addition, it follows easily from the approach of [CL2, CLW] that the $\mathfrak{u}$-homology groups (or Kazhdan-Lusztig polynomials in the sense of Vogan [Vo]) match perfectly between classical Lie superalgebras and the corresponding classical Lie algebras. This generalizes earlier partial results in this direction from Schur or Howe duality approach [CZ1, CK, CKW]. The super duality as outlined above is explained in Section 6.

Let us put the super duality work explained above in perspective. *Finite-dimensional* irreducible characters for $\mathfrak{gl}(m|n)$ have been also obtained earlier in two totally different approaches by [Sva] and [Br1]. The mixed algebraic and geometric approach of Serganova has been extended very recently in [GS] to obtain all irreducible finite-dimensional $\mathfrak{osp}$-characters. Brundan and Stroppel [BrS] also provided another approach to the main results of [Br1] and independently proved a special case of the super duality conjecture in type A as formulated in [CWZ]. All these approaches have brought new and different insights into super representation theory. Our super duality approach has the advantages of explaining the connection with classical Lie algebras and their Kazhdan-Lusztig polynomials, covering infinite-dimensional irreducible characters, and being extendable to general Kac-Moody Lie superalgebras.

A list of symbols is added at the end of the paper to facilitate the reading.

Let us end the Introduction with some remarks on the interrelations among the three dualities.

The $(\mathfrak{gl}(d), \mathfrak{gl}(n))$-Howe duality is equivalent to Schur duality. It follows from the Schur-Sergeev duality that the characters for irreducible polynomial $\mathfrak{gl}(m|n)$-modules are given by the so-called hook Schur functions. On the other hand, the irreducible character formulas for Lie superalgebras of types $\mathfrak{gl}$ or $\mathfrak{osp}$ obtained from Howe duality can be expressed in terms of infinite classical Weyl groups. The appearance of hook Schur functions and infinite Weyl groups in these formulas are conceptually explained from the viewpoint of super duality.

Super duality can be informally interpreted as a categorification of the standard involution on the ring of symmetric functions. It is well known that the ring of symmetric functions in infinitely many variables admits symmetries which are not observed in finitely many variables. Super duality is formulated precisely at the infinite rank limit. On the level of combinatorial parameterizations of highest weights, super duality manifests itself through (variation of) the conjugate of partitions.

Partly due to the time constraint of the lectures, we have left out many interesting topics on super representation theory. We refer to [BL, J] (and more recently [SZ]) for finite-dimensional irreducible characters of atypicality one, to [BKN, DS, Ma, Mu, Pe, PS] for geometric approaches, to [Br2, CWZ2] for further development of the Fock space approach of Brundan for the queer Lie superalgebra $\mathfrak{q}(n)$ and for $\mathfrak{osp}(2|2n)$, to [CK, CKW, CZ1, Ger, San, Sva, Zou] for some cohomological aspects, to [BrK, SW, WZ] for prime characteristic, to [JHKT, Su] for related combinatorial structures; also see [Gor, KW, Naz] for additional work on Lie superalgebras.

Acknowledgment. This paper is a modified and expanded written account of the 8 lectures given by the second author at the summer school of East China Normal University (ECNU), Shanghai, July 2009. We are grateful to Ngau Lam for his collaboration and insight. We thank Bin Shu at ECNU for hospitality and an enjoyable summer school.

1 Lie superalgebra ABC

A vector superspace V is understood as a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element $a \in V_i$ has parity $|a| = i$, and an element in $V_{\bar{0}}$ (respectively, $V_{\bar{1}}$) is called even (respectively, odd).

Definition 1.1. A Lie superalgebra is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with a bilinear bracket operation $[.,.]$ satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $i, j \in \mathbb{Z}_2$, and the following two axioms: for $\mathbb{Z}_2$-homogeneous $a, b, c \in \mathfrak{g}$,

(1) (Skew-supersymmetry) $[a, b] = -(-1)^{|a|\cdot|b|}[b, a]$.
(2) (Super Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|}[b, [a, c]]$.

Remark 1.2. (1) For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.
(2) (Sign Rule) As explained in Manin's book [Ma], there is a general heuristic sign rule for superalgebras as follows. *If in some formula for usual algebra there are monomials with interchanged terms, then in the corresponding formula for superalgebra every interchange of neighboring terms, say a and b, is accompanied by the multiplication of the monomials by the factor* $(-1)^{|a|\cdot|b|}$. This is already manifest in the definition of Lie superalgebra and will persist throughout the paper.

Example 1.3. (1) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra (i.e. $\mathbb{Z}_2$-graded). Then $(A, [.,.])$ is a Lie superalgebra, where for homogeneous elements $a, b \in A$, we define

$$[a, b] = ab - (-1)^{|a|\cdot|b|}ba.$$

(2) A Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_{\bar{1}} = 0$ is just a usual Lie algebra. A Lie superalgebra $\mathfrak{g}$ with purely odd part (i.e. $\mathfrak{g}_{\bar{0}} = 0$) has to be *abelian*, i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$.

1.1 Lie superalgebras of type A and the supertrace

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. Then $\mathrm{End}(V)$ is naturally an associative superalgebra. The Lie superalgebra $\mathfrak{gl}(V) := (\mathrm{End}(V), [.,.])$ from Example 1.3 (1) is called a *general linear Lie superalgebra.* If $V_{\bar{0}} = \mathbb{C}^m$ and $V_{\bar{1}} = \mathbb{C}^n$, we denote V by $\mathbb{C}^{m|n}$, and $\mathfrak{gl}(V)$ by $\mathfrak{gl}(m|n)$. Note that both $\mathfrak{gl}(m|0) \cong \mathfrak{gl}(0|m)$ are isomorphic to the usual Lie algebra $\mathfrak{gl}(m)$.

The Lie superalgebra $\mathfrak{gl}(m|n)$ consists of block matrices of size $m|n$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.1}$$

Throughout the paper, we choose to parameterize the rows and columns of the matrices by the set

$$I(m|n) = \{\bar{1}, \ldots, \overline{m}; 1, \ldots, n\}$$

with a total order

$$\bar{1} < \cdots < \overline{m} < 0 < 1 < \cdots < n \tag{1.2}$$

(where 0 is inserted for later convenience). Its even subalgebra consists of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

and is isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

Example 1.4. For $\mathfrak{g} = \mathfrak{gl}(1|1)$, let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$[e, f] = h_1 + h_2 \text{ (the identity matrix).}$$

The *supertrace*, denoted by str, of (1.1) is defined to be

$$\mathrm{str}(g) = \mathrm{tr}(a) - \mathrm{tr}(d).$$

The *special linear Lie superalgebra* is

$$\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) \mid \mathrm{str}(x) = 0\}.$$

The definitions of supertrace and of the Lie superalgebra $\mathfrak{sl}$ are justified by the following.

Exercise 1.5. Show that $\mathfrak{sl}(m|n) = [\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)]$ and in particular $\mathfrak{sl}(m|n)$ is a Lie subalgebra of $\mathfrak{gl}(m|n)$.

The notion of simple Lie superalgebras is defined in the same way as for Lie algebras. We note that $\mathfrak{sl}(n|n)$ is not a simple Lie superalgebra, as it contains a nontrivial center $\mathbb{C}I_{2n}$.

1.2 The bilinear form

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{gl}(m|n)$ consisting of all diagonal matrices. Note that $\mathfrak{h}$ is an even subalgebra of $\mathfrak{gl}(m|n)$.

Let E_{ij}, for $i, j \in I(m|n)$, denote the standard basis for $\mathfrak{gl}(m|n)$. We define a bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ by letting

$$(a, b) = \mathrm{str}(ab), \quad a, b \in \mathfrak{g}.$$

This restricts to a nondegenerate symmetric bilinear form on $\mathfrak{h}$: for $i, j \in I(m|n)$,

$$(E_{ii}, E_{jj}) = \begin{cases} 1 & \text{if } \overline{1} \le i = j \le \overline{m}, \\ -1 & \text{if } 1 \le i = j \le n, \\ 0 & \text{if } i \neq j. \end{cases}$$

Denote by $\{\delta_i, \epsilon_j\}_{i,j}$ the basis of $\mathfrak{h}^*$ dual to $\{E_{\overline{i}\,\overline{i}}, E_{jj}\}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$. Under the bilinear form $(\cdot, \cdot)$, we have the identification $\delta_i = (E_{\overline{i},\overline{i}}, \cdot)$ and $\epsilon_j = -(E_{jj}, \cdot)$. Whenever it is convenient we also use the notation

$$\epsilon_{\overline{i}} := \delta_i, \quad \text{for } 1 \le i \le m. \tag{1.3}$$

The form $(\cdot,\cdot)$ on $\mathfrak{h}$ induces a non-degenerate bilinear from on $\mathfrak{h}^*$, which will be denoted by the same notation, as follows: for $i, j \in I(m|n)$,

$$(\epsilon_i, \epsilon_j) = \begin{cases} 1 & \text{if } \overline{1} \le i = j \le \overline{m}, \\ -1 & \text{if } 1 \le i = j \le n, \\ 0 & \text{if } i \ne j. \end{cases} \tag{1.4}$$

1.3 The root system

For the Lie superalgebra $\mathfrak{gl}(m|n)$, we define the root space decomposition, a root system Φ, a set Φ^+ (respectively, Φ^-) of positive (respectively, negative) roots, a set Π of simple roots (in Φ^+), etc. As this can be done in the same way as for semisimple Lie algebras or $\mathfrak{gl}(m)$, we will merely write down the statements for later use.

Now let us make the super phenomenon explicit. A root α is *even* if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{0}}$, and it is *odd* if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{1}}$. Denote by $\Phi_{\bar{0}}$ (respectively, $\Phi_{\bar{1}}$) the set of all even (respectively, odd) roots in Φ. Denote

$$\Phi_i^{\pm} = \Phi_i \cap \Phi^{\pm}, \quad \Pi_i = \Phi_i \cap \Pi, \quad i \in \mathbb{Z}_2.$$

With respect to the Cartan subalgebra $\mathfrak{h}$ the Lie superalgebra $\mathfrak{gl}(m|n)$ admits a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

with a root system

$$\Phi = \{\epsilon_i - \epsilon_j \mid i, j \in I(m|n), i \ne j\}.$$

The standard set of simple roots is taken to be $\Pi = \Pi_{\bar{0}} \cup \Pi_{\bar{1}}$, where

$$\Pi_{\bar{0}} = \{\epsilon_{\bar{i}} - \epsilon_{\overline{i+1}}\}_{1 \le i \le m-1} \cup \{\epsilon_i - \epsilon_{i+1}\}_{1 \le i \le n-1}, \quad \Pi_{\bar{1}} = \{\epsilon_{\overline{m}} - \epsilon_1\},$$

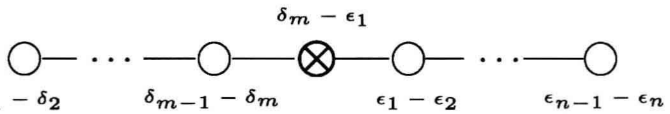
and the associated standard set of positive roots is

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i, j \in I(m|n), i < j\},$$

where the odd roots are $\epsilon_i - \epsilon_j$ with indices $i < 0 < j$. Clearly, $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}$. It follows by (1.4) that

$$(\delta_i - \epsilon_j, \delta_i - \epsilon_j) = 0,$$

for all the odd roots $\delta_i - \epsilon_j$, where $1 \le i \le m, 1 \le j \le n$. An odd root α with $(\alpha, \alpha) = 0$ is called *isotropic*. The standard Dynkin diagram is:



where we have used ⊗ to denote an isotropic odd simple root.

Remark 1.6. The notion of root systems and Dynkin diagrams makes sense for all the basic classical Lie superalgebras, which consist of $\mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{osp}(m|2n)$ and three exceptional ones (besides the simple Lie algebras).

1.4 Non-conjugate Borel subalgebras and $\epsilon\delta$-sequences

As we have seen above, the bilinear form on the real subspace $\mathfrak{h}_{\mathbb{R}}^*$ spanned by the ϵ_i's is not positive-definite (due to the supertrace), and moreover, there exist isotropic odd roots.

Another distinguished feature of Lie superalgebras is the existence of non-conjugate Borel subalgebras or non-isomorphic Dynkin diagrams (under the Weyl group action).

Lemma 1.7. *Let $\mathfrak{g}$ be a Lie superalgebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, which corresponds to the root system $\Phi = \Phi^+ \cup \Phi^-$. Let α be an odd isotropic simple root. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$. Then, $\Phi(\alpha)^+ := (\Phi^+ \backslash \{\alpha\}) \cup \{-\alpha\}$ is a new system of positive roots, whose corresponding set of simple roots is*

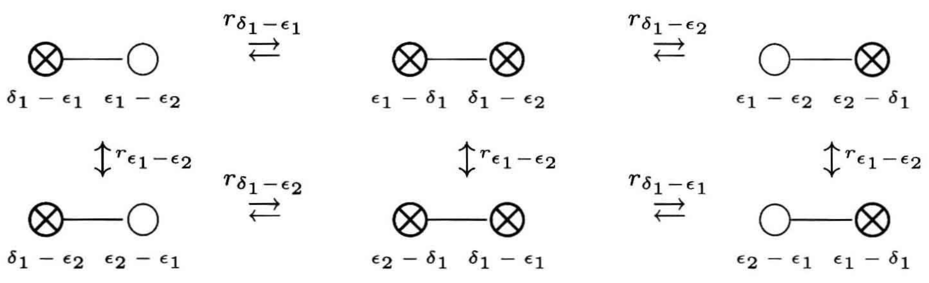
$$\Pi(\alpha) = \{\beta \in \Pi \mid (\beta, \alpha) = 0, \beta \neq \alpha\} \cup \{\beta + \alpha \mid \beta \in \Pi, (\beta, \alpha) \neq 0\} \cup \{-\alpha\}.$$

The new Borel subalgebra corresponding to $\Pi(\alpha)$ will be denoted by $\mathfrak{b}(\alpha)$.

Proof. Follows from a straightforward verification. □

The process of obtaining $\Pi(\alpha)$ from Π above will be referred to as an *odd reflection*, and will be denoted by r_α, in accordance with the usual notion of real reflections.

Example 1.8. Associated to $\mathfrak{gl}(1|2)$, we have $\Phi_{\bar{0}} = \{\pm(\epsilon_1 - \epsilon_2)\}$, and $\Phi_{\bar{1}} = \{\pm(\delta_1 - \epsilon_1), \pm(\delta_1 - \epsilon_2)\}$. There are 6 sets of simple roots, that are related by the real and odd reflections as follows. There are three conjugacy classes of Borel subalgebras, and each vertical pair corresponds to such a conjugacy class.



One convenient way to parameterize the conjugacy classes of Borel subalgebras of $\mathfrak{gl}(m|n)$ is via the notion of $\epsilon\delta$-sequences. Keeping (1.3) in mind, we list the simple roots associated to a given Borel subalgebra $\mathfrak{b}$ in order as $\epsilon_{i_1} - \epsilon_{i_2}, \epsilon_{i_2} - \epsilon_{i_3}, \ldots, \epsilon_{i_{m+n-1}} - \epsilon_{i_{m+n}}$, where $\{i_1, i_2, \ldots, i_{m+n}\} = I(m|n)$. Switching the ordered sequence $\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_{m+n}}$ to the $\epsilon\delta$-notation by (1.3) and then dropping the indices give us the *$\epsilon\delta$-sequence* associated to $\mathfrak{b}$. Note that the total number of δ's (respectively, ϵ's) is m (respectively, n).

For example, the three conjugacy classes of Borels for $\mathfrak{gl}(1|2)$ above correspond to the three sequences $\delta\epsilon\epsilon$, $\epsilon\delta\epsilon$, $\epsilon\epsilon\delta$, respectively. In more detail, the first sequence $\delta\epsilon\epsilon$ is obtained by removing the indices of $\delta_1\epsilon_1\epsilon_2$ (read off from the