JOHNSON & ZACCARO

MODERN INTRODUCTORY MATHEMATICS

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MODERN INTRODUCTORY MATHEMATICS

PREFACE

This book may be considered as an answer to the question: What mathematics would you give to a college freshman who is not planning to take the customary college mathematics course sequence? It has always seemed to the authors that the modern fundamental mathematics texts overemphasize logical explanations at the expense of problem-solving techniques and ignore many important topics of steadfast college algebra.

The work is intended to be a blending of the best of traditional college algebra and the so-called "modern" approach to mathematics, with its emphasis upon concept and its logical explanations for the memory rules of manipulation. Thus the book may be thought of as *college algebra brought up to date,* or *fundamentals of mathematics with emphasis upon college algebra.*

An attempt is made to acquaint the student with the various aspects of the word "mathematics." It must be borne in mind that the word is an arbitrary sound used to indicate roughly a certain type or range of man's endeavors. Do not look for an exact definition, but rather note what mathematicians do. Its meaning is conveyed gradually by continued practice in what mathematicians consider mathematical activity.

Chapter 1 consists of a condensed nonaxiomatic development of sets, a topic fundamental to virtually every subject. Chapter 2 outlines *intuitively* the procedures of mathematical proofs and mathematical systems. Chapter 3 fixes the real number system as a deductive postulational system with a degree of rigor judged reasonable for the freshman level. One point emphasized in Chapter 3 is the logical genesis of many of the manipulative rules of algebra. Among other things, this fills a need of teachers of arithmetic, who are thereby equipped to give full meaning to algebraic operations, their origins, and their uses.

The authors have found that the contents can be used to satisfy the needs of several groups. The following suggestions are some of the ways the book may be used:

1 Future elementary school teachers and secondary school teachers should cover at least Chapters 1 to 5.
2 For a three-hour course designed for nonscience students, the authors have found success using Chapters 1, 2, 3, 13, and 14. How-

ever, Chapters 1, 2, and 3, along with any suitable combination of the remaining chapters, would make a good fundamentals course. Another good combination consists of Chapters 1, 2, 3, 7, and 8. In choosing combinations, consider Chapters 11 and 12 as a unit. This is also true of Chapters 13 and 14.

3 A six-semester hour course can utilize any of the following chapter combinations: Chapters 1 to 10; Chapters 1 to 4 and 11 to 14; Chapters 1 to 3 and 7 to 12.

The traditional method of using examples to give specific interpretations of abstract ideas has been employed. In addition, *drill problems* have been included as a further aid to the student. These problems are intended to challenge the student's grasp of the material and provoke further thought. The answers to these drill problems are given where appropriate. In the case of a multiple-part problem, *the answers are given in an arbitrary order.*

For those studying Chapter 6, which deals with the complex numbers, it should be noted that from Section 6.4 on certain fundamentals of trigonometry are required. These prerequisites may be summarized as follows: (1) The two common systems of angle measure, radians and the sexagesimal system; (2) the definitions of the sine, cosine, and tangent of an angle; (3) that $\theta = \tan^{-1}(b/a)$ means $\tan\theta = b/a$; (4) the following identities: $\tan\theta = \sin\theta/\cos\theta$, $\sin^2\theta + \cos^2\theta = 1$, $\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi$, $\cos(\theta \pm \varphi) = \cos\theta\cos\varphi \pm \sin\theta\sin\varphi$.

Answers to most of the odd-numbered exercises are at the back of the book. The answers to most of the even-numbered exercises are contained in the Teacher's Manual.

The authors welcome any suggestions for improvement.

Wendell G. Johnson

Luke N. Zaccaro

TO THE STUDENT

The student of our culture who seeks the meaning of the word "mathematics" must first realize the futility of trying to funnel centuries of various activities into one simple phrase. Any such simplified statement would be either misleading or too general to convey meaning to the beginning student.

Mathematics is a "doing" subject. The student must participate in mathematics in order to understand it. For this reason, you will find many opportunities in each chapter to get the feel of the mathematics by doing it. The experience of facing a mathematical situation, thinking about it, and eventually devising a solution for it will result finally in understanding the meaning of the word.

This book was written with the assumption that you have already studied some algebra and some geometry, possibly a year of each, and so have had some experience of doing already. The authors have attempted here to build upon this experience and to approach mathematics from a more general, mature standpoint. The nature of mathematics as a whole will be studied first, followed by a topical sequence which exposes the fundamental subjects in mathematics for this level of training. Experience is a maturing influence for anyone; and as *you* gain mathematical experience, so the book takes an increasingly mature view.

The path of learning we have mapped out for you is not unlike the paths you followed in learning English, except that it is somewhat more formal. Before studying the grammar of the English language, you first had some experience learning words and phrases, during which their meanings became increasingly useful to you in expressing yourself. Then later, as you acquired maturity in the language, you consciously studied its structure and general nature. Your previous experience gave practical meaning to this more abstract approach. So it is with mathematics, and in particular with the approach adopted here.

The first three chapters, then, are devoted to the words, phrases, and "grammar." Chapter 1 is a brief introduction to the subject of sets, which underlie not only all of mathematics but all subjects to which logic may be applied. They are a mathematician's building blocks by which he seeks agreement with others on the meanings of his mathematical words and phrases. Chapter 2 describes the logical nature of mathematics but in terms that do not resort to the formalistic technology of mathematical logic. An attempt is made here to acquaint you with the logical setting of mathematics in our culture, just as the grammarian may show you how the words and phrases of English may be strung together to make meaningful statements without your having to analyze every sentence.

In Chapter 3 we begin to look into our number system, in particular the real numbers, and we examine the assumptions we must make in order to use numbers with precision. Many of the memory rules and manipulative methods of your precollege studies are confirmed, but as deductions from the information you have learned in the chapters on sets and numbers. It is hoped that the deductive steps taken from a few simple assumptions to an extensive group of theorems will help remove the aura of mystery surrounding the origins of these theorems and surrounding mathematics in general. To sum up, Chapter 2 sets the nature of a postulational or axiomatic system, and Chapter 3 presents the real numbers as just such a system.

These three chapters together deal with the logical nature of mathematics, and Chapter 4 follows with mathematical induction, how it arises, and how it is carried out. From then on, each chapter deals with a specific topic.

For the most part, these are topics of algebra which form a good base for continuing in mathematics; and they revolve around the study of number systems and their properties, using letters and symbols to stand for specified quantities. To begin with, Chapter 5 affords a brief review of algebra for those who may need it. Chapter 6 develops more ideas about numbers, in particular complex numbers.

Chapters 7, 8, and 9 deal with the very important subject of functions. They explain, in considerable detail, the various means for mathematically assigning a relationship to any variable that depends on another for its value in some precisely defined manner. To arrive at an understanding of this, you will study the topics of algebraic functions, trigonometric functions, and exponential and logarithmic functions.

Thereafter, the choice of topics is optional, usually depending upon the makeup of the class and the goals of the instructor. Several of the topics deal with the solution of equations by various methods. The last chapters deal with certain special cases having to do with the ways in which things may be combined and the chances of their being combined in specific ways.

As you read books and listen to lectures, bear in mind that you are viewing a polished finished product. In the early stages of learning a new subject there is bound to be a groping, trial-and-error striving. One topic may take minutes to learn while others may take many hours. Much depends upon one's interest and background.

The authors' sincere hope is that you will approach this study with an open and friendly mind. Do not be fearful of raising questions, no matter how elementary (chances are, incidentally, that over half your class would like to know the answer to the same question). Work so that you may acquire a *feeling* for the subject. This usually means applying yourself freely. At each step consider the summary of what you have had to that point. Note how each topic fits into the course and also how it fits into your experience in general.

Since mathematics is a special language, you must—of necessity—become involved with the symbols and words that have meaning in that language. Learn to appreciate their definitions. Expect to put out effort; and with the agreement between us that you will exert some effort, the authors feel assured not only that you will learn some important mathematics but that you will substantially develop both your intuition and your ability to reason.

CONTENTS

1

AN INFORMAL INTRODUCTION TO SETS

1.1 INTRODUCTION

One of the major areas of mathematical research during the past century has been in the foundations of mathematics. It is not easy to describe exactly what the foundations† of mathematics encompass, but one of the purposes of research in this area is to seek out the concepts upon which all mathematics can be based, concepts which permeate all mathematical thought. The simple concept of a set, or collection, of objects has been recognized as one of the most basic concepts, and one of the most useful in mathematics. It is the purpose of this chapter to present an informal introduction to the theory of sets, and ideas associated with this theory.

When we consider many objects we may consider them individually, or as a single entity. In the theory of sets a set of objects is considered as an entity itself. Many other mathematical terms are defined in terms of sets.

The concept of a set is taken as a primitive notion; that is, it is left as an undefined term. If it were required that every word be defined, we should necessarily be involved in circular definition, a situation

† For an account of the purposes and activity in this area of mathematics, see R. L. Wilder, "Foundation of Mathematics," John Wiley & Sons, Inc., New York, 1952, in particular, chaps. 1 and 3.

like having word A defined in terms of word B and word B defined in terms of word A. In order to avoid circular definition in a mathematical system, certain basic words are left undefined. This will be discussed more fully in Chap. 2.

We merely say that a particular set is defined when a rule is given by which it can be determined if an object does or does not belong to the set. Some property common to various objects is specified for identifying the set. For example, of all the triangles in the plane, attention may be given to those with two sides equal. The set designation is then given by the expression "isosceles triangles." The objects which belong to the set are called *elements* of the set.

A set may be specified by giving the rule, such as the set of even integers, or by a listing, such as the set whose elements are the numbers 1, 2, 3, 4, 5. Some examples of sets are the set of integers greater than 10, the set of present judges of the United States Supreme Court, the set of members of a given family, the set of countries in South America, the set of letters a, b, c, d, and the set of numbers 0, 1. In each case the set is uniquely specified; that is, it is clear what the elements of each set are.

Let A denote the set whose elements are 1, 2, 3, 4, 5. When a set is defined by a listing of its elements, it will be designated by the use of braces as follows. For the set A just mentioned, $A = \{1,2,3,4,5\}$. Thus the symbol $\{\cdot\cdot\cdot\}$ represents a set whose elements are the objects which appear inside the braces.

Suppose set $B = \{5,4,3,2,1\}$. Observe that sets A and B contain the same elements. In such cases the sets are said to be equal, $A = B$. The order of the listing of the elements of a set is not important.

■ DEFINITION 1.1 Two sets are equal if and only if they contain the same elements.

Consider the set of all universities in the United States which have granted doctorates in mathematics during the twentieth century. Even though one may not know which universities belong to the set, a rule is given by which one can decide; this is the important aspect of a set being defined by a rule or characteristic rather than a listing.

1.2 SPECIAL SETS

In any discussion where sets of elements are considered, usually there is more than one set. The set of all elements under discussion in any context is called the *universal set,* or universe of discourse, and is denoted by the symbol U. For example, in elementary algebra the universal set is often the set of real numbers. A particular set being discussed might be the set of positive numbers, and each element of this set is an element of U, the set of real numbers.

From the elements of U other sets can be specified. For example, let U be the set of all United States congressmen; the members of the Senate constitute a set, the representatives from Ohio constitute a set, and other sets could be designated. Consider the set of female United States senators from Ohio in 1964. Here a rule is given by which it can be determined if a person does or does not belong to the set. Even though there are no elements, we agree to consider this as a set and call it the *empty set.*

The *empty set,* or *null set,* is a set containing no elements, and is denoted by the symbol $\emptyset$. There is only one empty set; that is, it is of no importance whether the empty set contains no female senators from Ohio, no integers between 1 and 2, or no automobiles.

1.3 SUBSETS

Whenever we refer to a set A or a set B, we shall always understand that the universal set U is defined, that is, the set which contains all the elements of both A and B (and perhaps other elements).

Let A and B denote two sets. It may be that all the elements of A are also elements of B, in which case we say that A is a subset of B.

DEFINITION 1.2 Set A is a *subset* of set B provided that every element of A is an element of B. In symbols, this is denoted by $A \subseteq B$, read "A is a subset of B," or "A is contained in B." If $A \subseteq B$, we also write $B \supseteq A$, read "B contains A."

EXAMPLE 1 Let $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,4,5,6\}$. Hence $A \subseteq B$.

Since the null set $\emptyset$ contains no elements, there are no elements in $\emptyset$ which are not in A, for every set A. For this reason the null set is considered to be a subset of every set A.

$\emptyset \subseteq A$ for every set A

As the student follows the developments in the text, it is worthwhile for him to stop at certain places and take an active part by solving *drill problems.* These will serve as a brief test of his comprehension of the material that precedes. Answers are given when feasible, and when there are several parts to the problem, the answers are listed in an *arbitrary order.*

DRILL PROBLEM Find all the subsets of the set $\{1,2\}$.

ANSWER $\{1,2\}, \{1\}, \{2\}, \emptyset$

Since U contains all the elements under discussion, $A \subseteq U$ and $B \subseteq U$. Hence any set being considered in any one discussion is a subset of U. Furthermore, $A \subseteq A$ for any set A, for certainly every element of A is an element of A.

Suppose set $A = \{1,2,3\}$ and $B = \{2,1,3\}$. In this case $A = B$. Furthermore, A is a subset of B, since every element of A is an element of B. In like manner, $B \subseteq A$. Thus, if two sets are equal, then each set is a subset of the other. Conversely, if each of two sets is a subset of the other, the two sets are equal. In symbols, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

■ DEFINITION 1.3 Set A is a proper † subset of B if and only if every element of A is an element of B and there exists at least one element of B which is not an element of A. In symbols, $A \subset B$, read "A is a proper subset of B," or "B contains A properly."

EXAMPLE 2 Let $C = \{a,b,c\}$ and $D = \{a,b,c,d\}$. Hence, $C \subset D$; C is a proper subset of D. Does it hold that $C \subseteq D$?

The set of United States senators is a proper subset of the set of United States congressmen. The set of all lines in the plane parallel to a given line is a proper subset of the set of all lines in the plane.

EXERCISE 1.3

1 If $A \subset B$, show that $A \subseteq B$.
2 For what restrictions on A does it hold that $\emptyset \subset A$?
3 Which of the following are correct? Explain your answer.

a $\emptyset \subseteq \emptyset$ *b* $\emptyset \subset \emptyset$ *c* $\emptyset \subseteq A$ *d* $A \subset A$
e $A \subseteq A$

4 Designate two sets, one by stating the rule which defines the set, and one by listing the elements of the set.
5 Give an example of a set which has no proper subset.
6 Give a set and list the proper subsets.
7 List all the subsets of the set $A = \{a,b\}$.
8 List all the subsets of the set $B = \{a,b,c\}$.
9 List all the subsets of the set $C = \{a,b,c,d\}$.
10 On the basis of your answers to Probs. 7, 8, and 9, can you predict the number of subsets of the set $D = \{a,b,c,d,e\}$?
11 Let $A = \{a,b,c,d\}$ and $B = \{e,f,g\}$. What is the set of elements that are in both A and B?
12 Let A be the set of all male residents of New York City who are 21 years old or over and who own an automobile. What could the universal set U be?
13 Let $A = \{a,b,c,4\}$. *a* Could $U = \{a,b,c,d,e,f\}$? *b* What could U be?

1.4 SET OPERATIONS: UNION, INTERSECTION, COMPLEMENT

As indicated in previous sections, some of the basic ideas in mathematics can be defined in terms of sets. In this section we continue our informal discussion of sets, with emphasis on the set operations.

In the discussion which follows it is assumed that a universal set U and subsets A,B,C, etc., are given. It is possible to form new sets from the given sets in such a way that these new sets are also subsets of the same universal set U.

† The use of the word "proper" here is analogous to its use in the phrase "proper fraction."

■ DEFINITION 1.4 The *union* of two sets A and B is the set of all elements belonging to at least one of the two sets. The union of A and B is denoted by $A \cup B$, read "A union B."

First, we note that the union of two sets is a subset of U, since both A and B are subsets of U. Second, the operation of union (sometimes called set addition) is an operation which is defined for each pair of sets. The operation of union assigns a third set to each pair of sets. That is, for two sets A and B there corresponds a set $A \cup B$, and if A and B are defined, then $A \cup B$ is defined. The union operation is called a *binary* operation since it operates on pairs of sets. The set $A \cup B$ is unique; that is, given two sets A and B, there is exactly one set satisfying Definition 1.4. Lastly, we note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

EXAMPLE 1 Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3\}$, $B = \{7,8,9,10\}$, $C = \{2,3,7,8\}$. Find $A \cup B$, $A \cup C$, and $B \cup C$.

SOLUTION $A \cup B = \{1,2,3,7,8,9,10\}$, $A \cup C = \{1,2,3,7,8\}$, $B \cup C = \{2,3,7,8,9,10\}$.

Another way of forming a set from two given sets is given in the following definition.

■ DEFINITION 1.5 The *intersection* of two sets A and B is the set of all elements belonging to both A and B. The intersection is denoted by $A \cap B$, read "A intersection B."

As with the union $A \cup B$, the intersection of two sets $A \cap B$ (sometimes called set multiplication) is a subset of U, the operation of intersection is a binary operation, and the set $A \cap B$ is unique. The intersection of two sets is the set of elements common to the two sets. It follows that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

EXAMPLE 2 Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3\}$, $B = \{7,8,9,10\}$, and $C = \{2,3,7,8\}$. Find $A \cap B$, $A \cap C$, $B \cap C$, and $A \cap U$.

SOLUTION $A \cap B = \emptyset$, $A \cap C = \{2,3\}$, $B \cap C = \{7,8\}$, $A \cap U = \{1,2,3\} = A$.

In many applications of set theory it is necessary to discuss the set of elements of U which do not belong to a set A.

■ DEFINITION 1.6 The *complement* of a set A is the set of all elements included in U which are not included in A. The complement of A is denoted by A', read "the complement of A," or "A complement," or simply "A prime."

This is an operation performed on a single set. An operation such as this one is called a *unary* operation. For any set A, the set A' is defined, is unique, and is a subset of U. It follows from the definition that