H. Thomas (Ed.)

Nonlinear Dynamics in Solids

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Professor Dr. Harry Thomas
Institut für Physik, Universität Basel, Klingelbergstrasse 82,
CH-4056 Basel, Switzerland

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Nonlinear Dynamics in Solids

Preface

This volume contains the notes of lectures given at the school on "Nonlinear Dynamics in Solids" held at the Physikzentrum Bad Honnef, 2–6 October 1989 under the patronage of the Deutsche Physikalische Gesellschaft.

Nonlinear dynamics has become a highly active research area, owing to many interesting developments during the last three decades in the theoretical analysis of dynamical processes in both Hamiltonian and dissipative systems. Research has been focused on a variety of problems, such as the characteristics of regular and chaotic motion in Hamiltonian dynamics, the problem of quantum chaos, the formation and properties of solitary spatio-temporal structures, the occurrence of strange attractors in dissipative systems, and the bifurcation scenarios leading to complex time behaviour.

Until recently, predictions of the theory have been tested predominantly on instabilities in hydrodynamic systems, where many interesting experiments have provided valuable input and have led to a fruitful interaction between experiment and theory. Fluid systems are certainly good candidates for performing clean experiments free from disturbing influences: with fluids, compared to solids, it is simpler to prepare good samples, the relevant length and time scales are in easily accessible ranges, and it is possible to do measurements "inside" the fluid, because it can be filled in after the construction of the apparatus. Further, the theory describing the macroscopic dynamics of fluids is well established and contains only very few parameters, all of which have well-known values.

The dynamics of solids, on the other hand, is much richer and has a higher degree of physical interest. Moreover, for several solid-state systems, the methods of sample preparation and measuring techniques have reached a level where experiments on nonlinear dynamic behaviour of comparable quality and detail have become possible. It therefore appeared to be the right time to bring together a number of leading experts working on nonlinear dynamic properties of various solid-state systems to give introductory reports on the state of knowledge, problems incurred, and future prospects.

The volume starts with a presentation of the basic concepts of formation, symmetry and stability of dynamic structures, and a discussion of the analogies and differences to phase transitions in equilibrium systems. This is followed by a brief introduction to deterministic chaos and strange attractors, and an outline of methods for the characterization of chaotic motion and the reconstruction of attractors from experimental time series.

The next group of topics is concerned with nonlinear oscillations and chaos occurring in various solid-state systems: current instabilities and optical instabili-

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ties in semiconductors, driven Josephson junctions, and spin-wave instabilities in ferromagnets.

First, an introduction is given to various mechanisms of current instabilities in semiconductors, their theoretical description, and the methods used for their analysis. The current instability occurring in *p*-germanium due to avalanche breakdown gives rise to a rich variety of self-generated dynamical structures, as discussed in a contribution containing a detailed experimental analysis of oscillatory and chaotic states due to breathing of current filaments. Formation and dynamics of current filaments in semiconductor devices such as pin diodes is reviewed in a further contribution.

Next follows a discussion of optical bistabilities in passive semiconductors based on photo-thermal nonlinearities, and of self-oscillations in optical ring resonators with bistable elements.

A variety of nonlinear dynamical phenomena and chaos is expected to occur in Josephson junctions and devices driven by ac or dc currents. Such systems are the subject of a contribution reviewing experimental and numerical investigations of the characteristics of chaotic dynamics in various parts of parameter space.

Another class of solid-state systems exhibiting chaotic dynamics is ferromagnetic samples excited by strong microwave fields giving rise to spin-wave instabilities. The last contribution of this group reports and analyses complex multistable behaviour, self-oscillations, and bifurcation sequences leading to chaos found in magnetic resonance experiments on YIG (yttrium iron garnet) spheres.

A theme discussed in several of these contributions concerns the interplay between nonlinear dynamics and solid-state physics, and in particular the question to what extent it is possible to relate the details of the observed nonlinear dynamical effects – bifurcation scenarios, onset of chaos, characteristics of chaotic dynamics, etc. – to typical solid-state properties. Owing to the extreme sensitivity of the nonlinear dynamical properties to the detailed sample structure, this presents a serious problem: even samples cut from the same carefully grown material show differences in their nonlinear dynamic behaviour. On the other hand, this sensitivity is an important aspect for future development. It may well turn out that observation of nonlinear dynamical effects can be developed to become a most sensitive tool for the study of solid-state properties.

An interesting aspect of nonlinear dynamics, the formation of solitary spatiotemporal structures (kinks, domain walls), had actually already found experimental attention in solid-state physics at an early stage. Under certain conditions, these structures are expected to occur in thermodynamic equilibrium as a gas of nonlinear excitations; detailed investigations have been carried out in particular for quasi-one-dimensional magnetic systems, which are the subject of the next two contributions. The first of these reviews the theory of the formation, propagation and stability of such nonlinear excitations as well as their statistical mechanics, and the second reports results of inelastic neutron scattering, and NMR and ESR experiments for $CsNiF_3$ and TMMC (Tetramethylammonium Manganese Trichloride).

Macroscopic dynamical processes in solids are usually dissipative, and are therefore properly described in terms of the concepts of the dynamics of dissipative systems such as attractors and transients, with well-known exceptions of certain (nuclear and electron) spin systems with extremely small damping. However, semiconductor physics has reached a stage where mesoscopic semiconductor structures are now available in which electron transport is essentially dissipation-free, and which are therefore expected to exhibit nonlinear phenomena based on Hamiltonian dynamics. The relevance of Hamiltonian chaos and KAM theory for phenomena occurring in such systems is discussed in a contribution focusing on lateral surface superlattices.

It is an interesting question whether the concept of chaos may be extended to describe irregular behaviour in spatial dimensions, and whether spatial chaos is a useful concept for characterizing disordered or glassy structures. These questions are addressed in the last contribution of this volume, where simple models are introduced which have a multitude of spatially chaotic metastable states showing at least qualitatively some of the typical properties of glassy materials.

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H. Thomas

Contents

Dynamical Structures: Formation, Symmetry, Stability

H. Thomas

Institut für Physik der Universität Basel,
Klingelbergstrasse 82, CH-4056 Basel, Switzerland

1. Introduction

The subject of this lecture is the formation of dynamic structures in physical systems under the influence of external forces which drive currents through the system. It is a common feature of such "driven" systems that the driving force keeps them far from thermodynamic equilibrium, and that dissipation gives rise to the production of heat which has to be carried away by coupling the system to a heat sink (Fig. 1).

Our main interest is in systems under the influence of a *stationary* driving force. Here, one may distinguish two cases:

- Systems in which the force always gives rise to a current, for example a semiconductor in an electric field.
- Systems in which the occurrence of a current depends on the boundary conditions, for example a magnetic system in a static magnetic field: Here, thermodynamic equilibrium is possible because of the absence of magnetic charges, but continuous flow of magnetic flux may still occur if the field drives a domain wall between two oppositely magnetized domains.

Examples of driven systems in solid-state physics are treated in the other chapters of this book.

We shall concentrate on the behaviour of the system after the decay of transients. Of particular importance is the fact that, in contrast to thermodynamic systems which always approach a state of thermodynamic equilibrium, driven systems may either approach a stationary nonequilibrium state, or remain permanently time-dependent. It is the possiblity of the occurrence of such "dynamic structures" with a spontaneously broken time translation symmetry which makes the study of driven systems especially fascinating.

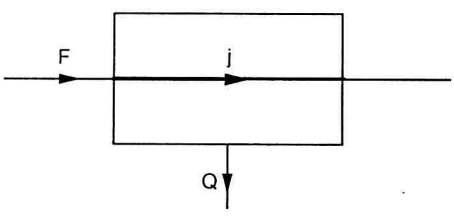

Fig. 1. The external force F drives a current j through the system; the dissipated energy Q is absorbed by a heat sink.

The formation of dynamic structures in driven systems bears a certain resemblance to the occurrence of phase transitions in thermodynamic equilibrium systems. In the latter case, symmetry aspects have proved to be of central importance. It appears therefore appropriate to extend the symmetry concept for application to driven systems, and to determine its significance for the prediction and classification of the types of dynamic structure which may occur in a given situation.

In this lecture, I shall give a simple introduction to the basic features of structure formation and its relation to symmetry. For more detailed presentations and discussions of other aspects, I refer to some standard texts [1–4] and a number of recent discussions of the subject [5–13].

2. Description of Driven Systems

2.1 Equation of Motion

We describe the dynamics of driven systems on a macroscopic level, similar to that used in Landau's theory of phase transitions: The *state* of the system is represented by a point $\boldsymbol{\theta}$ in a multidimensional state space (in general a differentiable manifold); the relevant external parameter is called the *control parameter* μ.

The state change in the course of time is described in terms of a velocity field $\boldsymbol{B}(\boldsymbol{\theta}, \mu)$ in state space, giving rise to an evolution equation

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} = \boldsymbol{B}(\boldsymbol{\theta}, \mu) \tag{2.1}$$

which is nonlinear and local in time (no memory!). In the case of a stationary control parameter μ, (2.1) is invariant with respect to time translations.

In dissipative systems, the vector field $\boldsymbol{B}$ is contracting, i.e. in the course of time any trajectory $\boldsymbol{\theta}(t)$ approaches an *attractor* in state space. The type of attractor may be characterized by a set of *Lyapunov exponents* (LE) which describe the asymptotic behaviour of the distance between two initially adjacent state points for $t \to \infty$. A trajectory $\gamma : \boldsymbol{\theta} = \boldsymbol{\theta}(t)$ has an LE λ if the distance $|\boldsymbol{\theta}(t) - \boldsymbol{\theta}_1(t)|$ between a point $\boldsymbol{\theta}(t)$ on γ and a neighboring point $\boldsymbol{\theta}_1(t)$ varies for $t \to \infty$ asymptotically as $\exp(\lambda t)$. In a state space of N dimensions, a trajectory has in general N LEs λ_i which satisfy $\sum_i \lambda_i < 0$ in dissipative systems. One distinguishes the following types of attractor:

- *Fixed points* in state space corresponding to stationary states $\boldsymbol{\theta}_\mathrm{s}(\mu)$, for which *all* Lyapunov exponents are negative, and which are found as stable solutions of
$$\boldsymbol{B}(\boldsymbol{\theta}_\mathrm{s}, \mu) = 0 \ . \tag{2.2}$$
- *Limit cycles* (1-*tori*) with a periodic time dependence, $\boldsymbol{\theta}_\mathrm{c}(t) = \boldsymbol{\theta}_\mathrm{c}(t + T)$, having 1 vanishing LE.

– Multiply periodic structures (*n-tori*) having n vanishing LEs.
– *Strange attractors* exhibiting *chaotic motion* characterized by 1 or more positive LEs.

Equation (2.1) represents a *deterministic* evolution equation. Fluctuations may be taken into account by adding a stochastic force $\boldsymbol{f}(\boldsymbol{\theta}, t)$ with $\langle \boldsymbol{f}(\boldsymbol{\theta}, t)\rangle = 0$ to the r.h.s of (2.1).

2.2 Symmetry

The set of transformations $g : \boldsymbol{\theta} \mapsto g\boldsymbol{\theta}$ of state space which leave the velocity field invariant forms the symmetry group G of the system:

$$G := \{ g \,|\, g\boldsymbol{B}(\boldsymbol{\theta}) = \boldsymbol{B}(g\boldsymbol{\theta}) \ \ \forall \boldsymbol{\theta} \} \ . \tag{2.3}$$

The symmetry group G_s of the stationary state $\boldsymbol{\theta}_s$ consists of all transformations $g \in G$ which leave $\boldsymbol{\theta}_s$ invariant:

$$G_s := \{ g \in G \,|\, g\boldsymbol{\theta}_s = \boldsymbol{\theta}_s \} \ . \tag{2.4}$$

The stationary state $\boldsymbol{\theta}_s$ may be fully symmetric (G_s coincides with G), or it may already have a broken symmetry (G_s is a genuine subgroup of G). In the latter case, there exists a set of symmetry-related stationary states corresponding to the left cosets of G_s in G.

Further, a stationary state is invariant under all time translations $\mathbf{T}(\tau) : t \mapsto t + \tau$ forming the full-time-translation group

$$\mathcal{I} := \{ \mathbf{T}(\tau) \,|\, \tau \in \mathbb{R} \} \ . \tag{2.5}$$

The *extended symmetry group* $\mathcal{G}_s$ of the stationary state $\boldsymbol{\theta}_s$ is defined as

$$\mathcal{G}_s := G_s \times \mathcal{I} \ . \tag{2.6}$$

It corresponds to the "H-group" in Landau's theory of phase transitions. Structure formation is associated with a breaking of symmetry: A structure bifurcating from $\boldsymbol{\theta}_s$ will be characterized by an "L-group", which is a subgroup of $\mathcal{G}_s$ (see Sect. 3.2).

3. Self-oscillations (Limit Cycles)

3.1 Bifurcation of Self-oscillations (Hopf Bifurcation)

3.1.1 Destabilization of a Stationary State. In order to obtain conditions for the formation of self-oscillations, we study the linear stability of the stationary state $\boldsymbol{\theta}_s(\mu)$ with respect to small perturbations $\boldsymbol{\vartheta}(t)$: $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_s + \boldsymbol{\vartheta}(t)$. Linearization of (2.1) about $\boldsymbol{\theta}_s$ yields the equation of motion for $\boldsymbol{\vartheta}$

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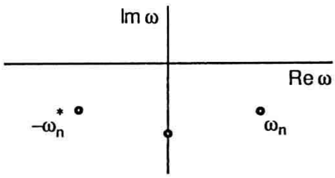

Fig. 2. Poles in the complex frequency plane with $\mathrm{Re}\,\omega_n \neq 0$ occur in pairs $(\omega_n, -\omega_n^*)$

$$\frac{\mathrm{d}\boldsymbol{\vartheta}}{\mathrm{d}t} = \mathbf{L}(\mu)\cdot\boldsymbol{\vartheta} \ , \tag{3.1}$$

where $\mathbf{L}(\mu)$ is a real time-independent matrix defined by

$$\mathbf{L} := \partial \boldsymbol{B}/\partial\boldsymbol{\theta}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_\mathrm{s}} \ . \tag{3.2}$$

The *normal modes* of the stationary state $\boldsymbol{\theta}_\mathrm{s}$ are solutions of (3.1) of the form

$$\boldsymbol{\vartheta}_n(t) = \boldsymbol{p}_n \mathrm{e}^{-\mathrm{i}\omega_n t} \ ; \tag{3.3}$$

their polarization amplitudes $\boldsymbol{p}_n$ and frequencies ω_n are determined by the linear eigenvalue problem

$$\mathbf{L}(\mu)\cdot\boldsymbol{p}_n = -\mathrm{i}\,\omega_n\boldsymbol{p}_n \ . \tag{3.4}$$

Because of the reality of $\mathbf{L}(\mu)$, normal modes with $\mathrm{Re}\,\omega_n \neq 0$ occur in pairs $\{(\boldsymbol{p}_n, \omega_n), (\boldsymbol{p}_n^*, -\omega_n^*)\}$ (see Fig. 2).

We assume that the stationary state $\boldsymbol{\theta}_\mathrm{s}$ is stable in a control-parameter range $\mu < \mu_\mathrm{c}$:

$$\mathrm{Im}\,\omega_n(\mu) < 0 \ \text{ for } \mu < \mu_\mathrm{c} \qquad \forall n \tag{3.5}$$

(except for zero-frequency modes),

and that at the stability threshold $\mu = \mu_\mathrm{c}$ there occurs an instability with respect to a single, possibly degenerate mode (single mode pair if $\mathrm{Re}\,\omega_1 \neq 0$):

$$\mathrm{Im}\,\omega_1(\mu_\mathrm{c}) = 0 , \qquad \left.\frac{\mathrm{d}\,\mathrm{Im}\,\omega_1}{\mathrm{d}\mu}\right|_{\mu=\mu_\mathrm{c}} > 0 \ . \tag{3.6}$$

Such an instability marks the bifurcation of a new structure.

3.1.2. The Order Parameter. Types of Bifurcating Structures. Close to the bifurcation threshold, the bifurcating structure is uniquely described by the projection $\boldsymbol{\phi}$ of the state vector $\boldsymbol{\theta}$ onto the space Ω spanned by the undamped normal modes. In analogy to Landau's theory of phase transitions, the projection $\boldsymbol{\phi}$ is called the "order parameter" (OP) associated with the instability. By adiabatic elimination of the other components of $\boldsymbol{\theta}$, one obtains from (2.1) a reduced equation of motion in OP space

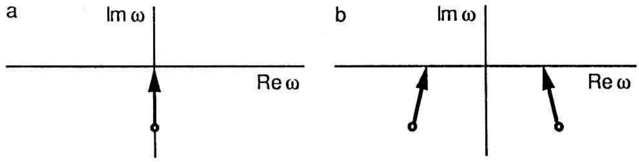

Fig. 3. **(a)** Soft-mode instability. **(b)** Hard-mode instability

$$\frac{d\phi}{dt} = -i\,\omega_1(\mu)\,\phi + \boldsymbol{N}(\phi) \;, \tag{3.7}$$

where $\boldsymbol{N}(\phi)$ contains all nonlinear terms. Symmetry requires that the terms in (3.7) transform equivariant with the OP. Therefore, the form of this equation may be constructed without actually carrying out the adiabatic elimimation, by writing $\boldsymbol{N}(\phi)$ as a sum of polynomial equivariants of increasing degree, with coefficients considered as model parameters.

The type of instability is determined by the real part $\omega_c := \mathrm{Re}\,\omega_1(\mu_c)$ of the mode frequency at threshold (see Fig. 3 and Table 1):

- $\omega_c = 0$: If a purely *relaxational* mode becomes undamped ("soft-mode instability"), one expects a bifurcation of a new *stationary state*.
- $\omega_c \neq 0$: If an *oscillating* mode becomes undamped ("hard-mode instability"), one expects a bifurcation of a *limit cycle* with period $T_c = 2\pi/\omega_c$ at threshold.

Table 1. Unstable modes and bifurcating structures

Type of Instability		Bifurcating Structure	
Soft Mode:	$\omega_c = 0$	Stationary:	$\phi = \text{const}$
Hard Mode:	$\omega_c \neq 0$	Oscillating:	$\phi(t+T) = \phi(t)$ $T_c = 2\pi/\omega_c$

These expectations are indeed confirmed by the formal bifurcation analysis based on an expansion of the solution of (3.7) (or even of the original equation (2.1)) in powers of the amplitude $\varepsilon = |\phi|$ of the OP. From such an analysis, one obtains the components of the OP, the control parameter μ, and in the case of a limit cycle its period T expressed in powers of ε,

$$\phi = \phi(\varepsilon) \quad \mu = \mu(\varepsilon) \,, \quad T = T(\varepsilon) \;. \tag{3.8}$$

The form of the function $\mu(\varepsilon)$ determines the existence range of the new structure (see Fig. 4):

- $\mu < \mu_c$ (subcritical bifurcation),
- $\mu > \mu_c$ (supercritical bifurcation),
- $\mu <> \mu_c$ (transcritical bifurcation).

The dependence of the amplitude of the OP, its orientation in OP space, and (if applicable) its period T on the control parameter μ are found from (3.8) by

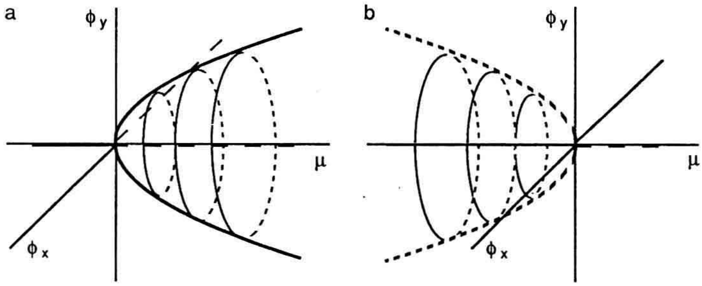

Fig. 4a,b. Hopf bifurcation. **(a)** supercritical, **(b)** subcritical

elimination of the amplitude parameter ε. Finally, the structures identified in this way have to be tested for stability in order to determine which of them can actually occur. It is found that close to the bifurcation threshold, subcritical branches are always unstable; only the supercritical branches are candidates for stable structures. If there is no symmetry at all, there either occurs a transcritical bifurcation with exchange of stability, or a Hopf bifurcation.

The bifurcation problem depends in an essential way on the degeneracy of the destabilized mode, i.e. on the dimension of the OP space Ω (assuming that the representation of the OP occurs with multiplicity 1) (Table 2):

- Nondegenerate soft mode, dim $\Omega = 1$: The direction of ϕ is fixed by symmetry; only its amplitude has to be determined from bifurcation analysis.
- Degenerate soft mode, dim $\Omega > 1$: Both the amplitude and the direction of ϕ have to be determined from bifurcation analysis.

Correspondingly:

- Nondegenerate hard mode, dim $\Omega = 2$ (Hopf bifurcation): The orientation of the limit cycle is fixed by symmetry, only its amplitude has to be determined from bifurcation analysis.
- Degenerate hard-mode, dim $\Omega > 2$ (degenerate Hopf bifurcation): Both the orientation of the limit cycle and its amplitude have to be determined from bifurcation analysis.

Note that in the case of a hard-mode instability dim Ω is even, i.e. the smallest dimension for a degenerate Hopf bifurcation is 4. Degenerate Hopf bifurcations with dim $\Omega = 4$ are studied in [6,7,9]; other examples are provided by wave bifurcations associated with degenerate modes (see Sects. 4, 5).

Table 2. OP repesentation and bifurcating structure

Instability type	dim Ω	Bifurcating Structure
Soft mode	dim $\Omega = 1$	uniquely determined by symmetry
	dim $\Omega \geq 2$	depending on higher-order terms
Hard Mode	dim $\Omega = 2$	uniquely determined by symmetry
	dim $\Omega \geq 4$	depending on higher-order terms

In most cases it is found that the direction of the OP satisfies a rule of "minimally broken symmetry":

> The symmetry group of the bifurcating structure is a maximal subgroup of the symmetry group of the state $\boldsymbol{\theta}_{\mathbf{s}}$,

which may serve as a first indication of the structures to be expected.

Other types of bifurcation which occur frequently but which cannot be found from the stability analysis of an "unperturbed" stationary state $\boldsymbol{\theta}_{\mathrm{s}}(\mu)$ include:

- "Saddle-node bifurcation": At a critical value of the control parameter there appears a pair of fixed points, one of them stable, the other unstable, but unconnected to any other fixed point.
- "Homoclinic bifurcation" of a limit cycle: At a critical value of the control parameter there appears a fixed point $\boldsymbol{\theta}_{\mathrm{h}}$ with a homoclinic orbit starting at $t = -\infty$ at $\boldsymbol{\theta}_{\mathrm{h}}$ and returning at $t = \infty$ to $\boldsymbol{\theta}_{\mathrm{h}}$ which develops into a limit cycle bifurcating with zero frequency but finite amplitude.

3.1.3 Symmetry Properties of the Order Parameter. The OP space Ω carries a representation of the symmetry group G_{s} of the parent state $\boldsymbol{\theta}_{\mathrm{s}}$. In analogy to Landau's theory of phase transitions, one expects that this representation contains important information on the bifurcation behaviour. Of particular interest is the question whether it can be predicted on the basis of symmetry alone if the instability gives rise to a stationary or an oscillating structure. The following result shows the extent to which this is the case [8]:

> The OP associated with the bifurcation of a *stationary state* transforms as a real irreducible representation of G_{s}
> The OP associated with a *limit cycle* bifurcation transforms
> - either as a physically irreducible representation consisting of two complex conjugate irreducible representations ("symmetry-induced limit-cycle bifurcation")
> - or as a reducible real representation consisting of two equivalent real irreducible representations of G_{s} ("coupling-induced limit-cycle bifurcation").

The two types of limit-cycle bifurcation may be illustrated for the case of a two-dimensional OP space by considering viscous motion in a potential $V(\phi_x, \phi_y) = -\frac{1}{2}\mu(\phi_x^2 + \phi_y^2) + V_0(\phi_x, \phi_y)$ under the action of an azimuthal force $\omega_0\{-\phi_y, \phi_x\}$,

$$\frac{\mathrm{d}\phi_x}{\mathrm{d}t} = \mu\phi_x - \omega_0\phi_y - \frac{\partial V_0}{\partial \phi_x}, \qquad \frac{\mathrm{d}\phi_y}{\mathrm{d}t} = \mu\phi_y + \omega_0\phi_x - \frac{\partial V_0}{\partial \phi_y}. \tag{3.9}$$

If $V(\phi_x, \phi_y)$ has n-fold symmetry, it will for sufficiently large values of the control parameter μ develop n minima separated by saddles with heights of $\mathrm{O}(|\phi_0|^n)$, where $|\phi_0|$ is the amplitude of the OP at minimum.

– For $n \geq 3$, the representation spanned by (ϕ_x, ϕ_x) is irreducible, and the azimuthal force can in fact drive the system across the saddles, independent of the value of the coupling constant ω_0 (symmetry-induced limit cycle).

– For $n = 2$, on the other hand, the representation decomposes into two equivalent real representations spanned by ϕ_x and ϕ_y. In this case, the coupling constant ω_0 has to exceed a critical value in order to drive the system across the saddle (coupling-induced limit cycle); for values of ω_0 smaller than the critical value, the system becomes trapped in a stationary state in one of the two potential troughs.

3.2 Symmetry of Cycles

A cycle breaks the symmetry of the stationary state which is described by its symmetry group G_s and the extended symmetry group $\mathcal{G}_s$ defined in (2.4,6). The symmetry group of the cycle $\boldsymbol{\theta}_c$ consists fo all transormations $g \in G_s$ which leave the *orbit* $[\boldsymbol{\theta}_c]$ of the cycle invariant:

$$G_c := \{ g \in G_s \,|\, [g\boldsymbol{\theta}_c] = [\boldsymbol{\theta}_c] \} \ . \tag{3.10}$$

Further, the cycle $\boldsymbol{\theta}_c(t) = \boldsymbol{\theta}_c(t+T)$ is left invariant under *discrete* time translations $\mathbf{T}(nT)$ forming a *time lattice*

$$\mathcal{I}_T := \{ \mathbf{T}(nT) \,|\, n = 0, \pm 1, \dots \} \ . \tag{3.11}$$

The *extended symmetry group* $\mathcal{G}_c$ of the cycle consists of all transformations $g \in \mathcal{G}_s$ which leave the cycle invariant:

$$\mathcal{G}_c := \{ g \in \mathcal{G}_s \,|\, g\boldsymbol{\theta}_c(t) = \boldsymbol{\theta}_c(t) \ \forall t \} \ . \tag{3.12}$$

It consists of a product of the time lattice $\mathcal{I}_T$ and a group L which is isomorphic to G_c:

$$\mathcal{G}_c = L \times \mathcal{I}_T \quad \text{with} \quad L \cong G_c \ . \tag{3.13}$$

According to the rule of minimally broken symmetry, G_c is a maximal subgroup of G_s. The extended symmetry group $\mathcal{G}_c$ corresponds to the space group of a spatial structure, and the group G_s to its point group.

The time lattice $\mathcal{I}_T$ gives rise to a *Brillouin-zone* (BZ) structure on the $\mathrm{Re}\,\omega$ axis. In terms of the reciprocal lattice vector $\Omega := 2\pi/T$, the first BZ is given by

$$-\Omega < \mathrm{Re}\,\omega \le \Omega \tag{3.14}$$

(see Fig. 5). The analog of Bragg scattering by spatial lattices is the occurrence

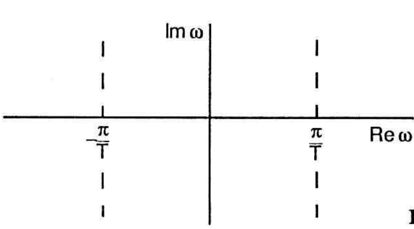

Fig. 5. "Brillouin zone" on the real frequency axis

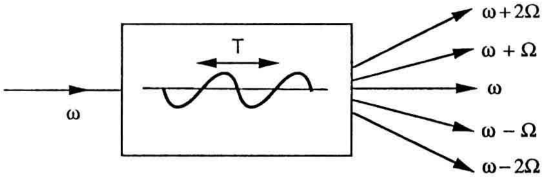

Fig. 6. "Bragg scattering" at a time lattice: Linear response to a force at frequency ω gives rise to side bands at frequencies $\omega \pm n\Omega$ where $\Omega = 2\pi/T$

of *side-bands* at frequencies $\omega \pm n\Omega$ in the linear response of the cycle to an external force of frequency ω (Fig. 6).

3.3 Destabilization of a Cycle

The linear stability of the cycle $\boldsymbol{\theta}_{\rm c}(t)$ is determined by the time evolution of small perturbations $\boldsymbol{\vartheta}(t)$: $\boldsymbol{\theta}_{\rm c}(t) + \boldsymbol{\vartheta}(t)$. Linearization of (2.1) about $\boldsymbol{\theta}_{\rm c}(t)$ yields the equation of motion

$$\frac{{\rm d}\boldsymbol{\vartheta}}{{\rm d}t} = \mathbf{L}(t,\mu)\cdot\boldsymbol{\vartheta} \ , \tag{3.15}$$

where $\mathbf{L}(t)$ is a periodically time-dependent matrix of period T,

$$\mathbf{L}(t) = \partial \boldsymbol{B}/\partial\boldsymbol{\theta}\big|_{\boldsymbol{\theta} = \boldsymbol{\theta}_{\rm c}(t)} = \mathbf{L}(t+T) \ . \tag{3.16}$$

According to the Floquet–Bloch theorem, the normal modes of the cycle are solutions of (3.15) of the form

$$\boldsymbol{\vartheta}_n(t) = \boldsymbol{\psi}_n(t)\,{\rm e}^{-{\rm i}\omega_n t} \ , \tag{3.17}$$

where $\boldsymbol{\psi}_n(t)$ is T-periodic, and ${\rm Re}\,\omega_n$ lies in the first BZ,

$$\boldsymbol{\psi}_n(t+T) = \boldsymbol{\psi}_n(t)\,, \qquad -\frac{\pi}{T} < {\rm Re}\,\omega_n \le \frac{\pi}{T} \ . \tag{3.18}$$

The eigenvalues ω_n and eigenfunction $\boldsymbol{\psi}_n(t)$ have to be found as solutions of the eigenvalue problem

$$\frac{{\rm d}\boldsymbol{\psi}_n}{{\rm d}t} - \mathbf{L}(t,\mu)\cdot\boldsymbol{\psi}_n = {\rm i}\,\omega_n(\mu)\,\boldsymbol{\psi}_n \ . \tag{3.19}$$

We assume that the cycle $\boldsymbol{\theta}_{\rm c}(t)$ is stable up to a second threshold $\mu_{\rm c2}$, where an instability occurs with respect to a single mode:

$${\rm Im}\,\omega_1(\mu_{\rm c2}) = 0\,, \qquad \frac{{\rm d}\,{\rm Im}\,\omega_1}{{\rm d}\mu}\bigg|_{\mu = \mu_{\rm c2}} > 0 \ . \tag{3.20}$$

The type of instability is again determined by the real part of the eigenvalue ω_1 at threshold (see Fig. 7):

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