How to Think Like a Mathematician

A Companion to Undergraduate Mathematics

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CAMBRIDGE

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How to Think Like a Mathematician

Looking for a head start in your undergraduate degree in mathematics? Maybe you've already started your degree and feel bewildered by the subject you previously loved? Don't panic! This friendly companion will ease your transition to real mathematical thinking.

Working through the book you will develop an arsenal of techniques to help you unlock the meaning of definitions, theorems and proofs, solve problems, and write mathematics effectively. All the major methods of proof – direct method, cases, induction, contradiction and contrapositive – are featured. Concrete examples are used throughout, and you'll get plenty of practice on topics common to many courses such as divisors, Euclidean Algorithm, modular arithmetic, equivalence relations, and injectivity and surjectivity of functions.

The material has been tested by real students over many years so all the essentials are covered. With over 300 exercises to help you test your progress, you'll soon learn how to think like a mathematician.

Essential for any starting undergraduate in mathematics, this book can also help if you're studying engineering or physics and need access to undergraduate mathematics topics, or if you're taking a subject that requires logic such as computer science, philosophy or linguistics.

To Mum and Dad – Thanks for everything.

Preface

Question: How many months have 28 days?
Mathematician's answer: All of them.

The power of mathematics

Mathematics is the most powerful tool we have. It controls our world. We can use it to put men on the moon. We use it to calculate how much insulin a diabetic should take. It is hard to get right.

And yet. And yet . . . And yet people who use or like mathematics are considered geeks or nerds.[1] And yet mathematics is considered useless by most people – throughout history children at school have whined 'When am I ever going to use this?'

Why would anyone want to become a mathematician? As mentioned earlier mathematics is a very powerful tool. Jobs that use mathematics are often well-paid and people do tend to be impressed. There are a number of responses from non-mathematicians when meeting a mathematician, the most common being 'I hated maths at school. I wasn't any good at it', but another common response is 'You must be really clever.'

The concept

The aim of this book is to divulge the secrets of how a mathematician actually thinks. As I went through my mathematical career, there were many instances when I thought, 'I wish someone had told me that earlier.' This is a collection of such advice. Well, I hope it is more than such a collection. I wish to present an attitude – a way of thinking and doing mathematics that works – not just a collection of techniques (which I will present as well!)

If you are a beginner, then studying high-level mathematics probably involves using study skills new to you. I will not be discussing generic study skills necessary for success – time management, note taking, exam technique and so on; for this information you must look elsewhere.

I want you to be able to think like a mathematician and so my aim is to give you a book jam-packed with practical advice and helpful hints on how to acquire skills specific to

[1] Add your own favourite term of abuse for the intelligent but unstylish.

thinking like a mathematician. Some points are subtle, others appear obvious when you have been told them. For example, when trying to show that an equation holds you should take the most complicated side and reduce it until you get to the other side (page 143). Some advice involves high-level mathematical thinking and will be too sophisticated for a beginner – so don't worry if you don't understand it all immediately.

How to use this book

Each part has a different style as it deals with a different idea or set of ideas. The book contains a lot of information and, like most mathematics books, you can't read it like a novel in one sitting.

Some friendly advice

And now for some friendly advice that you have probably heard before – but is worth repeating.

- *It's up to you* – Your actions are likely to be the greatest determiner of the outcome of your studies. Consider the ancient proverb: The teacher can open the door, but you must enter by yourself.
- *Be active* – Read the book. Do the exercises set.
- *Think for yourself* – Always good advice.
- *Question everything* – Be sceptical of all results presented to you. Don't accept them until you are sure you believe them.
- *Observe* – The power of Sherlock Holmes came not from his deductions but his observations.
- *Prepare to be wrong* – You will often be told you are wrong when doing mathematics. Don't despair; mathematics is hard, but the rewards are great. Use it to spur yourself on.
- *Don't memorize – seek to understand* – It is easy to remember what you truly understand.
- *Develop your intuition* – But don't trust it completely.
- *Collaborate* – Work with others, if you can, to understand the mathematics. This isn't a competition. Don't merely copy from them though!
- *Reflect* – Look back and see what you have learned. Ask yourself how you could have done better.

To instructors and lecturers – a moment of your valuable time

One of my colleagues recently complained to me that when a student is given a statement of the form A implies B to prove their method of proof is generally wholly inadequate. He jokingly said, the student assumes A, works with that for a bit, uses the fact that B is true and so concludes that A is true. How can it be that so many students have such a hard time constructing logical arguments that form the backbone of proofs?

I wish I had an answer to this. This book is an attempt at an answer. It is not a theoretical manifesto. The ideas have been tried and tested from years of teaching to improve mathematical thinking in my students. I hope I have provided some good techniques to get them onto the path of understanding.

If you want to use this book, then I suggest you take your favourite bits or pick some techniques that you know your own students find hard, as even I think that students cannot swallow every piece of advice in this book in a single course. One aim in my own teaching is to be inspirational to students. Mathematics should be exciting. If the students feel this excitement, they are motivated to study and, as in the proverb quoted above, will enter by themselves. I aim to make them free to explore, give them the tools to climb the mountains, and give them their own compasses so they can explore other mathematical lands. Achieving this is hard, as you know, and it is often not lack of time, resources, help from the university or colleagues that is the problem. Often, through no fault of their own, it is the students themselves. Unfortunately, they are not taught to have a questioning nature, they are taught to have an answering nature. They expect us to ask questions and for them to give the answers because that is they way they have been educated. This book aims to give them the questions they need to ask so they don't need me anymore.

I'd just like to thank...

This book has had a rather lengthy genesis and so there are many people to thank for influencing me or my choice of contents. Some of the material appeared in a booklet of the same name, given to all first-year Mathematics students at the University of Leeds, and so many students and staff have given their opinions on it over the years. The booklet was available on the web, and people from around the world have sent unsolicited comments. My thanks go to Ahmed Ali, John Bibby, Garth Dales, Tobias Gläßer, Chris Robson, Sergey Klokov, Katy Mills, Mike Robinson and Rachael Smith, and to students at the University of Leeds and at the University of Warwick who were first subjected to my wild theories and experiments (and whose names I have forgotten). Many thanks to David Franco, Margit Messmer, Alan Slomson and Maria Veretennikova for reading a preliminary draft. Particular thanks to Margit and Alan with whom I have had many fruitful discussions. My thanks to an anonymous referee and all the people at the Cambridge University Press who were involved in publishing this book, in particular, Peter Thompson.

Lastly, I would like to thank my gorgeous wife Carol for putting up with me while I was writing this book and for putting the sunshine in my life.

Contents

PART I

Study skills for mathematicians

CHAPTER 1

Sets and functions

Everything starts somewhere, although many physicists disagree.
Terry Pratchett, *Hogfather*, 1996

To think like a mathematician requires some mathematics to think about. I wish to keep the number of prerequisites for this book low so that any gaps in your knowledge are not a drag on understanding. Just so that we have some mathematics to play with, this chapter introduces sets and functions. These are very basic mathematical objects but have sufficient abstraction for our purposes.

A set is a collection of objects, and a function is an association of members of one set to members of another. Most high-level mathematics is about sets and functions between them. For example, calculus is the study of functions from the set of real numbers to the set of real numbers that have the property that we can differentiate them. In effect, we can view sets and functions as the mathematician's building blocks.

While you read and study this chapter, think about *how* you are studying. Do you read every word? Which exercises do you do? Do you, in fact, do the exercises? We shall discuss this further in the next chapter on reading mathematics.

Sets

The set is the fundamental object in mathematics. Mathematicians take a set and do wonderful things with it.

Definition 1.1

A ***set*** *is a well-defined collection of objects.*[1]

The objects in the set are called the ***elements*** *or* ***members*** *of the set.*

We usually define a particular set by making a list of its elements between brackets. (We don't care about the ordering of the list.)

[1] The proper mathematical definition of set is much more complicated; see almost any text book on set theory. This definition is intuitive and will not lead us into many problems. Of course, a pedant would ask what does well-defined mean?

If x is a member of the set X, then we write $x \in X$. We read this as 'x is an element (or member) of X' or 'x is in X'.[2] If x is not a member, then we write $x \notin X$.

Examples 1.2

(i) The set containing the numbers 1, 2, 3, 4 and 5 is written $\{1, 2, 3, 4, 5\}$. The number 3 is an element of the set, i.e. $3 \in \{1, 2, 3, 4, 5\}$, but $6 \notin \{1, 2, 3, 4, 5\}$. Note that we could have written the set as $\{3, 2, 5, 4, 1\}$ as the order of the elements is unimportant.

(ii) The set {dog, cat, mouse} is a set with three elements: dog, cat and mouse.

(iii) The set {1, 5, 12, {dog, cat}, {5, 72}} is the set containing the numbers 1, 5, 12 and the sets {dog, cat} and {5, 72}. Note that sets can contain sets as members. Realizing this now can avoid a lot of confusion later.

It is vitally important to note that $\{5\}$ and 5 are not the same. That is, we must distinguish between being a set and being an element of a set. Confusion is possible since in the last example we have $\{5, 72\}$, which is a set in its own right but can also be thought of as an element of a set, i.e. $\{5, 72\} \in \{1, 5, 12, \{\text{dog}, \text{cat}\}, \{5, 72\}\}$.

Let's have another example of a set created using sets.

Example 1.3

The set $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$ has five elements. It has the the four elements, 1, 2, dog, mouse; and the other element is the set $\{3, 4\}$. We can write $1 \in X$, and $\{3, 4\} \in X$. It is vitally important to note that $3 \notin X$ and $4 \notin X$, i.e. the numbers 3 and 4 are not members of X, the set $\{3, 4\}$ is.

Some interesting sets of numbers

Let's look at different types of numbers that we can have in our sets.

Natural numbers

The set of **natural numbers** is $\{1, 2, 3, 4, \dots\}$ and is denoted by $\mathbb{N}$. The dots mean that we go on forever and can be read as 'and so on'.

Some mathematicians, particularly logicians, like to include 0 as a natural number. Others say that the natural numbers are the counting numbers and you don't start counting with zero (unless you are a computer programmer). Furthermore, how natural is a number that was not invented until recently?

On the other hand, some theorems have a better statement if we take $0 \in \mathbb{N}$. One can get round the argument by specifying that we are dealing with non-negative integers or positive integers, which we now define.

[2] Of course, to distinguish the x and X we read it out loud as 'little x is an element of capital X.'

Integers

The set of **integers** is $\{\dots, -4, -3, -2, 0, 1, 2, 3, 4, \dots\}$ and is denoted by $\mathbb{Z}$. The $\mathbb{Z}$ symbol comes from the German word Zahlen, which means number. From this set it is easy to define the **non-negative integers**, $\{0, 1, 2, 3, 4, \dots\}$, often denoted $\mathbb{Z}^+$. Note that all natural numbers are integers.

Rational numbers

The set of **rational numbers** is denoted by $\mathbb{Q}$ and consists of all fractional numbers, i.e. $x \in \mathbb{Q}$ if x can be written in the form p/q where p and q are integers with $q \neq 0$. For example, $1/2$, $6/1$ and $80/5$. Note that the representation is not unique since, for example, $80/5 = 16/1$. Note also that all integers are rational numbers since we can write $x \in \mathbb{Z}$ as $x/1$.

Real numbers

The **real numbers**, denoted $\mathbb{R}$, are hard to define rigorously. For the moment let us take them to be any number that can be given a decimal representation (including infinitely long representations) or as being represented as a point on an infinitely long number line.

The real numbers include all rational numbers (hence integers, hence natural numbers). Also real are π and e, neither of which is a rational number.[3] The number $\sqrt{2}$ is not rational as we shall see in Chapter 23.

The set of real numbers that are not rational are called **irrational numbers**.

Complex numbers

We can go further and introduce **complex numbers**, denoted $\mathbb{C}$, by pretending that the square root of -1 exists. This is one of the most powerful additions to the mathematician's toolbox as complex numbers can be used in pure and applied mathematics. However, we shall not use them in this book.

More on sets

The empty set

The most fundamental set in mathematics is perhaps the oddest – it is the set with no elements!

[3] The proof of these assertions are beyond the scope of this book. For π see Ian Stewart, *Galois Theory*, 2nd edition, Chapman and Hall 1989, p. 62 and for e see Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill 1976, p. 65.

Definition 1.4

The set with no elements is called the **empty set** *and is denoted* $\emptyset$.

It may appear to be a strange object to define. The set has no elements so what use can it be? Rather surprisingly this set allows us to build up ideas about counting. We don't have time to explain fully here but this set is vital for the foundations of mathematics. If you are interested, see a high level book on set theory or logic.

Example 1.5

The set $\{\emptyset\}$ is the set that contains the empty set. This set has one element. Note that we can then write $\emptyset \in \{\emptyset\}$, but we *cannot* write $\emptyset \in \emptyset$ as the empty set has, by definition, no elements.

Definition 1.6

Two sets are **equal** *if they have the same elements. If set* X *equals set* Y*, then we write* $X = Y$*. If not we write* $X \neq Y$.

Examples 1.7

(i) The sets $\{5, 7, 15\}$ and $\{7, 15, 5\}$ are equal, i.e. $\{5, 7, 15\} = \{7, 15, 5\}$.
(ii) The sets $\{1, 2, 3\}$ and $\{2, 3\}$ are not equal, i.e. $\{1, 2, 3\} \neq \{2, 3\}$.
(iii) The sets $\{2, 3\}$ and $\{\{2\}, 3\}$ are not equal.
(iv) The sets $\mathbb{R}$ and $\mathbb{N}$ are not equal.

Note that, as used in the above, if we have a symbol such as $=$ or $\in$, then we can take the opposite by drawing a line through it, such as $\neq$ and $\notin$.

Definition 1.8

If the set X *has a finite number of elements, then we say that* X *is a* **finite set**. *If* X *is finite, then the number of elements is called the* **cardinality** *of* X *and is denoted* $|X|$.

If X has an infinite number of elements, then it becomes difficult to define the cardinality of X. We shall see why in Chapter 30. Essentially it is because there are different sizes of infinity! For the moment we shall just say that the cardinality is undefined for infinite sets.

Examples 1.9

(i) The set $\{\emptyset, 3, 4, \text{cat}\}$ has cardinality 4.
(ii) The set $\{\emptyset, 3, \{4, \text{cat}\}\}$ has cardinality 3.

Exercises 1.10

What is the cardinality of the following sets?

(i) $\{1, 2, 5, 4, 6\}$	(ii) $\{\pi, 6, \{\pi, 5, 8, 10\}\}$
(iii) $\{\pi, 6, \{\pi, 5, 8, 10\}, \{\text{dog}, \text{cat}, \{5\}\}\}$	(iv) $\emptyset$
(v) $\mathbb{N}$	(vi) $\{\text{dog}, \emptyset\}$
(vii) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$	(viii) $\{\emptyset, \{20, \pi, \{\emptyset\}\}, 14\}$

Now we come to another crucial definition, that of being a subset.

Definition 1.11

*Suppose X is a set. A set Y is a **subset** of X if every element of Y is an element of X. We write $Y \subseteq X$.*

This is the same as saying that, if $x \in Y$, then $x \in X$.

Examples 1.12

(i) The set $Y = \{1, \{3, 4\}, \text{mouse}\}$ is a subset of $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$.
(ii) The set of even numbers is a subset of $\mathbb{N}$.
(iii) The set $\{1, 2, 3\}$ is not a subset of $\{2, 3, 4\}$ or $\{2, 3\}$.
(iv) For any set X, we have $X \subseteq X$.
(v) For any set X, we have $\emptyset \subseteq X$.

Remark 1.13

It is vitally important that you distinguish between being an *element* of a set and being a *subset* of a set. These are often confused by students. If $x \in X$, then $\{x\} \subseteq X$. Note the brackets. Usually, and I stress usually, if $x \in X$, then $\{x\} \notin X$, but sometimes $\{x\} \in X$, as the following special example shows.

Example 1.14

Consider the set $X = \{x, \{x\}\}$. Then $x \in X$ and $\{x\} \subseteq X$ (the latter since $x \in X$) but we also have $\{x\} \in X$.

Therefore we *cannot* state any simple rule such as 'if $a \in A$, then it would be wrong to write $a \subseteq A$', and vice versa.

If you felt a bit confused by that last example, then go back and think about it some more, until you really understand it. This type of precision and the nasty examples that go against intuition, and prevent us from using simple rules, are an important aspect of high-level mathematics.

Definition 1.15

*A subset Y of X is called a **proper subset** of X if Y is not equal to X. We denote this by $Y \subset X$. Some people use $Y \subsetneq X$ for this.*

Examples 1.16

(i) $\{1, 2, 5\}$ is a proper subset of $\{-6, 0, 1, 2, 3, 5\}$.
(ii) For any set X, the subset X is not a proper subset of X.
(iii) For any set $X \neq \emptyset$, the empty set $\emptyset$ is a proper subset of X. Note that, if $X = \emptyset$, then the empty set $\emptyset$ is *not* a proper subset of X.
(iv) For numbers, we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Note that we can use the symbols $\nsubseteq$ to denote 'not a subset of' and $\not\subset$ to denote 'not a proper subset of'.

Now let's consider where the notation came from. It is is obvious that for a finite set the two statements

$$\text{If } X \subseteq Y, \text{ then } |X| \leq |Y|,$$