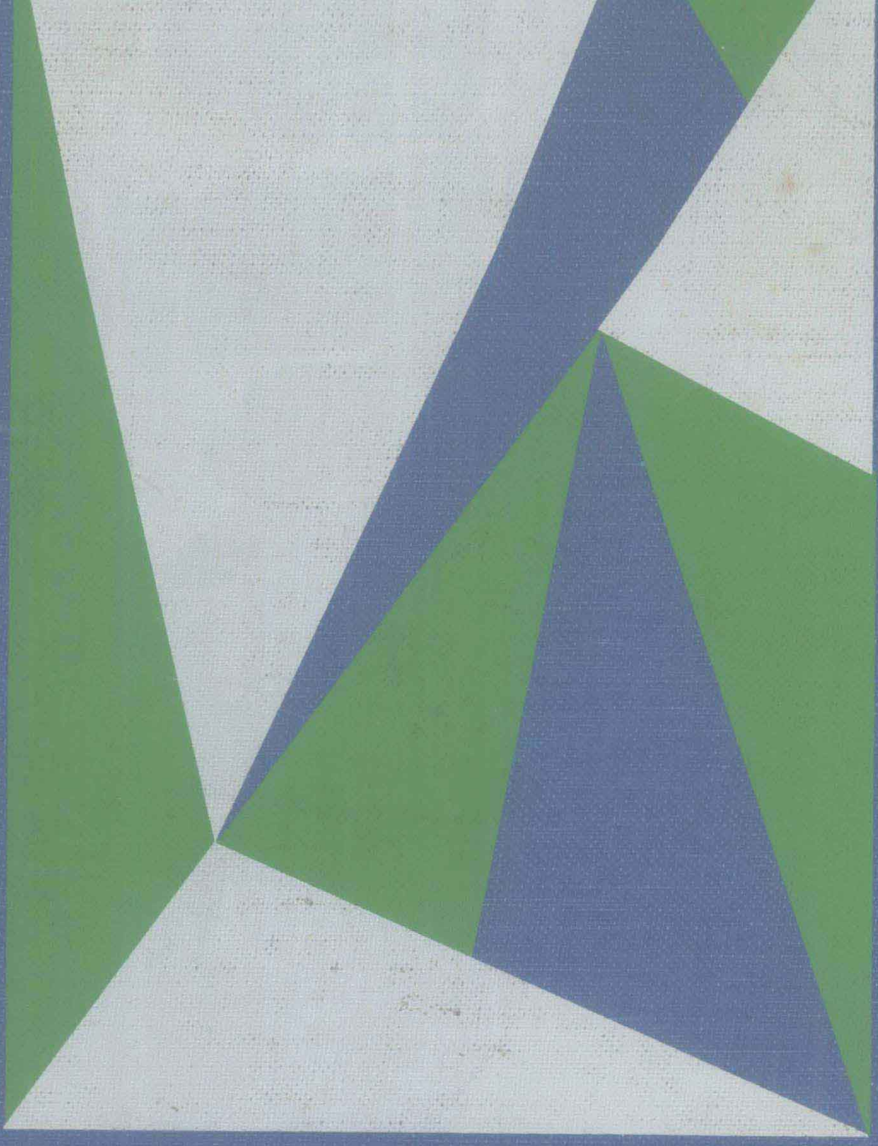

essentials of trigonometry

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trigonometry

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ESSENTIALS OF TRIGONOMETRY

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Preface

This textbook has been written for college classes. It contains essential material on the trigonometric functions, including some of the traditional applications.

The emphasis of the book is on the trigonometric functions *per se*. The definition and discussion of the idea of function, as a set of ordered pairs, which appears in the first chapter, not only gives clarity to a fundamental concept, but makes possible straightforward accounts of various topics; for example, of inverse functions.

The sine and cosine are defined in terms of the coordinates of points on the unit circle. The definitions and graphs of these and of the other "direct" trigonometric functions are discussed in considerable detail. If an easy familiarity with these graphs can be attained, many useful results become evident; for example, reduction formulas used in interpreting tables of values of the trigonometric functions.

The formula for $\cos (A - B)$ is obtained and many of the other usual addition formulas are then deduced. A chapter on identities puts more emphasis than is customary on the equivalence between trigonometric and corresponding algebraic identities obtained by applying to the former the definitions of the trigonometric functions. The Law of Cosines is proved and is then used in obtaining the Law of Sines.

Logarithm functions are introduced as inverses of the corresponding exponential functions. The use of logarithms in computation is treated and there are brief references to natural logarithms and to continuously compounded interest.

The last chapter applies the inverse functions in solving trigonometric equations and then turns to polynomial equations of the form $x^n + a = 0$. This motivates an outline of the construction of the complex numbers, based on assumed properties of the reals. De Moivre's theorem for integral exponents is derived, but in an informal way, since mathematical induction is not used.

There are twenty eight sections in the book, providing enough material for a one-semester course meeting two to three times a week, or its quarter-course equivalent. Answers to most of the even-numbered problems are given at the end of the book.

Trigonometry is a venerable subject and the trigonometric functions are ingenious and of enduring usefulness. Fresh presentations of this material seem worthwhile because of changing attitudes on the part of mathematics teachers and changing backgrounds which students are apt to bring to trigonometry courses. The stress on the idea of function and some of the material, such as on the construction of the complex numbers, might have seemed excessive to students approaching a trigonometry course as recently as a dozen years ago. Improvements in school courses in mathematics have, however, been profound in the last few years and students beginning the study of trigonometry now, are more critical and better prepared than formerly. This textbook, in the balance of intuition and demonstration application and theory, attempts to capitalize upon that excellent progress.

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Contents

CHAPTER 1

Coordinates and Functions

This chapter includes some material, such as that on rectangular coordinate systems, which may be familiar to many readers. The content of Section 1-3, however, will probably be new to most who approach the study of trigonometry. That section contains the definition of *function*, a fundamental mathematical term, and the distinctions and notations introduced there occur repeatedly in the book. This is followed by material on the measurement of angles, thus preparing the way for some of the classical applications of the trigonometric functions to the geometry of triangles. [TRIGONOMETRY: Gr. *trigōnon* triangle + *-metry* science of measuring.] A brief section on the analytic geometry of the circle concludes the chapter. It provides background for the definitions of the trigonometric functions which are given in Chapter 2.

1-1 THE RECTANGULAR COORDINATE SYSTEM

In establishing a rectangular coordinate system in the plane, two perpendicular straight lines are assumed, one generally sketched as horizontal. The lines intersect at a point, O, the *origin* of the coordinate system, and the lines themselves are *coordinate axes*.

A point, I, is selected to the right of the origin on the horizontal axis (the *x-axis*), and the distance of I from the origin is regarded as one unit.

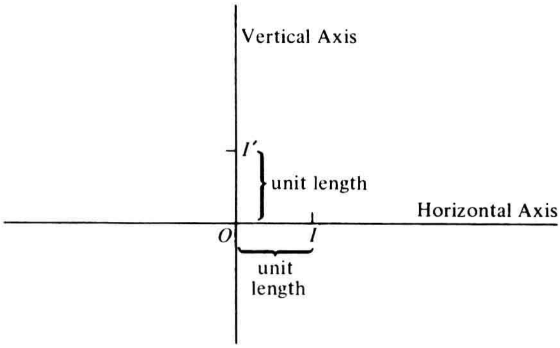

Figure 1-1

Corresponding to each point of this axis to the right of the origin there is a positive number which represents the distance, in terms of the unit length, of the point from the origin. Conversely, to each positive real number, p, there corresponds exactly one point on the axis to the right of the origin whose distance from the origin is p units. Similarly, the negative real numbers are put into correspondence with points of the x-axis to the left of the origin so that the point q units to the left has correlated with it the negative number $-q$. Finally, the origin is a point on this axis whose distance from itself is zero and, hence, associated with that point is the number 0.

In a similar way, a $1-1$ correspondence is established between points of the vertical axis (the *y-axis*) and the numbers. (In this textbook "number," if unqualified, means "real number.") Usually, as here, the same length is taken as unit length on the vertical axis as on the horizontal and the unit point I' on the vertical axis is located above the origin. Points on the y-axis above the origin then have positive numbers associated with them, while those below have negative numbers associated with them. The origin, considered as a point of the y-axis has the number 0 associated with it.

As described, the points on each of the two coordinate axes have been put into $1-1$ correspondence with the real numbers. In terms of these correspondences, it is then possible to establish a correspondence between points of the plane and so-called "ordered pairs" of numbers. Let P be a point of the plane not on either coordinate axis (Figure 1-2 illustrates), and, through P, construct a line parallel to the y-axis. That line intercepts the x-axis at precisely one point with which there is identified a unique real number, established previously in the correspondence between points of that axis and the numbers. Suppose the number is a. In the same fashion construct a line through P parallel to the x-axis. That line intercepts the y-axis at a point to which, in the correspondence between points of that axis and the real numbers, there has been assigned some number b. With P, then, associate the ordered pair of numbers, (a, b). The number a is the *first*

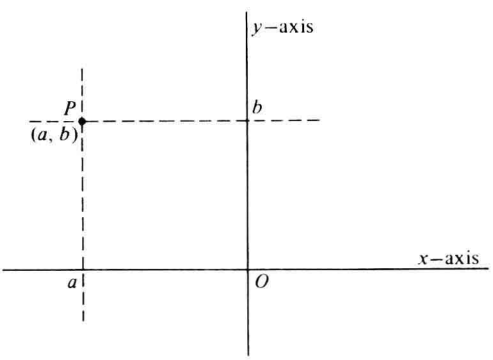

Figure 1-2

coordinate, the *x-coordinate*, or the *abscissa* of P; while b is the *second coordinate*, the *y-coordinate*, or the *ordinate* of P.

If P were a point on the horizontal axis, and if to that point the number c had been correlated, then associate with P the ordered pair $(c, 0)$. Finally, if P were a point on the vertical axis to which had been correlated the number d, then associate the ordered pair $(0, d)$ with P.

It should seem clear that these conventions correlate with each point of the plane exactly one ordered pair of numbers, once the correspondences

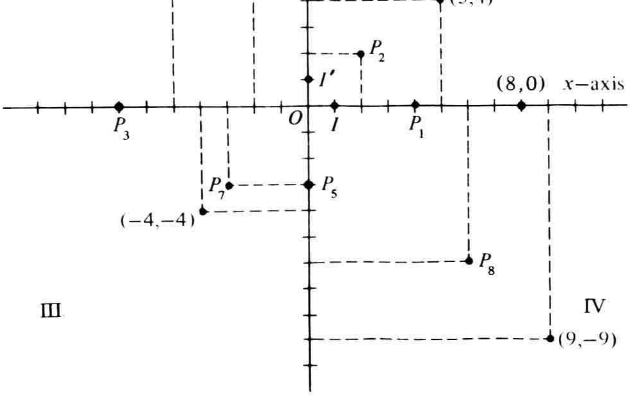

Figure 1-3

have been established on the two axes. Conversely, by this scheme, corresponding to each ordered pair of numbers (a, b), there is precisely one point in the plane whose first coordinate is a and whose second is b. (To see this, ask where a point must be if its first coordinate is to be a; then where it must be if its second coordinate is to be b.)

In Figure 1-3, various points are plotted and some are labeled with the ordered pairs which correspond. The two coordinate axes divide the plane into four *quadrants* denoted I, II, III, IV as indicated. Leaving aside the axes themselves, a point in quadrant I has both coordinates positive; in quadrant II the first coordinate is negative and the second positive. What conditions of this sort characterize points in quadrant III? In quadrant IV?

If a point P corresponds to the ordered pair (2, 3), for example, it is common to designate the point as "$P(2, 3)$;" again, to refer to P as "the point (2, 3)."

PROBLEMS

1. In which quadrant is the point corresponding to
   (a) (3, 2)?
   (b) (−3, 2)?
   (c) (6, −2)?
   (d) (−3, −4)?

2. In Figure 1-3, using the distance from O to I as unit length, find the ordered pair corresponding to each of the points $P_1, P_2, P_3, \ldots P_8$.

3. Construct a rectangular coordinate system and plot each of the points:
   (a) (2, 3).
   (b) (−4, 6).
   (c) (7, −3).
   (d) (6, 0).
   (e) (0, −5).
   (f) (−4, −1).
   (g) (0, 4).
   (h) (−2, 0).

4. (a) Find a value for b so that $(3, b)$ will lie in quadrant IV.
   (b) Can a number b be found so that $(3, b)$ will lie in quadrant III?
   (c) Find a value for a so that $(a, 2)$ will lie in quadrant II.
   (d) Can a number a be found so that $(-a, -a)$ will lie in quadrant I?

5. On a rectangular coordinate system
   (a) plot each of the points: $P_1(8, 0)$, $P_2(6, 6)$, $P_3(0, 8)$, $P_4(-6, 6)$, $P_5(-8, 0)$, $P_6(-6, -6)$, $P_7(0, -8)$, $P_8(6, -6)$, $P_9(6, 0)$, $P_{10}(4, 4)$, $P_{11}(0, 6)$, $P_{12}(-4, 4)$, $P_{13}(-6, 0)$, $P_{14}(-4, -4)$, $P_{15}(0, -6)$ and $P_{16}(4, -4)$.
   (b) Connect, using straight lines, P_1 with P_2, then P_2 with P_3, P_3 with P_4, and so on, until P_{15} connects with P_{16}. (A sense of symmetry may enable you to determine the correctness of your sketch.)

6. (a) Plot the ten points: $P_0(6, 6)$, $P_1(-6, 6)$, $P_2(-6, -6)$, $P_3(6, -6)$, $P_4(5, 5)$, $P_5(-5, 5)$, $P_6(-5, -5)$, $P_7(5, -5)$, $P_8(4, 4)$, $P_9(-4, 4)$.
   (b) Using straight lines, connect each of the points beginning with P_0, to the next point (P_1 in this case), ending the process with P_9.

(c) Following the suggested pattern, find the coordinates of the points P_{10}, P_{11}, P_{12}.

7. Plot the points $P_1(a_1, b_1)$, $P_2(a_2, b_2)$ in (a) through (j), and in each case compute $\dfrac{b_2 - b_1}{a_2 - a_1}$, where defined.

(a) $P_1(1, 1)$, $P_2(3, 3)$.
(b) $P_1(-1, -1)$, $P_2(2, -2)$.
(c) $P_1(0, -3)$, $P_2(3, 0)$.
(d) $P_1(0, 0)$, $P_2(1, 8)$.
(e) $P_1(0, 0)$, $P_2(0, 5)$.
(f) $P_1(1, -2)$, $P_2(-2, 10)$.
(g) $P_1(-1, -2)$, $P_2(-2, -10)$.
(h) $P_1(3, 4)$, $P_2(-3, 4)$.
(i) $P_1(3, 4)$, $P_2(3, -4)$.
(j) $P_1(-1, -8)$, $P_2(2, 16)$.

8. (a) Suppose a point $(h, 3/5)$ is known to be in quadrant I and that, $h^2 + (3/5)^2 = 1$. Find h.

(b) Can numbers a, b be found such that the point (a, b) lies in quadrant III while, at the same time, $a + b = 3$? Explain.

1-2 ABSOLUTE VALUE

In Section 1-1, the point $(a, 0)$, for a a positive number, was located a units to the right of point $(0, 0)$. On the other hand, if a were a negative number, $(a, 0)$ would be located $-a$ units to the left of $(0, 0)$. Suppose that $(a, 0)$, $(b, 0)$ are points on the x-axis. Is there a simple formula for the distance between them? The answer is evidently affirmative and is easily given in terms of the notion of absolute value.

DEFINITION. If x is a number, then the *absolute value of* x, denoted $|x|$, is a number defined as follows:

(i) $|x| = x$ if x is positive or zero;
(ii) $|x| = -x$ if x is negative.

Thus $|3| = 3$, according to part (i) of the definition; $|-3| = -(-3) = 3$, according to part (ii); $|0| = 0$, according to part (i). If x is any number then parts (i) and (ii) ensure that $|x|$ cannot be negative.

If A is a non-negative number, then $\sqrt{A}$ is by definition the non-negative number whose square is A. (It is frequently said that "$\sqrt{A}$ is the *principal square root of* A.") According to this convention, there is no ambiguity, for example, as to the meaning of "$\sqrt{16}$;" that symbol stands for the non-negative square root of 16 and hence for 4. Again, $\sqrt{4} = 2$, $\sqrt{100} = 10$, $\sqrt{0} = 0$. Of course, there continue to be two square roots of 16, one of which is 4 while the other is -4. The first of these is $\sqrt{16}$. (The other is $-\sqrt{16}$.) Again, it follows that $\sqrt{(-4)^2} = 4$, because the non-negative square root of $(-4)^2$, (that is, the non-negative square root of 16) is 4.

Similarly, $\sqrt{(-2)^2} = 2$, $\sqrt{(-7)^2} = 7$, $\sqrt{2^2} = 2$, $\sqrt{7^2} = 7$. In algebra courses it is argued that if A, B are non-negative numbers, then $\sqrt{A \cdot B} = \sqrt{A} \cdot \sqrt{B}$; also, if $B \neq 0$, $\sqrt{A/B} = \sqrt{A}/\sqrt{B}$. Again, if A is non-negative, then $(\sqrt{A})^2 = A$.

A useful relation between the notions "principal square root" and "absolute value" is expressed as follows.

THEOREM. If x is any number, then $|x| = \sqrt{x^2}$.

For if x is positive or zero, then $|x| = x$, while also $\sqrt{x^2} = x$. On the other hand, if x is negative, $|x| = -x$ while $\sqrt{x^2}$, the non-negative square root of x^2, is also $-x$. For example, $|2| = \sqrt{2^2} = 2$; $|-3| = \sqrt{(-3)^2} = 3$.

The above theorem has a certain utility, as the proofs of the following three corollaries show.

COROLLARY. If x is any number, then $|x| = |-x|$.

Proof. $|x| = \sqrt{x^2} = \sqrt{(-x)^2} = |-x|$.

COROLLARY. If x, y are numbers, then $|x \cdot y| = |x| \cdot |y|$.

Proof. $|x \cdot y| = \sqrt{(xy)^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|$.

COROLLARY. If x is a number, then $|x|^2 = x^2$.

Proof. $|x|^2 = (\sqrt{x^2})^2 = x^2$.

The absolute value notation may be used to write a convenient formula for the distance between points of the x-axis:

The distance between $A(a, 0)$ and $B(b, 0) = |b - a|$.

A little reflection on how distances may be measured using a unit of length such as a standard foot makes this formula seem correct. By the formula, the distance between A and B is always a non-negative number. As $|a - b| = |b - a|$, the distance between A and B is the same as that between B and A. This common value is also termed the *length* of the line segment determined by the two points.

To rationalize the formula, first suppose that the positions of $A(a, 0)$, $B(b, 0)$, $O(0, 0)$ are as indicated in Figure 1-4.

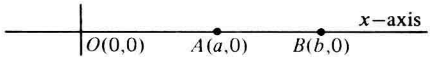

Figure 1-4

Here a, b are positive and B is to the right of A. The distance between A and B is apparently $b - a$. Since $b - a$ is positive, $b - a = |b - a|$, and the formula appears correct in this case.

But there are other arrangements. Consider Figure 1-5:

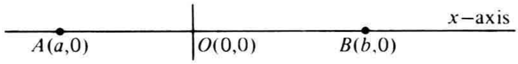

Figure 1-5

Here A is $|a|$ units to the left of O and the distance between A and B should be given as $|a| + b$. However, as a is negative and $b - a$ is positive,

$$|a| + b = -a + b = b - a = |b - a|.$$

Again, suppose that both A and B are to the left of O as in Figure 1-6.

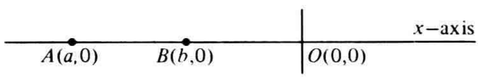

Figure 1-6

Now a, b are both negative; A is $|a|$ units and B is $|b|$ units to the left of O. In this case, the distance between A and B should be $|a| - |b|$. However, for a, b, negative $|a| - |b| = -a - (-b) = b - a$. But, with A to the left of B, $b - a$ is positive, whence $b - a = |b - a|$.

In the cases examined, where the point A is to the left of B, the number $|b - a|$ seems correct for the distance between A and B. Also, however, if three cases like the above were to be considered, except that B were to the left of A, the same considerations would give $|a - b|$ for the distance between the points. This, of course, is the same as $|b - a|$.

A few additional observations should be made on the distance formula. Thus, suppose that A and B were to coincide, as would occur were $a = b$. Then $|b - a| = |b - b| = 0$, which is appropriate for the distance between a point and itself. [Conversely, if $|b - a| = 0$, then $A(a, 0) = B(b, 0)$.] Again, one of the points, say $B(b, 0)$ might coincide with $O(0, 0)$. Then $|b - a| = |-a| = |a|$, in agreement with the way the first coordinate of A was defined.

In adopting the formula above for the distance between two points on the x-axis, it is possible to show that if P, Q, R are any three points of that axis, then the distance between P and R cannot exceed the sum of the distance between P and Q and that between Q and R. See Problem 7 at the end of this section and its extension in Problem 8.

The formula for the distance between points on the x-axis may be extended to give the distance between any two points, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, in the plane. The extension is based on the requirement that geometrically congruent line segments have the same length — that is, the distance between