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Studies in Integer Programming

Edited by

P. L. Hammer, E. L. Johnson,
B. H. Korte and G. L. Nemhauser

ANNALS OF DISCRETE MATHEMATICS 1

STUDIES IN INTEGER PROGRAMMING

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PREFACE

This volume constitutes the proceedings of the Workshop on Integer Programming that was held in Bonn, September 8–12, 1975. The Workshop was organized by the Institute of Operations Research (Sonderforschungsbereich 21), University of Bonn and was generously sponsored by IBM Germany. In all, 71 participants from 13 different countries took part in the Workshop.

Integer programming is one of the most fascinating and difficult areas of mathematical optimization. There are a great many real-world problems of large dimension that urgently need to be solved, but there is a large gap between the practical requirements and the theoretical development. Since combinatorial problems in general are among the most difficult in mathematics, a great deal of theoretical research is necessary before substantial advances in the practical solution of problems can be expected. Nevertheless the rapid progress of research in this field has produced mathematical results significant in their own right and has also borne substantial fruit for practical applications. We believe that this will be adequately demonstrated by the papers in this volume.

The 37 papers appearing in this volume cover a wide spectrum of topics in integer programming. The volume includes works on the theoretical foundations of integer programming, on algorithmic aspects of discrete optimization, on specific types of integer programming problems, as well as on some related questions on polytopes and on graphs and networks.

All the papers have been carefully referred. We express our sincere thanks to all authors for their cooperation, to the referees for their useful support, to numerous participants for stimulating discussions, and to the editors of the Annals of Discrete Mathematics for their willingness to include this volume in their new series.

Bonn, 1976

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REDUCTION AND DECOMPOSITION OF INTEGER PROGRAMS OVER CONES

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We consider the problem

$$(\dagger)\qquad \begin{aligned} &\min c'x\\ &\text{s.t. } Nx + By = b,\\ &\quad x \in \mathbf{N}^r,\ y \in \mathbf{Z}^n \end{aligned}$$

where N is an (m, r), B an (m, n) integer matrix, and $b \in \mathbf{Z}^m$. In Section 2 we characterize all solutions $x \in \mathbf{Z}^r$ of (†) by an explicit formula and give as a corollary a minimal group representation of equality restricted integer programs, where some of the nonnegativity restrictions are relaxed. In Section 3 we discuss decomposing integer programs over cones in case the matrix N has special structure.

1. Introduction

We consider the problem

$$\begin{aligned} &\min c'x\\ &\text{s.t. } Nx + By = b \\ &\quad x \in \mathbf{N}^r, y \in \mathbf{Z}^n \end{aligned} \tag{1.1}$$

where N is an (m, r) and B an (m, n) integer matrix. As B is an arbitrary (m, n) integer matrix, the convex hull of the feasible set of (1.1) is a generalized corner polyhedron, that is an equality restricted integer program, where the nonnegativity restriction of some of the variables are relaxed. To give a group representation of the problem, we reformulate (1.1) as a congruence problem,

$$\begin{aligned} &\min c'x\\ &\text{s.t. } Nx \equiv b \mod B \\ &\quad x \in \mathbf{N}^r \end{aligned} \tag{1.2}$$

where we define $Nx \equiv b \pmod B$, iff there is a $\lambda \in \mathbf{Z}^n$, such that $Nx - b = B\lambda$ holds. To set this definition in a more general framework we have to introduce the concepts of Smith and Hermite normal form.

Definition. If B is an (m, n) integer matrix, we denote by $\mathrm{S}(B)$ and $\mathrm{H}(B)$ the Smith and Hermite normal form of B, $\mathrm{S}^*(B)$ and $\mathrm{H}^*(B)$ denotes the nonsingular part of $\mathrm{S}(B)$, $\mathrm{H}(B)$ resp. The unimodular matrices which transform B into Smith normal form are denoted by U_B, K_B and the projection matrices, which eliminate the nonsingular part $\mathrm{S}^*(B)$ of $\mathrm{S}(B)$ are denoted by W_B, V_B. Thus we have $\mathrm{S}^*(B) = W_B U_B B K_B V_B$.

Sometimes it is advantageous to look at congruences from an algebraic point of view, that is to look at the definition of $a := x(\equiv \bmod \alpha)$[1] as an image of the function $a := h_\alpha(x) = x - \alpha[x/\alpha]$ (where "$[x]$" denotes the integer part of x). For (m, n) matrices B with $\mathrm{rank}(B) \in \{m, n\}$ the scalar α is replaced in the above formula and we get the generalized form as

$$h_B(x) := x - \bar{B}[\bar{B}^\dagger x]$$

where $\bar{B}$ denotes the Hermite form $\mathrm{H}(B)V_B$ of B (the zero colums of $\mathrm{H}(B)$ are omitted) and where $B^\dagger$ denotes the Moore–Penrose inverse of B. In fact we have

Proposition (1.3). *Let G be an additive subgroup of $\mathbf{Z}^m$. The map $h_B : G \to h_B(G)$ is a homomorphism onto $(h_B(G), \oplus)$ with kernel $(h_B) = \{x \in G \mid x = B\lambda, \lambda \in \mathbf{Z}^n\}$, and $x \oplus y := h_B(x + y)$.*

Remark (1.4). Obviously

$$\begin{aligned} & a = x(\equiv \bmod B) \\ & \iff a - x = B\lambda \quad \text{for some } \lambda \in \mathbf{Z}^n \\ & \iff a - x \in \mathrm{kernel}(h_B) \quad \text{holds} \end{aligned}$$

and so problem (1.1) is equivalent to

$$\begin{aligned} & \min c'x \\ & \bigoplus_{i=1}^{r} h_B(N_i)\cdot x_i = h_B(b), \qquad (1.5) \\ & x_i \in \mathbf{N}, \end{aligned}$$

where N_i denotes the ith column of the matrix N and " = " is the group equation in the group $G(B) := h_B(\mathbf{Z}^m)$.

Proof of Proposition (1.3). Since $\bar{B}$ has maximal column range, $\bar{B}'\bar{B}$ is regular, and we have

[1] ':=' means that the left side of the equation will be defined.

$$\bar{B}^\dagger B = (\bar{B}'\bar{B})^\dagger \bar{B}'\bar{B} = I^r.$$

So we conclude

$$\begin{aligned} h_B(x) \oplus h_B(y) &= h_B(x) + h_B(y) - \bar{B}[\bar{B}^\dagger(x+y) - ([\bar{B}^\dagger x] + [\bar{B}^\dagger y])] \\ &= x + y - \bar{B}[\bar{B}^\dagger(x+y)] \\ &= h_B(x+y), \end{aligned}$$

hence h_B is a homomorphism. Let $x \in \text{kernel}(h_B)$, that means $x = \bar{B}[\bar{B}^\dagger x]$. If we denote $b := [\bar{B}^\dagger x] \in \mathbf{Z}^r$ and $a := (b', 0'_{n-r})'$ we conclude $x = H(B)a$ and $x = Bc$ where $c = Ka$, here K denotes the unimodular right multiplicator of $H(B)$. Let now $x = Ba$ with $a \in \mathbf{Z}^n$, that means $x = \bar{B}b$, $b \in \mathbf{Z}^r$. With $\bar{B}^\dagger x = b$ we conclude $h_B(x) = x - \bar{B}[\bar{B}^\dagger x] = \bar{B}b - \bar{B}b = 0$ which completes the proof.

Clearly problem (1.5) is a group problem over the group $G(B)$, which is not necessarily of finite order (it depends obviously on the rank of B). If we follow the usual definition of equivalent matrices (cf. (5)), that is the (m, n) integer matrix A and the (r, s) integer matrix B are equivalent iff they have the same invariant factors (apart from units), we get a slight generalization of a well known fact:

Remark (1.6). The groups $G(A)$ and $G(B)$ are isomorphic, iff the matrices A and B are equivalent and m-rank$(A) = r$-rank(B)holds.

Using this result it is easy to give a formula for the number of different (nonisomorphic) groups $G(B)$, where the product of invariant factors of the (m, n) matrices B is fixed. This number is well known for regular (m, n) integer matrices B. Here we are going to treat the general case.

Definition. Let B be an (m, n) integer matrix. We call the product of the invariant factors of B the *invariant* of B (inv(B)) which coincides with the determinant of B in case B is a square nonsingular matrix.

If $d = \prod_{j=1}^{k} \pm P_j^{\varepsilon_j}$ is a representation of $d = \text{inv}(B)$ as a product of prime factors and p a function from $\mathbf{N}^2$ into $\mathbf{N}$ defined recursively as

$$p(n, m) := \begin{cases} p(n, m), & 1 \leq n \leq m, \\ p(n, m-1) + p(n-m, m), & n \geq m \geq 1, \end{cases}$$

$p(0, m) := 1, p(n, 0) := 0 (n, m \in \mathbf{N})$, we define

$$K(d) := \sup_{m \in \mathbf{N}} \prod_{j=1}^{k} p(\varepsilon_j, m)$$

$$L(d, m) := \sum_{i=1}^{m} \prod_{j=1}^{k} p(\varepsilon_j, i).$$

Proposition (1.7). *The number of nonisomorphic groups $G(B)$, where B varies over all (m, n) integer matrices $(m, n \in \mathbf{N})$ with maximal row rank and invariant d, equals the integer number $K(d)$.*

The number of nonisomorphic groups $G(B)$, where A varies over all (m, n) integer matrices $(n \in \mathbf{N})$ with $\operatorname{rank}(B) \in \{m, n\}$ *and invariant d, equals $L(d, m)$.*

Notice that $K(d)$ is a finite number, though we consider all (m, n) integer matrices B with $m, n \in \mathbf{N}$. If we compute the numbers $K(d)$ and $L(d, m)$ for d's between 1 and 10^5, we note that $0 \leq K(d) \leq 10$ in 95% of the cases, that is the group $G(B)$ is more or less determined by $d = \operatorname{inv}(B)$.

Proof of Proposition (1.7). Two groups are isomorphic iff the generating matrices are equivalent and the rank condition holds (cf. Remark (1.6)). Proving the first part of the proposition we have only to deal with maximal row rank matrices and using Remark (1.4) we can restrict ourselves to square matrices, because $h_B(x)$ is defined in terms of $\mathrm{H}^*(B)$ and this an (m, n) integer matrix with $\det \mathrm{H}^*(B) = \operatorname{inv}(B)$. Because of the divisibility property of the invariant factors of an (m, m) integer matrix it suffices now to compute the number of different representations of the exponents of a prime factor presentation of the determinant $d = \det B$ as a sum of m nonnegative integers. In fact this number equals $p(\varepsilon_j, m)$ (cf. (2)) and moreover $\mathrm{H}(d)$ is finite because

$$\varepsilon_{j_0} := \max_{j=1}^{k} \varepsilon_j$$

leads to

$$\prod_{j=1}^{k} p(\varepsilon_j, \varepsilon_{j_0} + k) = \prod_{j=1}^{k} p(\varepsilon_j, \varepsilon_{j_0}) \quad (k \in \mathbf{N}).$$

To prove the second part of the proposition we first note that $\operatorname{rank}(B) \leq m$. Since two groups $G(A)$ and $G(B)$ with matrices having both less than m columns, cannot be isomorphic, the second statement follows obviously from the first one.

2. Minimal group representation

We have seen that (1.5) is a group problem, namely of the group $G(B)$. In fact this is the group which will usually be considered in the asymptotic integer programming approach (cf. (3)), whereas the actual underlying group of (1.5) is the group

$$G(N/B) := \{h_B(x)/x = N\lambda, \lambda \in \mathbf{Z}^r\}$$

which is a subgroup of $G(B)$ generated by the columns of the matrix N. From a computational point of view the group $G(N/B)$ is more difficult to handle than the group $G(B)$ (though it has less elements), because there is no proper respresentation of $G(N/B)$. From this reason here we are going to find a $\delta \in \mathbf{N}^m$ which will be defined in terms of N and B, such that the group $G(N/B)$ is isomorphic to

G (diag(δ)). Clearly this is a minimal group representation of problem (1.5) and as a corollary we get the order of $G(N/B)$ by

$$\prod_{i=1}^{m} \delta_i .$$

First we want to give some results concerning congruences which will be used later, they seem to be of general interest, though.

Theorem (2.1). *Let B be an (m, n) integer matrix with rank $(B) = m$, N an (m, s) integer matrix, $b \in \mathbf{Z}^m$ and $A := (N, B)$. The system of congruences*

$$Nx \equiv Nb \quad \text{mod } B$$

$$x \text{ integer}$$

has a solution iff $S^*(A)^{-1} V_A U_A b$ is integer. In this case, all solutions are of the form

$$x \equiv b \quad \text{mod } H$$

$$x \text{ integer}$$

where $H := (K_M V_M W_M L, R)$. Here we denote by $L := S^*(A)^{-1} U_A N$, $M := S^*(A)^{-1} U_A B$ and R denotes the last $s - k$ columns of K_M, where $k :=$ rank(N).

Proof. Without loss of generality we set $b = 0$. It is easy to see that $S^*(M, L)$ equals an (m, m) identity matrix I^m, so we conclude

$$S(S(M),\ U_M L) = (I^m, 0_{m,n}).$$

With diag$(t_1, \ldots, t_k) := S^*(M)$, $t_{k+i} := 0$ $(i = 1, \ldots, m - k)$ and $D := U_M L$ we get immediately

(†) $\quad \gcd(t_i, d_i) = 1, \quad i = 1, \ldots, m,$

where $d_i := \gcd(D_{ij} / j = 1, \ldots, n)$ $(i = 1, \ldots, m)$.

Obviously the system

$$Nx \equiv 0 \quad \text{mod } B$$

$$x \text{ integer}$$

is equivalent to the system

$$\begin{pmatrix} S^*(M) & 0_{m,s-k} \\ 0_{m-k,k} & \end{pmatrix} y \equiv 0 \quad \text{mod } U_M L$$

$$y \text{ integer},$$

and using (†) it is also equivalent to

$$(S^*(M), 0_{m,s-k})\, y \equiv 0 \quad \text{mod } W_M U_M L.$$

$$y \text{ integer}.$$

Let $y = (y_1', y_2')'$ be a $(k, s-k)$-partition of y, then we get

$$S^*(M)y_1 \equiv 0 \mod W_M U_M L.$$

$$y_1, y_2 \text{ integer}.$$

Let $K_i (i = 1, \ldots, k)$ be unimodular matrices, which transform the ith row of $\hat{D} := W_M U_M L$ into $(d_i, 0, \ldots, 0)$. Using

$$E_i := K_i \operatorname{diag}(1, \ldots, 1, t_i^{-1}, 1, \ldots, 1) K_i^{-1}$$

$i = 1, \ldots, m$ we define

$$E := \prod_{i=k}^{1} E_i.$$

By induction on i one can easily show that

$$\operatorname{diag}(1, \ldots, t_{i+1}, \ldots, t_m) y_1 = \hat{D} \prod_{j=i}^{1} E_j z$$

$$y_2, \prod_{j=i}^{1} E_j z \text{ integer}$$

is equivalent (for all $i = 1, \ldots, m$) to

(*) $$S^*(M)y_1 \equiv 0 \mod B$$

$$y_1, y_2 \quad \text{integer}$$

so that

$$y_1 = \hat{D}Ez$$

$$y_2, Ez \quad \text{integer}$$

is equivalent to (*).

Since E^{-1} is an integer matrix and $x = K_M y$, the equation

$$x = (K_M V_M y_1 + R y_2)$$

completes the proof.

Theorem (2.2). *With the notations of theorem* (2.1) *we get*

(i) $S^*(L) = S(A)^{-1} U_A U_B^{-1} S^*(B)$

(ii) $S^*(H) = I^{s-k} \dot{+} \operatorname{diag}(t_{m-k+1}, \ldots, t_m)$

where $S^*(L) =: \operatorname{diag}(t_1, \ldots, t_m)$.

Proof. Because of

$$L = S^*(A)^{-1} U_A U_B^{-1} U_B B,$$

(i) follows immediately from the equation

$$S^*(L) = S^*(LK_B) = S^*(LK_B V_B).$$

Let

$$P := \begin{pmatrix} 0_{k,s-k} \\ I^{s-k} \end{pmatrix}$$

where I^{s-k} denotes an $((s-k),(s-k))$ identity matrix. Because of $H = K_M(W_M U_M L, P)$, we conclude $S^*(H) = S^*(W_M U_M L, P)$, that is

$$S^*(H) = \begin{pmatrix} I^{s-k} & 0_{s-k,k} \\ 0_{k,s-k} & S^*(QL) \end{pmatrix}$$

where Q denotes the first k rows of U_M.

From the proof of theorem (2.1) we know that

$$S^*(L) = S^*(H(U_M L)) = \operatorname{diag}(t_1, \ldots, t_m),$$

so

$$S^*(QL) = \operatorname{diag}(t_{m-k+1}, \ldots, t_m)$$

which completes the proof.

Now we are able to give an isomorphic representation of the subgroup $G(N/B)$.

Theorem (2.3). *Let B be an (m, n) and N an (m, r) integer matrix with* $\operatorname{rank}(B) = m$. *Then we get*

$$G(N/B) \simeq G(S^*(E)),$$

that means the group $G(N/B)$ is isomorphic to the group $G(S^*(E))$, where $E := W_M U_M L$ and $L := S^*(N, B)^{-1} U_{(N,B)} N$, $M := S^*(N,B)^{-1} U_{(N,B)} B$.

Corollary (2.4).

$$\Theta := U_E\, S^*(M)^{-1} W_M U_M\, S^*(N, B)^{-1} U_{(\bar{N},B)}$$

is an isomorphism from $G(N/B)$ to $G(S^*(E))$.

Corollary (2.5). *The order of $G(N/B)$ equals*

$$\frac{\operatorname{inv}(B)}{\det(S^*(N, B))}.$$

Proof of Theorem (2.3). Let K be a unimodular matrix, so that NK is up to permutations of rows in Hermite normal form. Let $\bar{N}$ be the matrix NK without the zero columns. Obviously we have $G(N/B) = G(\bar{N}/B)$. Let

$$\{\bar{N}\} := \{x \in \mathbf{Z}^m \mid x = \bar{N}y \quad \text{for a} \quad y \in \mathbf{Z}^k\}$$

be a subgroup of $(\mathbf{Z}^m, +)$. Because $h_B : \{\bar{N}\} \to h_B(\{\bar{N}\})$ is a homomorphism (Proposition 1.3) $G(\bar{N}/B)$ is isomorphic to the factor group

$$\{\bar{N}\}/\mathrm{kernel}(h_B)$$

where $\mathrm{kernel}(h_B) = \{x \in \{\bar{N}\} \mid x \equiv 0 \mod B\}$.

With Theorem (2.1) we conclude

$$\mathrm{kernel}(h_B) = \{x \in \mathbf{Z}^m \mid x = \bar{N}y,\ y \equiv 0 \mod K_M W_M U_M L \text{ for a } y \in \mathbf{Z}^k\}.$$

Let

$$f := S^*(M)^{-1} W_M U_M B L^{-1}.$$

Then

$$f : \{\bar{N}\} \to \mathbf{Z}^k$$

is an isomorphism and $f(\mathrm{kernel}(h_B)) = \{z \in \mathbf{Z}^k \mid z \equiv 0 \mod W_M U_M L\}$. Thus we get

$$\{\bar{N}\}/\mathrm{kernel}(h_B) \simeq \mathbf{Z}^k/\mathrm{kernel}(h_E)$$

and because U_E is also an isomorphism we get the isomorphism

$$G(\bar{N}/B) \simeq G(S^*(E)).$$

The corollaries follow immediately from Theorem (2.3) in conjunction with Theorem (2.2).

3. Partitioning of integer programs over cones

The computational effort to solve the problem

$$\begin{aligned} &\min c'x \\ \text{s.t. } &Nx + By = b \\ &x \in \mathbf{N}^r,\ y \in \mathbf{Z}^n \end{aligned} \tag{3.1}$$

usually grows rapidly according to the determinant of B. It is therefore sometimes advantageous to decompose the problem into smaller subproblems and to link the optima of the subproblems to a solution of the masterproblem. We give now two examples of decomposing problem (3.1) in case the matrix N is of the form

$$N = \begin{bmatrix} & N_2 & & 0 \\ & & \cdot & \\ N_1 & & & \cdot \\ & & & & \cdot \\ & 0 & & & N_r \end{bmatrix} \tag{3.2}$$

or

$$N = \begin{bmatrix} A_1, \ldots & \ldots, A_r \\ N_1 & \\ & \ddots & \\ 0 & & N_r \end{bmatrix} \qquad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_r \end{bmatrix} \tag{3.3}$$

To simplify notation let $B = S^*(B)$, i.e. B is given as a diagonal matrix. (Otherwise we have to impose some special structure on U_B.)

Let us denote the set of feasible solutions of problem (3.1) by

$$SG(N, b/B) := \{x \in \mathbf{N}^r \mid Nx - b \in \operatorname{kernel}(h_B)\}.$$

Let N be an (m, r) integer matrix of form (3.2), let $b_i(x) := h_B(b - N_1 x)_{I_i}$, where I_i corresponds to the row indices of the submatrix N_i and let us denote by

$$z(b_i(y)) := \begin{cases} \infty & \text{if } b_i(y) \notin G(N_i / B_{I_i}), \\ \min c_i' x, & \\ x \in SG(N_i, b_i(y)/B_{I_i}) & \text{otherwise}, \end{cases}$$

the optimal value of the subproblems.

Proposition (3.4). *The programs*

$$\min c'x \qquad x \in SG(N, b/B), \tag{3.5}$$

$$\min c_1 y + \sum_{i=2}^{r} z(b_i(y)) \qquad y \in \mathbf{N} \tag{3.6}$$

are equivalent.

Proof. Let $r_i(y)$ be the minimard corresponding to the optimal value $z(b_i(y))$. Let y be optimal in (3.6) and assume that there is an $\hat{x} \in SG(N, b/B)$, $(\hat{x} \neq x := (y, r_2(y), \ldots, r_r(y))$ such that $c'\hat{x} < c'x$.

Let $\hat{x} := (\hat{y}_1, \hat{x}_2, \ldots, \hat{x}_r)$, where $\hat{y}_1$ are the components corresponding to N_1. Because $\hat{x}_i$ are feasible, we get

$$c_i'\hat{x}_i \geq \min c_i x_i = c'\bar{x}_i \qquad i = 2, \ldots, r$$
$$x_i \in SG(N_i, b_i(\hat{y}_i)/B_{I_i})$$

and the contradiction

$$c'\hat{x} \geq c_1\hat{y}_1 + \sum_{i=2}^{r} c_i'\hat{x}_i \geq c'x = \min\left\{c_1'y + \sum_{i=2}^{r} z(b_i(y)) \,\middle|\, y \in \mathbf{N}\right\}$$

proves one part of the proposition, however the reverse direction is trivial.

Let again N be an (m, r) integer matrix which has form (3.3) and define $z_1(x_2, \ldots, x_r) := \min c_1 x_1$
s.t.

$$x_1 \in SG\left[\begin{pmatrix} A_1 \\ N_1 \end{pmatrix}, \begin{pmatrix} b_0 - \sum_{i=2}^{r} A_i x_i \\ b_1 \end{pmatrix} \Big/ B_{I_1}\right],$$

$$z_i(x_i, \ldots, x_r) := \min c_i x_i + z_{i-1}(x_i, \ldots, x_r)$$
$$x_i \in SG(N_i, b_i / B_{I_i}), \quad i = 2, \ldots, r,$$

as the optimal value of the subproblems.