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To Marlene and Esteban

Foreword

I met Henri Anciaux in 2000, at a time when I was reading an interesting paper where he solved partially a conjecture of Yong-Geun Oh. Since that time I have followed his scientific development and I have observed his skills to make his results more understandable by illustrating them with many examples.

In my opinion this book is a consequence of his particular vision of geometry. It also fills a long-standing gap, because many texts of pseudo-Riemannian geometry make use of physics to approach their topics, which sometimes implies that to learn basic concepts is a hard job for graduate students.

The introduction to pseudo-Riemannian and pseudo-Kähler geometries is enjoyable and easy to follow. The treatment made by the author about minimal, complex and Lagrangian submanifolds is clear and it will be useful for young researchers interested in these topics. The great quantity of examples not only helps to make clear the theory, but also allows an easier comprehension and a pleasant approach to the topic.

F. Urbano

Preface

About 1755, the Turinese mathematician Lagrange derived the differential equation that a function satisfies when its graph minimizes the area among all surfaces with the same boundary. This achievement may be considered as the birth of the theory of minimal submanifolds, although Euler had discovered a few years before the first non-planar example of minimal surface, the catenoid. This is a surface of revolution (actually the only non-planar, minimal one) which owes its name to the fact that its generating curve is the catenary, the curve obtained by hanging freely a chain with uniform weight. The next step was taken by Meusnier who gave a geometric characterization of the minimal surface equation: the sum of the two principal curvatures of the surface vanishes at any point. He also re-discovered the catenoid[1] and did discover the helicoid, the surface made up by the trajectory of a straight line subject to a helical motion.

Since then this subject has been enjoying an enduring —although not at a constant rate— development until today. It has become an important one, with connections not only with the analysis of partial differential equations, but more surprisingly with complex analysis and even algebraic geometry, and has received contributions of major mathematicians, such as Poisson, Riemann, Weierstrass, Calabi... to name a few. While even some problems regarding the original setting, i.e. two-dimensional surfaces in Euclidean three-dimensional space, have proved hard to handle (to give an example, it was proved only in 2005 that the only non-planar minimal surface of Euclidean three-dimensional space which is embedded, complete and simply connected is the helicoid, see [Meeks, Rosenberg (2005)]), the theory has been generalized in several directions: instead of surfaces of Euclidean three-dimensional space, one can consider higher dimensional submanifolds,

[1]In those times free of "publish or perish" ideology, successive discoveries were frequent.

and replace the multi-dimensional Euclidean space by an arbitrary Riemannian manifold. Ultimately, one may observe that the assumption that the metric tensor is positive is unnecessary for most of the aspects of the topic, and that it can be dropped: the most general framework in which to address the study of minimal submanifolds is therefore that of pseudo-Riemannian geometry[2].

Although there is a huge literature on Riemannian geometry, and in particular on the theory of minimal submanifolds (without intending to be complete, we refer to [Osserman (1969)], [Chen (1973)], [Spivak (1979)], [Nitsche (1989)], [do Carmo (1992)], [Xin (2003)]), there are not many books about pseudo-Riemannian geometry, and those we know are focused on global analysis and/or physical applications rather than submanifold theory (see [O'Neill (1983)], [Kriele (1999)], [Palomo, Romero (2006)], [Alekseevsky, Baum (2008)]). The purpose of this book is twofold. We first give a basic introduction to the theory of minimal submanifolds, set from the beginning in the pseudo-Riemannian framework. This includes the important first variation formula, i.e. the generalization of Meusnier observation stating that a minimal submanifold has vanishing mean curvature vector. Our second aim is to present a selection of important results, ranging from classical ones, suitably generalized to the pseudo-Riemannian case (such as the Weierstrass representation, the classification of ruled minimal surfaces and the minimality of complex submanifolds) to more elaborate ones, including the classification of equivariant minimal hypersurfaces and a detailed study of Lagrangian submanifolds. It is hoped that this book, despite its imperfections, will be useful for graduate and postgraduate students, and researchers interested in this growing, exciting field.

The text is organized as follows: The first chapter provides a set of definitions and facts about pseudo-Riemannian geometry and submanifold theory, ending with the proof of the first variation formula. We only assume from the reader some knowledge of basic manifold theory (including the notion of vector fields, submanifolds, integration), but of course some acquaintance with Riemannian geometry, or at least with the classical theory of curves and surfaces, will ease the reading of the whole book. All the necessary material can be found, for example, in [Kühnel (2000)],

[2]Minimal surfaces have also been introduced in two close but different fields, namely in affine geometry and discrete geometry. Strictly speaking, these concepts are variants (interesting ones!) and not generalizations of the classical one discussed here. We refer the interested reader to [Simon (2000)] and [Bobenko, Schröder, Sullivan, Ziegler (2008)].

[do Carmo (1976)], [do Carmo (1992)]. The second chapter is devoted to the case of surfaces (two-dimensional submanifolds) in pseudo-Euclidean space. We first describe a variety of examples and give a first global result: the classification of ruled minimal surfaces. We also derive a generalized form of the classical Weierstrass representation formulae, a very important tool in the study of minimal surfaces. The third chapter is more technical: we introduce the simplest examples of non-flat pseudo-Riemannian manifolds, the *space forms,* and the notion of equivariant hypersurface. We classify minimal hypersurfaces of pseudo-Euclidean space and of space forms which are equivariant with respect to some natural group actions. The fourth chapter forgets for a while the subject of submanifolds and is devoted to the description of an important class of manifolds which enjoy a triple structure: pseudo-Riemannian, complex, and symplectic. Such manifolds are called *pseudo-Kähler manifolds* and generalize the concept of Kähler manifold. We describe some examples, such as the complex counter-parts of the *space forms,* and show that the tangent bundle of a pseudo-Kähler manifold is itself pseudo-Kähler. In the fifth chapter we come back to the core of the subject, focusing on two special classes of submanifolds appearing in pseudo-Kähler geometry, the complex and the Lagrangian ones. It is easily seen that a complex submanifold is always minimal, and the rest of the chapter is devoted to the study of Lagrangian submanifolds. In particular, equivariant minimal Lagrangian submanifolds of complex pseudo-Euclidean space and of complex space forms are classified, with the method already used in Chapter 3. The last chapter raises briefly the important question of whether a minimal submanifold, which is, by definition, a critical point of the volume, is actually an extremum of the volume functional, or not. We give both necessary and sufficient conditions for this to happen.

The notations used throughout the text should be transparent to the reader familiar with current mathematical textbooks. A word written *in italics* is being defined in the statement in which it appears, and the expression $A := B$ means that the mathematical quantity A is defined to be equal to B. The symbol $\Box$ marks the end of a proof.

Most of this book was written in Tralee, while I was a post doctoral fellow of the SFI (Science Foundation of Ireland). I had the opportunity to give two mini courses based on the material of this book. The first one, in January 2009, took place at the Technische Universität of Berlin, where I benefited an Elie Cartan Scholarship (Stiftung Luftbrückendank), while the second one was given in June 2010 at the Federal University of São Carlos, thanks to the support of the FAPESP (Fomento de Amparo à Pesquisa do Estado de São Paulo). I am grateful to both Mike Scherfner and Guillermo Lobos for taking care of everything in Berlin and São Carlos respectively. I warmly thank Kwong Lai Fun, from World Scientific Publishing, who has supported me along the process of preparing and editing the manuscript. I am also greatly indebted to Ildefonso Castro, Benoît Daniel, Brendan Guilfoyle and Pascal Romon, all of them both colleagues and friends, who carefully read earlier versions of this work and whose remarks reduced significantly, I hope, the number of the typos and imprecisions of the final text. I am honoured that Francisco Urbano kindly accepted to write the foreword of this book and it is my pleasure to thank him. This book is dedicated to my wife Marlene and my son Esteban.

H. Anciaux

Contents

Chapter 1

Submanifolds in pseudo-Riemannian geometry

1.1 Pseudo-Riemannian manifolds

1.1.1 *Pseudo-Riemannian metrics*

A *pseudo-Riemannian* structure on a differentiable manifold $\mathcal{M}$ is simply a smooth, bilinear 2-form, called the *metric,* which is non-degenerate in the following sense: given a tangent vector X at some point x,

$$\text{if } g(X,Y) = 0, \forall\, Y \in T_x\mathcal{M}, \text{ then } X = 0.$$

If in addition the metric satisfies $g(X,X) > 0$ for any non-vanishing tangent vector X, we say that the metric is *positive,* and that we have a *Riemannian* structure. Hence Pseudo-Riemannian geometry is simply a generalization of Riemannian geometry. A number of properties of positive metrics are no longer true in the general case, such as Cauchy-Schwartz inequality.

Remark 1. The non-degeneracy assumption implies the following important fact: given a tangent vector X in $T_x\mathcal{M}$, if we know the value of $g(X,Y), \forall Y \in T_x\mathcal{M}$, then we can uniquely determine X.
In practice, we need only compute $g(X,X_i)$, where $(X_1,...,X_m)$ is a basis of $T_x\mathcal{M}$: setting $g_{ij} = g(X_i,X_j), 1 \leq i,j \leq m$, the fact that the metric is non-degenerate implies that the matrix $[g_{ij}]_{1\leq i,j\leq m}$ is invertible. Denoting the coefficients of the inverse matrix by g^{ij} and writing $X = \sum_{i=1}^m \lambda_i X_i$, we find that $g(X,X_j) = \sum_{i=1}^m \lambda_i g(X_i,X_j) = \sum_{i=1}^m \lambda_i g_{ij}$. Multiplying by g^{ij}, we get $\lambda_i = \sum_{j=1}^m g^{ij} g(X,X_j)$, hence

$$X = \sum_{i,j=1}^m g^{ij} g(X,X_j) X_i. \tag{1.1}$$

A non-vanishing tangent vector X will be called

- *positive* (or *spacelike*) if $g(X, X) > 0$;
- *negative* (or *timelike*) if $g(X, X) < 0$;
- *null* (or *lightlike*) if $g(X, X) = 0$.

These are the three possible *causal characters* of a vector. The terms in parenthesis come from relativity, the theory which first made use of pseudo-Riemannian geometry. More precisely the pseudo-Riemannian metric $\langle .,. \rangle_1 := -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ was introduced on the space $\mathbb{R}^4$, as a model of the space-time in special relativity[1]. In modern terms the pair $(\mathbb{R}^4, \langle .,. \rangle_1)$, sometimes written in abbreviated form $\mathbb{R}^4_1$ or $\mathbb{R}^{3,1}$, is called the Minkowski space (this space and its generalizations will be described in depth later on).

By Sylvester's theorem, at any point x of $\mathcal{M}$, there exists an *orthonormal* basis $(e_1, ..., e_m)$ of $T_x\mathcal{M}$, in the sense that $g(e_i, e_j) = 0$ if $i \neq j$ and $|g(e_i, e_i)| = 1$ (we shall say that the e_is are *unit* vectors). Moreover, the number p of vectors of the basis which are negative (and hence the number $m - p$ of those which are positive) does not depend of the basis, nor on the point x. The pair $(p, m - p)$ is called the *signature* of g. For example, if the signature is $(0, m)$, the metric is Riemannian; if neither p nor $m - p$ vanish, or equivalently if there exist null vectors, we say that the metric is *indefinite*. The metric of Minkowski space has signature $(1, 3)$. More generally, a pseudo-Riemannian manifold of signature $(1, m - 1)$ is referred to as a *Lorentzian* manifold. In the following, we shall set $\epsilon_i := g(e_i, e_i) = 1$ or -1 whenever we speak of an orthonormal basis $(e_1, ..., e_m)$. Formula (1.1) takes a much simpler form in the case of an orthonormal basis:

$$X = \sum_{i=1}^{m} \epsilon_i g(X, e_i) e_i. \qquad (1.2)$$

The non-degeneracy assumption of the metric allows to define the important concept of the *trace* of a bilinear form with respect to g. Let b be a bilinear form (possibly degenerate) on $T_x\mathcal{M}$, valued on any vector space F. Given a basis $(X_1, ..., X_m)$ of $T_x\mathcal{M}$, we claim that the quantity $\sum_{i,j=1}^m g^{ij} b(X_i, X_j) \in F$ depends only on g and b, not on the choice of the basis: if $(Y_1, ..., Y_m)$ is another basis of $T\mathcal{M}$, there exist real constants $a_{ij}, 1 \leq i, j \leq n$ such that $Y_i = \sum_{k=1}^m a_{ik} X_k$. Thus we have $b(Y_i, Y_j) = \sum_{k,l=1}^m a_{ik} a_{jl} h(X_i, X_j)$. On the other hand, setting $\bar{g}_{ij} := g(Y_i, Y_j)$, we check that $\bar{g}^{ij} = a^{ik} a^{jl} g^{kl}$, where $[a^{ik}]$ is the inverse matrix of $[a_{ik}]$.

[1]Soon after, the theory of general relativity replaced this model by a more general pseudo-Riemannian manifold, thus triggering broad interest in the subject.

Therefore,

$$\sum_{i,j=1}^{m} \bar{g}^{ij} b(Y_i, Y_j) = \sum_{k,l=1}^{m} g^{kl} b(X_k, X_l).$$

Hence, the next definition makes sense:

Definition 1. The *trace* of a bilinear form b with respect to g is the quantity

$$\mathrm{tr}(b) := \sum_{i,j=1}^{m} g^{ij} b(X_i, X_j).$$

Remark 2. Given an orthonormal basis $(e_1, ..., e_m)$, we have

$$\mathrm{tr}(b) = \sum_{i=1}^{m} \epsilon_i b(e_i, e_i).$$

1.1.2 *Structures induced by the metric*

A metric is a very rich structure, in the sense that it induces several other structures in a canonical way. In the following we are going to review them quickly. We refer to [Kriele (1999)] or [O'Neill (1983)] for further details.

1.1.2.1 *Volume*

A pseudo-Riemannian structure induces a volume structure, that is a n-density dV defined by

$$dV(X_1, ..., X_m) = |\det([g(X_i, X_j)]_{1\leq i,j\leq m})|^{1/2},$$

where $(X_1, ..., X_m)$ are m tangent vectors to $\mathcal{M}$ at the point x. In particular, we may define the volume (possibly infinite) of the manifold $\mathcal{M}$, simply by integrating dV on it:

$$\mathrm{Vol}(\mathcal{M}) := \int_{\mathcal{M}} dV.$$

1.1.2.2 *The Levi-Civita connection*

The differentiable structure of $\mathcal{M}$ allows to define the differentiation of a real function f in the direction of a tangent vector $X \in T_x\mathcal{M}$, denoted by $X(f)(x)$ or $df_x(X)$: we set

$$X(f)(x) := \frac{d}{dt} f(\gamma(t))\Big|_{t=0},$$

where $\gamma(t)$, $t \in I$ is a parametrized curve a curve $\gamma(t)$, such that $\gamma(0) = x$ and $\gamma'(0) = X$. To be rigorous, we should check that this definition does not depend on the choice of the curve γ but only on the vector X, i.e. that if $\tilde{\gamma}(0) = \gamma(0)$ and $\tilde{\gamma}'(0) = \gamma'(0)$, then $\frac{d}{dt} f(\gamma(t))\big|_{t=0} = \frac{d}{dt} f(\tilde{\gamma}(t))\big|_{t=0}$. This easy task is left to the reader.

An *affine connection* D or *covariant derivative* on $\mathcal{M}$ is, roughly speaking, a "way of differentiate a vector field" Y of $\mathcal{M}$ along a parametrized curve $\gamma(t)$. The result is again a vector field defined along the curve. As in the case of the differentiation of real functions, this quantity does not actually depend on the curve γ, but rather on its velocity $\gamma'(t)$. We therefore denote it by $D_X Y$, where $X = \gamma'(t)$. Of course the expression $D_X Y$ makes sense if X and Y are two vector fields defined on an open subset of $\mathcal{M}$, the result being itself a vector field. We require furthermore that D satisfies the two following rules:

$$D_X(fY) = f D_X Y + X(f)Y,$$
$$D_{fX} Y = f D_X Y,$$

where f is a real function on $\mathcal{M}$.

A parametrized curve $\gamma(t)$ is said to be a *geodesic* with respect to the connection D if $D_{\gamma'(t)}\gamma'(t) = 0, \forall t \in I$. Writing this equation in a local system of coordinates and using the theorem of existence for second order systems of ordinary differential equations, we get the local existence of geodesics: given a point $x \in \mathcal{M}$ and a tangent vector $X \in T_x\mathcal{M}$, there exists a real number t_0 and a unique geodesic $t \mapsto \gamma_{x,X}(t)$ defined on the interval $(-t_0, t_0)$, such that $\gamma_{x,X}(0) = x$ and $\gamma'_{x,X}(0) = X$.

There are many different affine connections on an arbitrary differentiable manifold, however the next result states that a pseudo-Riemannian structure comes with a canonical one:

Theorem 1. *There exists a unique affine connection D on a pseudo-Riemannian manifold $(\mathcal{M}, g)$ satisfying*

(i) *D has no torsion,* i.e.

$$D_X Y - D_Y X = [X, Y];$$

(ii) *g is parallel with respect to D,* i.e.

$$Z\big(g(X,Y)\big) = g(D_Z X, Y) + g(X, D_Z Y).$$

This unique connection is called the Levi-Civita connection of g.