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Numerical Methods for Delay Differential Equations

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and
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To Nevy and Doris

PREFACE

In this book we cover initial value problems for Volterra functional differential equations characterized by the presence of discretely distributed delays, possibly of neutral type, usually called "delay differential equations".

The interest of applied mathematicians for the numerical solution of such a class of problems has increased considerably in the last decades and the number of papers dealing with different aspects of their numerical integration now amounts to several hundreds. Nevertheless, there is no book describing, analyzing, unifying and, where necessary, extending and improving the various approaches and techniques appearing in the literature.

Indeed, this is the main aim of this book, which is intended for a wide variety of readers, including mathematicians, physicists, engineers, economists and other scientists ranging from those who are most interested in the theoretical aspects of numerical methods for ordinary and delay differential equations to those who are just looking for a suitable technique in order to simulate their own model by numerically solving some specific equations. The second aim is to bridge the gap between the basic knowledge on discrete and continuous numerical methods for ordinary differential equations in view of possible applications to dense output, error estimation, discontinuous equations, problems with driving equations and, of course, delay differential equations and more general Volterra differential and integro-differential equations.

Although we have reported many concepts and results on accuracy and stability of numerical methods for ordinary differential equations (mainly in Chapters 3, 5 and 8), for a fruitful reading of the book, at least a basic level knowledge of these topics is recommended.

In the introductory Chapter 1 we formalize the classes of problems treated and focus on some of the most significant qualitative differences between delay equations and ordinary equations, as well as on how such differences reflect in their numerical treatment.

In Chapter 2 we discuss the regularity of the solutions by analyzing how the discontinuities, caused by different types of delays, propagate along the solutions and the impact this lack of smoothness has on the design of efficient numerical methods. Far from being exhaustive, the chapter also reports some existence and uniqueness results for the most known classes of delay equations.

Chapter 3 deals with a general formulation and convergence results for discrete and continuous methods for ordinary equations, followed by an introductory review of the most used numerical methods for delay equations, including a fast "historical" excursus.

Chapter 4 is devoted to the error (convergence) analysis of the most usual technique, based on the use of a discrete method for ordinary equations endowed with some interpolant, which we call "the standard approach". It is also shown that, despite the fact that any discrete method is, in principle, suitable for the standard approach, one-step methods (essentially Runge-Kutta methods) are preferable to multistep methods. At the end, a list of available codes is given.

Chaper 5 is a self-contained presentation of the "continuous Runge-Kutta methods". Besides being the bricks for the construction of the standard approach for delay equations, as developed in the rest of the book, this chapter provides, on a small scale, one of the first systematic collections of results on continuous Runge-Kutta methods useful for treating discontinuous differential equations, systems with driving equations, dense output and more general Volterra functional differential equations.

Chapter 6 specializes the theory presented in Chapter 4 for the class of continuous Runge-Kutta methods and provides several additional results on superconvergence for "constrained mesh" methods.

Chapter 7 treats the stepsize control mechanism. As with ordinary differential equation solvers, this issue plays a central role in the production of efficient numerical algorithms for delay equations. In particular, we discuss the importance of the choice of the continuous extension in the estimation of the local error and its influence on the response of the global error to the user supplied tolerance. Finally, we illustrate the implementation of RADAR5, the most recent code designed for a very general class of stiff delay equations, for two critical examples.

The last chapters, Chapters 8, 9 and 10, are entirely devoted to the important issue of stability. They are largely theoretical, discussing quite technical aspects; nevertheless, in order to produce efficient numerical algorithms for particular classes of "stiff" delay problems, knowledge of most of their contents is crucial.

Chapter 8 is relevant to ordinary differential equations and provides an essential review of stability concepts and results for Runge-Kutta methods, including the first systematic presentation of the so-called "stability with respect to forcing terms". Chapters 9 and 10 address the stability analysis of some classes of test delay equations and of Runge-Kutta methods for their stable integration, respectively. In particular, due to the many-sided stability requirement and to the large variety of significantly different test equations, several definitions of stability for numerical methods are introduced in Chapter 10. In the corresponding stability analysis of continuous Runge-Kutta methods, some order barriers are proved that make the standard approach unsuitable for the most general delay-dependent asymptotic stability property of linear delay systems. A completely different approach, based on restating the delay equation as an "abstract Cauchy problem" and, equivalently, as a partial differential equation with suitable initial/boundary conditions, is then considered. These alternative approaches, still under investigation, overcome the mentioned order barriers and show good potential for the stable integration of an even larger class of Volterra functional differential equations. Chapter 10 ends with some specific additional stability issues which have been developed in the literature or are still in progress.

Throughout the book we provide many examples and discuss the results by means of illustrative numerical experiments. We also supply many algorithms (often in the form of pseudo-codes), so that the interested reader can write his own computer programs without too much effort.

We wish to thank all who have helped us either by useful suggestions and discussions or by providing material for examples, illustrations, bibliographic references, etc. and, in general, by encouraging us to pursue the realization of this book (in particular, C.T.H. Baker, H. Brunner, N. Guglielmi, E. Hairer, V. Kolmanovskii and S. Maset). We want to give particular thanks to K. Burrage, who gave a detailed reading of an advanced version of the book.

Finally, we want to thank our very understanding wives for their support and great deal of patience while drafting the book.

Trieste A. B.
July 2002 M. Z.

CONTENTS

1

INTRODUCTION

Many real-life phenomena in physics, engineering, biology, medicine, economics, etc. can be modeled by an *initial value problem* (IVP), or *Cauchy problem*, for *ordinary differential equations* (ODEs) of the type

$$\begin{cases} y'(t) = g\big(t, y(t)\big), & t \geq t_0, \\ y(t_0) = y_0, \end{cases} \tag{1.0.1}$$

where the function $y(t)$, called the state variable, represents some physical quantity that evolves over time.

However, in order to make the model more consistent with the real phenomenon, it is sometimes necessary to modify the right-hand side of (1.0.1) to include also the dependence of the derivative y' on past values of the state variable y. The most general form of such models is given by the *retarded functional differential equation*

$$y'(t) = f(t, y_t), \quad t \geq t_0,$$

where $y_t = y(t+\theta)$, $\theta \in [-r, 0]$, is a function belonging to the Banach space $C = C^0([-r, 0], \mathbb{R}^d)$ of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^d$, and $f : \Omega \longrightarrow \mathbb{R}^d$ is a given function of the set $\Omega \subset \mathbb{R} \times C$ into $\mathbb{R}^d$.

In this context $y'(t)$ stands for the right-hand derivative $y'(t)^+$, and the initial value problem is

$$\begin{cases} y'(t) = f(t, y_t), & t \geq t_0, \\ y_{t_0} = y(t_0 + \theta) = \Phi(\theta), \end{cases} \tag{1.0.2}$$

where $\Phi(\theta) \in C$ represents the *initial point* or the *initial data*.

Equation (1.0.2), also called the *Volterra functional differential equation*, includes both *distributed delay differential equations*, where f depends on y computed on a continuum, possibly unbounded ($r = +\infty$), set of past values, and *discrete delay differential equations*, where only a finite number of past values of the state variable y are involved. Despite the latter being special cases of the former, they are suitable to describe a wide class of phenomena in many branches of applied mathematics and we shall confine our interest to them. Throughout the book they will be referred to as *delay differential equations* (DDEs) or *difference differential equations*.

The general theory of DDEs is widely developed and we refer the reader to the classical books by Bellman and Cooke [39], Hale [120], Driver [77], El'sgol'ts and Norkin [80] and to the more recent books by Hale and Verduyn Lunel [122], Kolmanovskii and Myshkis [171], Kolmanovskii and Nosov [172], Diekmann, van Gils, Verduyn-Lunel and Walter [74] and Kuang [182], which also include many real-life examples of DDEs and more general retarded functional differential equations.

1.1 DDEs versus ODEs: some examples

Throughout the book, the initial value problem (1.0.2) will be expressed in a more friendly manner by

$$\begin{cases} y'(t) = f\big(t, y(t-\tau_1), \ldots, y(t-\tau_n)\big), & t \geq t_0, \\ y(t) = \phi(t), & t \leq t_0. \end{cases} \tag{1.1.1}$$

Here, according to the complexity of the phenomenon, the *delays* (or *lags*) τ_i, which always are non-negative, may be just constants (the *constant delay* case), or functions of t, $\tau_i = \tau_i(t)$ (the *variable* or *time dependent delay* case), or even functions of t and y itself, $\tau_i = \tau_i(t, y(t))$ (the *state dependent delay* case). In order to simplify the notation, the function $\phi(t)$ is understood to be defined in $[\rho, t_0]$, where

$$\rho = \min_{1 \leq i \leq n} \left\{ \min_{t \geq t_0}(t - \tau_i) \right\}.$$

In particular, for state dependent delays, the bound ρ cannot be determined a priori.

An interesting and quite common case is given by $n = 2$ and $\tau_1 \equiv 0$ for which (1.1.1) takes the standard form

$$\begin{cases} y'(t) = f\big(t, y(t), y(t-\tau)\big), & t \geq t_0, \\ y(t) = \phi(t), & t \leq t_0. \end{cases} \tag{1.1.2}$$

Since for some $t \geq t_0$ it can be that $t - \tau < t_0$, a first difference between equations (1.0.1) and (1.1.2) is that the solution of the latter is usually determined by an *initial function* $\phi(t)$ rather than by a simple initial value y_0, as happens for the former. In general, the right-hand derivative $y'(t_0)^+$, that is $f(t_0, \phi(t_0), \phi(t_0 - \tau))$, does not equal the left-hand derivative $\phi'(t_0)^-$ and hence the solution y is not smoothly linked to the initial function $\phi(t)$ at the point t_0, where only C^0-continuity can be assured. Moreover, such a derivative jump discontinuity propagates (see Chapter 2) from the initial point t_0 along the integration interval and gives rise to subsequent discontinuity points where the solution is smoothed out more and more. As a consequence, even if the functions $f(t, y, x)$, $\tau(t, y)$ and $\phi(t)$ in (1.1.2) are C^∞-continuous, in general the solution $y(t)$ is simply C^1-continuous in $[t_0, t_f]$.

Example 1.1.1 Consider the equation

$$\begin{cases} y'(t) = -y(t-1), & t \geq 0, \\ y(t) = 1, & t \leq 0, \end{cases} \tag{1.1.3}$$

whose solution is depicted in Figure 1.1. Since $y'(0)^- = 0$ and $y'(0)^+ = -y(-1) = -1$, the derivative function $y'(t)$ has a jump at $t = 0$. The second derivative $y''(t)$ is given by

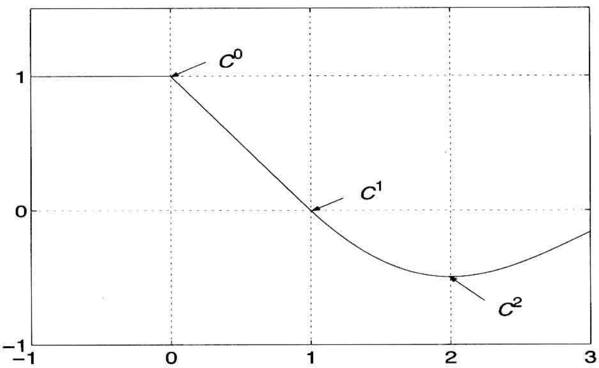

FIG. 1.1. Solutions of (1.1.3).

$$y''(t) = -y'(t-1),$$

and therefore it has a jump at $t = 1$. The third derivative $y'''(t)$ is given by

$$y'''(t) = -y''(t-1) = y'(t-2),$$

and hence it has a jump at $t = 2$, and so forth at multiples of the delay $t = 3, 4, \ldots$
$\diamond$

The presence of an initial function in the problem (1.1.2) has various other unexpected consequences on the solutions. Some of them are illustrated by the following examples.

Example 1.1.2 Unlike the ordinary equations, there is no longer injectivity between the set of initial data and the set of solutions $y(t)$, $t \geq t_0$. In fact, the equation

$$y'(t) = y(t-1)\big(y(t) - 1\big), \qquad t \geq 0,$$

has the constant solution $y(t) = 1$ in $[0, +\infty)$ for any initial function $\phi(t)$ defined in $[-1, 0]$ such that $\phi(0) = 1$. $\diamond$

The next two examples show that, in the state dependent delay case, the lack of regularity of the initial function $\phi(t)$ may cause a loss of uniqueness for the solution of (1.1.2) or its termination after some bounded interval.

Example 1.1.3 As an example of non-uniqueness, consider the equation

$$\begin{cases} y'(t) = y\big(t - |y(t)| - 1\big) + \frac{1}{2}, & t \geq 0, \\ y(t) = \phi(t), & t \leq 0, \end{cases} \qquad (1.1.4)$$

where

$$\phi(t) = \begin{cases} 1, & t < -1, \\ 0, & -1 \leq t \leq 0. \end{cases} \qquad (1.1.5)$$

It is easy to see that in $[0, 2]$ both functions

$$y(t) = \tfrac{3}{2}t$$

and

$$y(t) = \tfrac{1}{2}t$$

are solutions of (1.1.4). ◇

Example 1.1.4 (see El'sgol'ts and Norkin [80]) As an example of termination of the solution, consider the equation

$$\begin{cases} y'(t) = -y\big(t - 2 - y(t)^2\big) + 5, & t \geq 0, \\ y(t) = \phi(t), & t \leq 0, \end{cases} \tag{1.1.6}$$

where

$$\phi(t) = \begin{cases} \frac{9}{2}, & t < -1, \\ -\frac{1}{2}, & -1 \leq t \leq 0. \end{cases} \tag{1.1.7}$$

The solution in $[0, \frac{125}{121}]$ is given by

$$y(t) = \begin{cases} \frac{1}{2}(t-1), & 0 \leq t \leq 1, \\ \frac{11}{2}(t-1), & 1 \leq t \leq \frac{125}{121}. \end{cases} \tag{1.1.8}$$

It is not difficult to see that the solution cannot be continued beyond the point $t = \frac{125}{121}$. In fact, at $t = \frac{125}{121}$ the deviated argument $t - 2 - y(t)^2$ is equal to -1 and therefore, in a right neighborhood of such a point, $y(t - 2 - y(t)^2)$ is given by one of the two values of $\phi(t)$. Thus the solutions of (1.1.6) should take the form

$$y(t) = c\Big(t - \tfrac{125}{121}\Big) + \tfrac{2}{11},$$

with

$$c = \tfrac{1}{2} \quad \text{if } t - 2 - y(t)^2 < -1$$

and

$$c = \tfrac{11}{2} \quad \text{if } t - 2 - y(t)^2 \geq -1.$$

Now, each choice of c leads to a solution $y(t)$ that contradicts the assumption made on $t - 2 - y(t)^2$ and hence the solution does not exist for $t > \frac{125}{121}$. It is worth remarking that, from a numerical point of view, termination of the solution is a very delicate issue. In fact, it may result in surprising and misleading behavior in the implementation of the numerical method. For example, in a right neighborhood of the termination point $t_N = \frac{125}{121}$, where $y_N \approx \frac{2}{11}$, the *forward Euler* method reads

$$y_{n+1} = y_n + h_{n+1}(-\tfrac{9}{2} + 5) \quad \text{if } t_n - 2 - y_n^2 < -1$$

and

$$y_{n+1} = y_n + h_{n+1}(\tfrac{1}{2} + 5) \quad \text{if } t_n - 2 - y_n^2 \geq -1,$$

and for no reason it stops integrating at any $n \geq N$. The resulting approximation is plotted in Figure 1.2 where, for $t \geq \frac{125}{121}$, a *ghost solution* appears that