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SUB-HARDY HILBERT SPACES IN THE UNIT DISK

DONALD SARASON

UNIVERSITY OF ARKANSAS LECTURE NOTES IN THE MATHEMATICAL SCIENCES

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SUB-HARDY HILBERT SPACES IN THE UNIT DISK

University of Arkansas Lecture Notes in the Mathematical Sciences

WILLIAM H. SUMMERS, Series Editor

FOREWORD

I have lived the greater part of my mathematical life in the unit disk of the complex plane. From afar the disk may seem a constraining environment, but to me, as to many predecessors, its terrain has offered unending fascination.

These notes focus on a family of Hilbert spaces that live inside the Hardy space H^2 of the disk. The spaces emerge from a viewpoint developed by Louis de Branges and were originally investigated around 1966 by de Branges in collaboration with James Rovnyak [14]. My own acquaintance with de Branges's viewpoint began, somewhat belatedly, with a 1984 lecture of his I heard. One of de Branges's key ideas is the notion of a complementary space, a generalization of the notion of orthogonal complement. In an effort to understand this notion I tried to see how it applies in a setting with which I was familiar, which took me back to the spaces of de Branges and Rovnyak. These spaces, and their vector-valued analogues, are the setting for the operator model theory of de Branges and Rovnyak [15]. My initial motivation was to understand the basic structure of the (scalar-valued) de Branges-Rovnyak spaces, and to understand the relation between the de Branges-Rovnyak model theory and the better known model theory of B. Sz.-Nagy and C. Foias (again, in the scalar-valued case). It soon became clear that, besides possessing a fascinating internal structure, the spaces of de Branges and Rovnyak have a role to play in several questions in function theory I had previously considered, and several additional ones as well. The aim of these notes is to describe some of what has been learned thus far about the structure of the de Branges–Rovnyak spaces and about their function-theoretic connections.

Chapter I introduces several Hilbert space notions, including that of a complementary space, needed in the sequel. The remaining chapters are devoted to an exploration of the structure of the spaces of de Branges and Rovnyak, which are introduced in Chapter II. There is one of these spaces associated with each nonconstant function b in the unit ball of H^∞; the space associated with b is denoted by $\mathcal{H}(b)$. A related space, denoted $\mathcal{H}(\bar{b})$, arises naturally in its study. The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under the backward shift operator, and the restrictions of the backward shift to the spaces $\mathcal{H}(b)$ comprise the simplest class of de Branges-Rovnyak model operators.

In Chapter III, Cauchy integral representations of the spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are derived. In the case of $\mathcal{H}(b)$, the more complicated of the two cases, the representation gives an isometry between $\mathcal{H}(b)$ and the H^2 space of the measure on the unit circle whose Poisson integral is the real part of the function $\dfrac{1+b}{1-b}$.

The structures of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are sensitive in crucial ways to whether b is or is not an extreme point of the unit ball of H^∞. For example, these spaces are invariant under the forward shift operator if and only if b is not an extreme point. Chapter IV is devoted to the case where b is not an extreme point, and Chapter V to the case where it is an extreme point.

In Chapter VI it is shown that two classical theorems in function theory, C. Carathéodory's theorem on angular derivatives, and the theorem of A. Denjoy and J.Wolff on iteration, fit naturally within the context of the spaces $\mathcal{H}(b)$. Chapter VII sketches a partial extension of Carathéodory's theorem to higher derivatives.

Chapters VIII–X address a variety of questions concerning the spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. For example, the conditions under which $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ coincide are determined, and the connection between $\mathcal{H}(b)$ and so-called rigid functions in H^1 is explained. The concluding Chapter XI contains brief mention of a few additional topics of current interest.

A large portion of the results presented here are already in the literature, and references to original sources are provided. Most references are confined to the Notes sections at the ends of most chapters. The treatment here, it is hoped, offers improvements over previous ones, thereby making the subject more accessible.

A preliminary version of these notes, under the title *Function Theory in the Unit Disk from a Hilbert Space Perspective*, was completed in 1991 and circulated to a few colleagues and students. That version has been revised to take account of recent developments and the comments of Wiley's reviewers. The author expresses his gratitude to the reviewers for their very helpful suggestions. He is also grateful to José Barría for detecting numerous corrections in the preliminary version. So as to avoid the task of renumbering the list of references, the references added after the completion of the preliminary version have been put in a supplementary list.

Besides possessing a standard background in real and complex analysis, functional analysis, and operator theory, the reader of these notes is assumed to be familiar with the theory of Hardy spaces in the unit disk. The material in the initial chapters of any of the standard references on Hardy spaces, such as the books of P. L. Duren [S2] and J. B. Garnett [27], will be adequate. Basic properties of Toeplitz operators will be needed from time to time. The book of R. G. Douglas [S1] is a good reference. Someone with the preceding prerequisites will find the treatment here reasonably self-contained.

These notes grew out of my 1989 lectures in the University of Arkansas Annual Lecture Series in the Mathematical Sciences. I am deeply indebted to the University of Arkansas for its splendid hospitality. Special thanks are due to the conference organizers: John Akeroyd, John Duncan, Daniel Luecking, Itrel Monroe, and William Summers.

Berkeley, California
December 22, 1993

CONVENTIONS

The following standard conventions in notation and terminology are used in these notes.

1. All Hilbert spaces considered are assumed to be complex and separable.
2. Subspaces of a Hilbert space are assumed to be closed. The terms "vector subspace" and "linear manifold" are used to designate possibly nonclosed subspaces.
3. Hilbert space operators are assumed to be linear.
4. A scalar multiple of the identity operator on a Hilbert space is identified notationally with the corresponding scalar.
5. If H is a Hilbert space, then $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$ denote the inner product and norm in H. The subscript will be modified in certain cases.
6. The open unit disk in the complex plane is denoted by D and its boundary, the unit circle, by ∂D.
7. L^p denotes the standard Lebesgue space with respect to normalized Lebesgue measure on ∂D. The corresponding Hardy space is denoted by H^p; in the usual way, it will be regarded either as a subspace of L^p or as a space of holomorphic functions in D, as convenience dictates. The space of functions in H^p that vanish at the origin is denoted by H_0^p. (The cases $p = 1, 2, \infty$ are the main ones of interest here.)
8. The inner product and norm in L^2 are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$.
9. The shift operator on H^2 is denoted by $S : (Sf)(z) = zf(z)$. It's adjoint, the backward shift, is given by $(S^*f)(z) = (f(z) - f(0))/z$.
10. If u and v are vectors in the Hilbert space H, then $u \otimes v$ denotes the rank-one operator on H that sends the vector x to the vector $\langle x, v\rangle_H\, u$.

SUB-HARDY HILBERT SPACES IN THE UNIT DISK

CONTENTS

INDEX OF NOTATIONS

The notations are listed in the order of appearance. Each is followed by the number of the section in which it is introduced.

CHAPTER I

HILBERT SPACES INSIDE HILBERT SPACES

This chapter contains some general facts about Hilbert spaces and Hilbert space operators that are needed in the investigations to follow. The theme is Hilbert spaces that live inside larger Hilbert spaces. A Hilbert space contained boundedly in a larger Hilbert space can be realized as an operator range. The complementary space of an operator range is defined when the operator is a contraction. The approach to complementary spaces employed here emphasizes the operator viewpoint, whereas the original approach of de Branges is more geometric. The two approaches are reconciled in one of the notes at the end of the chapter.

A basic question one encounters in dealing with contained Hilbert spaces is that of recognizing when a given vector in the containing space also lies in the contained one. An often useful criterion will be given, which, for the complementary space associated with a given contraction, relates the question to the analogous one for the adjoint of the contraction.

Another basic issue is the relation between factorization of a contraction and decomposition of its associated complementary space. A general result along these lines will be established.

(I-1) Bounded and Contractive Containment. If H is a Hilbert space, one says that another Hilbert space is contained boundedly in H if it is a vector subspace of H and if the inclusion map of it into H is bounded. If the inclusion map is a contraction, one says that the second Hilbert space is contained contractively in H.

Examples come readily to mind. Every subspace of H is contained contractively (in fact, isometrically) in it. If H is somehow supplied with a second inner product giving an equivalent norm, then H equipped with the new inner product is contained boundedly in H equipped with the original one, and vice versa. If μ and ν are positive measures on the same sigma-algebra and μ dominates ν (i.e., is at least as large on every measurable set), then $L^2(\mu)$ is contained contractively in $L^2(\nu)$.

(I-2) Operator Ranges. If A is a bounded operator from the Hilbert space H_1 into the Hilbert space H, then we define $\mathcal{M}(A)$ to be the range of A with the Hilbert space structure that makes A a coisometry from H_1 onto $\mathcal{M}(A)$. Thus, if x and y are vectors in H_1 and if they are orthogonal to the kernel of A (or even if only one of them is orthogonal to the kernel of A), then

$$\langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x, y\rangle_{H_1}.$$

The space $\mathcal{M}(A)$ is contained boundedly in H, and if A is a contraction it is contained contractively in H. Every Hilbert space contained boundedly in H is such an operator range; it is, namely, the range of the inclusion map of it into H.

(I-3) Transfer of Linear Functionals. Suppose H, H_1 and A have the meanings above, and let y be a vector in H. The restriction to $\mathcal{M}(A)$ of the linear functional on H induced by y is then a bounded linear functional on $\mathcal{M}(A)$. It is thus induced, relative to the inner product in $\mathcal{M}(A)$, by a vector in $\mathcal{M}(A)$. That vector is AA^*y, as one sees from the calculation

$$\langle Ax, y\rangle_H = \langle x, A^*y\rangle_{H_1} = \langle Ax, AA^*y\rangle_{\mathcal{M}(A)}.$$

(I-4) Douglas's Criterion. The following criterion of R. G. Douglas is often useful in establishing containment relations among spaces and in showing that an operator maps one space into another: *Let H, H_1 and H_2 be Hilbert spaces, and let A and B be bounded operators from H_1 and H_2, respectively, into H. Then the operator inequality $AA^* \leq BB^*$ is necessary and sufficient for the existence of a factorization $A = BR$ with R a contraction from H_1 into H_2.*

That the factorization implies the inequality is obvious. To establish the other half of the criterion one argues just as in the proof of the polar decomposition theorem. Namely, if the inequality holds, one first defines an operator Q from the range of B^* to the range of A^* by setting $QB^*x = A^*x$ ($x \in H$). The inequality implies that the definition makes sense and that Q does not increase norms. Thus Q extends by continuity to a contraction from the closure of the range of B^* into H_1. We can finally extend Q to a contraction from H_2 into H_1 by letting it be the zero operator on the orthogonal complement of the range of B^*. The operator $R = Q^*$ then has the desired properties.

(I-5) Consequences. The following conclusions are immediate consequences of Douglas's criterion.

(i) The space $\mathcal{M}(A)$ is contained contractively in the space $\mathcal{M}(B)$ if and only if $AA^* \leq BB^*$.

(ii) The spaces $\mathcal{M}(A)$ and $\mathcal{M}(B)$ coincide as Hilbert spaces if and only if $AA^* = BB^*$. In particular, $\mathcal{M}(A) = \mathcal{M}((AA^*)^{1/2})$.

(iii) The space $\mathcal{M}(A)$ is an ordinary subspace if and only if A is a partial isometry.

(I-6) Complementary Spaces. If A is a Hilbert space contraction, then the space $\mathcal{M}((1-AA^*)^{1/2})$ is called the complementary space of $\mathcal{M}(A)$ and is denoted by $\mathcal{H}(A)$. If $\mathcal{M}(A)$ is an ordinary subspace, in other words, if A is a partial isometry, then AA^* and $1-AA^*$ are complementary projections, and $\mathcal{H}(A)$ is the ordinary orthogonal complement of $\mathcal{M}(A)$. In the contrary case the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$, called here an overlapping space, is nontrivial, as will be seen shortly.

(I-7) Intertwining Relation. *If A is a Hilbert space contraction, then $A(1-A^*A)^{1/2} = (1-AA^*)^{1/2}A$.*

The proof starts from the obvious equality $A(1-A^*A) = (1-AA^*)A$, which can be iterated to give $A(1-A^*A)^n = (1-AA^*)^n A$ for every positive integer n. Hence, if p is any polynomial, then $Ap(1-A^*A) = p(1-AA^*)A$. Now take a sequence $(p_n)_1^\infty$ of polynomials that converges uniformly on the interval $[0,1]$ to the square-root function. Then $p_n(1-A^*A) \to (1-A^*A)^{1/2}$ in norm and $p_n(1-AA^*) \to (1-AA^*)^{1/2}$ in norm, and the desired equality follows.

(I-8) Relation Between $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$. *Let A be a contraction from the Hilbert space H_1 into the Hilbert space H. Then the vector x in H belongs to $\mathcal{H}(A)$ if and only if A^*x belongs to $\mathcal{H}(A^*)$. If x_1 and x_2 are two vectors in $\mathcal{H}(A)$, then*

$$\langle x_1, x_2 \rangle_{\mathcal{H}(A)} = \langle x_1, x_2 \rangle_H + \langle A^*x_1, A^*x_2 \rangle_{\mathcal{H}(A^*)}.$$

In fact, the inclusion $A^*\mathcal{H}(A) \subset \mathcal{H}(A^*)$ follows immediately from the intertwining relation (I-7). Suppose on the other hand that x is a vector in H such that A^*x is in $\mathcal{H}(A^*)$, say $A^*x = (1-A^*A)^{1/2}y$, where y is in H_1. Then the equality $x = (1-AA^*)x + AA^*x$ can, in virtue of the intertwining relation, be rewritten as

$$x = (1-AA^*)^{1/2}[(1-AA^*)^{1/2}x + Ay],$$

which shows that x is in $\mathcal{H}(A)$.

To obtain the expression for the inner product, let x_1 and x_2 be two vectors in $\mathcal{H}(A)$, and for $j = 1, 2$ let y_j be the vector in H_1 that is orthogonal to the kernel of $1 - A^*A$ and satisfies $A^*x_j = (1 - A^*A)^{1/2}y_j$. For each j we then have

$$x_j = (1 - AA^*)^{1/2}[(1 - AA^*)^{1/2}x_j + Ay_j]$$

and, because of the way y_j was chosen, the vector in square brackets is orthogonal to the kernel of $1 - AA^*$, as one easily verifies. Hence

$$\langle x_1, x_2\rangle_{\mathcal{H}(A)} = \langle(1 - AA^*)^{1/2}x_1 + Ay_1, (1 - AA^*)^{1/2}x_2 + Ay_2\rangle_H.$$

When one expands the inner product on the right side one obtains four terms. One term is $\langle(1 - AA^*)^{1/2}x_1, (1 - AA^*)^{1/2}x_2\rangle_H$, which is the same as $\langle x_1, x_2\rangle_H - \langle A^*x_1, A^*x_2\rangle_{H_1}$. Another is $\langle Ay_1, Ay_2\rangle_H$. There are then two "cross-product" terms, of which one is $\langle(1 - AA^*)^{1/2}x_1, Ay_2\rangle_H$. Because of the intertwining relation (I-7) this can be rewritten as

$$\langle A^*x_1, (1 - A^*A)^{1/2}y_2\rangle_{H_1},$$

which equals $\langle A^*x_1, A^*x_2\rangle_{H_1}$. The other cross-product term has the same value. All together, then, we have

$$\begin{aligned}
\langle x_1, x_2\rangle_{\mathcal{H}(A)} &= \langle x_1, x_2\rangle_H + \langle A^*x_1, A^*x_2\rangle_{H_1} + \langle Ay_1, Ay_2\rangle_H \\
&= \langle x_1, x_2\rangle_H + \langle(1 - A^*A)^{1/2}y_1, (1 - A^*A)^{1/2}y_2\rangle_{H_1} + \langle Ay_1, Ay_2\rangle_H \\
&= \langle x_1, x_2\rangle_H + \langle y_1, y_2\rangle_{H_1} \\
&= \langle x_1, x_2\rangle_H + \langle A^*x_1, A^*x_2\rangle_{\mathcal{H}(A^*)},
\end{aligned}$$

as desired.

(I-9) Description of the Overlapping Space. *If A is as above, then $\mathcal{M}(A) \cap \mathcal{H}(A) = A\mathcal{H}(A^*)$.*

This follows immediately from (I-8) (with the roles of A and A^* reversed). Notice that the overlapping space is trivial if and only if $A(1-A^*A)^{1/2} = 0$. If that happens then also $A(1-A^*A) = 0$, so $A = AA^*A$. Then $AA^* = (AA^*)^2$, which means AA^* is a projection (that is, A is a partial isometry), and $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are ordinary subspaces of H, orthogonal complements of each other.