RELATIVISTIC QUANTUM THEORY

Volume 4 of Course of Theoretical Physics

Part 1

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PREFACE

IN ACCORDANCE with the general plan of this *Course of Theoretical Physics*, the present volume deals with relativistic quantum theory in the broad sense: the theory of all phenomena which depend upon the finite velocity of light, including the whole of the theory of radiation.

This branch of theoretical physics is still far from completion, even as regards its basic physical principles, and this is particularly true of the theory of strong and weak interactions. But even quantum electrodynamics, despite the remarkable achievements of the last twenty years, still lacks a satisfactory logical structure.

In the choice of material for this book we have considered only results which appear to be reasonably firmly established. In consequence, of course, the greater part of the book is devoted to quantum electrodynamics. We have tried to give a realistic exposition, with emphasis on the physical hypotheses used in the theory, but without going into details of justifications, which in the present state of the theory are in any case purely formal.

In the discussion of specific applications of the theory, our aim has been not to include the whole vast range of effects but to select only the most fundamental of them, adding some references to original papers which contain more detailed studies. We have often omitted some of the intermediate steps in the calculations, which in this subject are usually very lengthy, but we have always sought to indicate any non-trivial point of technique.

The discussion in this book demands a higher degree of previous knowledge on the part of the reader than do the other volumes in the *Course*. Our assumption has been that a reader whose study of theoretical physics has extended as far as the quantum theory of fields has no further need of predigested material.

The division of the book into two parts has no fundamental significance, and is due only to the large amount of matter to be treated. The second part will deal with radiative corrections in electrodynamics, the theory of weak interactions, and certain problems in the theory of strong interactions.

This book has been written without the direct assistance of our teacher, L. D. Landau. Yet we have striven to be guided by the spirit and the approach to theoretical physics which characterised his teaching of us and which he embodied in the other volumes. We have often asked ourselves what would be the attitude of 'Dau to this or that topic, and sought the answer prompted by our many years' association with him.

Our thanks are due to V. N. Baĭer, who gave great help in compiling §§59 and 94, and to V. I. Ritus for great help in writing §98. We are grateful to B. É. Meĭerovich for assistance with calculations, and also to A. S. Kompaneets, who made available his notes of L. D. Landau's lectures on quantum electrodynamics, given at Moscow State University in the academic year 1959–60.

This book is the result of a joint undertaking, and all the authors have discussed the contents of the entire book. In the first instance, the work was allocated as follows: Chapters I and V were written by Lifshitz; II, III and VII by Berestetskiĭ and Lifshitz; IV, VI and IX by Lifshitz and Pitaevskiĭ; and VIII and X by all three authors jointly.

June 1967 V. B. Berestetskiĭ, E. M. Lifshitz, L. P. Pitaevskiĭ

NOTATION

Four-dimensional

Four-dimensional tensor indices are denoted by Greek letters $\lambda, \mu, \nu, \ldots$, taking the values 0, 1, 2, 3.

A 4-metric with signature $(+ - - -)$ is used. The metric tensor is

$$g_{\mu\nu}\ (g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1).$$

Components of a 4-vector are stated in the form $a^\mu = (a^0, \boldsymbol{a})$.

To simplify the formulae, the index is often omitted in writing the components of a 4-vector.† The scalar products of 4-vectors are written simply as (ab) or ab; $ab \equiv a_\mu b^\mu = a_0 b_0 - \boldsymbol{a}.\boldsymbol{b}$.

The 4-radius-vector is $x^\mu = (t, \boldsymbol{r})$. The 4-volume element is d^4x.

The operator of differentiation with respect to the 4-coordinates is $\partial_\mu = \partial/\partial x^\mu$.

The antisymmetric unit 4-tensor is $e^{\lambda\mu\nu\rho}$, with $e^{0123} = -e_{0123} = +1$.

The four-dimensional delta function is $\delta^4(a) = \delta(a_0)\delta(\boldsymbol{a})$.

Three-dimensional

Three-dimensional tensor indices are denoted by Latin letters $i, k, l, \ldots$, taking the values x, y, z.

Three-dimensional vectors are denoted by letters in bold italic type.

The three-dimensional volume element is d^3x.

Operators

Operators are denoted by letters in roman type.

ψ-operators are denoted by the letter $\boldsymbol{\psi}$ in bold type.

Commutators or anticommutators of two operators are written $\{\mathrm{f}, \mathrm{g}\}_\pm = \mathrm{fg} \pm \mathrm{gf}$.

The transposed operator is $\tilde{\mathrm{f}}$.

The Hermitian conjugate operator is f^+.

The charge conjugation operator is C.

The spatial inversion operator is P (for "parity").

The time reversal operator is T; this letter also denotes time-ordering of products of operators.

† This way of writing the components is often used in recent literature. It is a compromise between the limited resources of the alphabet and the demands of physics, and means, of course, that the reader must be particularly attentive.

Matrix elements

The matrix element of the operator F for a transition from initial state i to final state f is F_{fi} or $\langle f|F|i\rangle$.

The notation $|i\rangle$ is used as an abstract symbol for a state independently of any specific representation in which its wave function may be expressed. The notation $\langle f|$ denotes a final ("complex conjugate") state.†

Correspondingly, $\langle s|r\rangle$ denotes the coefficients in the expression of a set of states with quantum numbers r as superpositions of states with quantum numbers s: $|r\rangle = \sum_s |s\rangle\langle s|r\rangle$.

The reduced matrix elements of spherical tensors are $\langle f\|F\|i\rangle$.

Dirac's equation

The Dirac matrices are γ^μ, with $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$. The matrix $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\beta = \gamma^0$. The expressions in the spinor and standard representations are (21.3), (21.16) and (21.20).

$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $(\gamma^5)^2 = 1$; see (22.18).

$\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$; see (28.2).

The product of a 4-vector with the Dirac matrices is $\hat{a} = (a\gamma) \equiv a_\mu\gamma^\mu$.

Dirac conjugation is expressed by $\overline{\psi} = \psi^*\gamma^0$.

The Pauli matrices are $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, defined in §20.

The 4-spinor indices are $\alpha, \beta, \ldots$ and $\dot{\alpha}, \dot{\beta}, \ldots$, taking the values 1, 2 and $\dot{1}, \dot{2}$.

The bispinor indices are $i, k, l, \ldots$, taking the values 1, 2, 3, 4.

Fourier expansion

Three-dimensional:

$$f(\boldsymbol{r}) = \int f(\boldsymbol{k})\, e^{i\boldsymbol{k}.\boldsymbol{r}}\, \frac{d^3k}{(2\pi)^3}, \qquad f(\boldsymbol{k}) = \int f(\boldsymbol{r})\, e^{-i\boldsymbol{k}.\boldsymbol{r}}\, d^3x,$$

and similarly for the four-dimensional expansion.

Units

Except where otherwise specified, *relativistic units* are used, with $\hbar = 1$, $c = 1$. In these units, the square of the unit charge is $e^2 = 1/137$.

Atomic units have $e = 1$, $\hbar = 1$, $m = 1$. In these units, $c = 137$. The atomic units of length, time and energy are $\hbar^2/me^2$, $\hbar^3/me^4$ and $me^4/\hbar^2$; the quantity $Ry = me^4/2\hbar^2$ is called a *rydberg*.

Ordinary units are given in the absolute (Gaussian) system.

Constants

Velocity of light $c = 2.997925 \times 10^{10}$ cm/sec.
Unit charge $e = 4.803 \times 10^{-10}$ CGS electrostatic units.
Electron mass $m = 9.110 \times 10^{-28}$ g.
Planck's constant $\hbar = 1.055 \times 10^{-27}$ erg. sec.
Fine-structure constant $\alpha = e^2/\hbar c$; $1/\alpha = 137.04$.
Bohr radius $\hbar^2/me^2 = 5.292 \times 10^{-9}$ cm.

† This notation is due to Dirac.

Classical electron radius $r_e = e^2/mc^2 = 2.818 \times 10^{-13}$ cm.
Compton wavelength of the electron $\hbar/mc = 3.862 \times 10^{-11}$ cm.
Electron rest energy $mc^2 = 0.5110 \times 10^6$ eV.
Atomic energy unit $me^4/\hbar^2 = 4.360 \times 10^{-11}$ erg $= 27.21$ eV.
Bohr magneton $e\hbar/2mc = 9.274 \times 10^{-21}$ erg/G.
Proton mass $m_p = 1.673 \times 10^{-24}$ g.
Compton wavelength of the proton $\hbar/m_p c = 2.103 \times 10^{-14}$ cm.
Nuclear magneton $e\hbar/2m_p c = 5.051 \times 10^{-24}$ erg/G.

References to earlier volumes in the *Course of Theoretical Physics*:

Mechanics = Vol. 1 (*Mechanics*, second English edition, 1969).
Fields = Vol. 2 (*The Classical Theory of Fields*, third English edition, 1970).
QM or *Quantum Mechanics* = Vol. 3 (*Quantum Mechanics*, second English edition, 1965).

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INTRODUCTION

§1. The uncertainty principle in the relativistic case

THE quantum theory described in Volume 3 (*Quantum Mechanics*) is essentially non-relativistic throughout, and is not applicable to phenomena involving motion at velocities comparable with that of light. At first sight, one might expect that the change to a relativistic theory is possible by a fairly direct generalisation of the formalism of non-relativistic quantum mechanics. But further consideration shows that a logically complete relativistic theory cannot be constructed without invoking new physical principles.

Let us recall some of the physical concepts forming the basis of non-relativistic quantum mechanics (*QM*, §1). We saw that one fundamental concept is that of *measurement*, by which is meant the process of interaction between a quantum system and a classical object or *apparatus*, causing the quantum system to acquire definite values of some particular dynamical variables (coordinates, velocities, etc.). We saw also that quantum mechanics greatly restricts the possibility that an electron† simultaneously possesses values of different dynamical variables. For example, the uncertainties Δq and Δp in simultaneously existing values of the coordinate and the momentum are related by the expression‡ $\Delta q \Delta p \sim \hbar$; the greater the accuracy with which one of these quantities is measured, the less the accuracy with which the other can be measured at the same time.

It is important to note, however, that any of the dynamical variables of the electron can individually be measured with arbitrarily high accuracy, and in an arbitrarily short period of time. This fact is of fundamental importance throughout non-relativistic quantum mechanics. It is the only justification for using the concept of the wave function, which is a basic part of the formalism. The physical significance of the wave function $\psi(q)$ is that the square of its modulus gives the probability of finding a particular value of the electron coordinate as the result of a measurement made at a given instant. The concept of such a probability clearly requires that the coordinate can in principle be measured with any specified accuracy and rapidity, since otherwise this concept would be purposeless and devoid of physical significance.

The existence of a limiting velocity (the velocity of light, denoted by c) leads to new fundamental limitations on the possible measurements of various physical quantities (L. D. Landau and R. E. Peierls, 1930).

In *QM*, §44, the following relationship has been derived:

$$(v'-v)\Delta p \Delta t \sim \hbar, \tag{1.1}$$

† As in *QM*, §1, we shall, for brevity, speak of an "electron", meaning any quantum system.
‡ In this section, ordinary units are used.

relating the uncertainty Δp in the measurement of the electron momentum and the duration Δt of the measurement process itself; v and v' are the velocities of the electron before and after the measurement. From this relationship it follows that a momentum measurement of high accuracy made during a short time (i.e. with Δp and Δt both small) can occur only if there is a large change in the velocity as a result of the measurement process itself. In the non-relativistic theory, this showed that the measurement of momentum cannot be repeated at short intervals of time, but it did not at all diminish the possibility, in principle, of making a single measurement of the momentum with arbitrarily high accuracy, since the difference $v'-v$ could take any value, no matter how large.

The existence of a limiting velocity, however, radically alters the situation. The difference $v'-v$, like the velocities themselves, cannot now exceed c (or rather $2c$). Replacing $v'-v$ in (1.1) by c, we obtain

$$\Delta p \Delta t \sim \hbar/c, \tag{1.2}$$

which determines the highest accuracy theoretically attainable when the momentum is measured by a process occupying a given time Δt. In the relativistic theory, therefore, it is in principle impossible to make an arbitrarily accurate and rapid measurement of the momentum. An exact measurement ($\Delta p \to 0$) is possible only in the limit as the duration of the measurement tends to infinity.

Equally fundamental changes occur in regard to the measurability of the coordinate. In the relativistic theory, the coordinate cannot be measured with an accuracy better than a certain limit. The concept of localising the electron is thereby further restricted in its physical significance.

In the mathematical formalism of the theory, this situation is shown by the fact that an accurate measurement of the coordinate is incompatible with the assertion that the energy of a free particle is positive. It will be seen later that the complete set of eigenfunctions of the relativistic wave equation of a free particle includes, as well as solutions having the "correct" time dependence, also solutions having a "negative frequency". These functions will in general appear in the expansion of the wave packet corresponding to an electron localised in a small region of space.

It will be shown that the wave functions having a "negative frequency" correspond to the existence of antiparticles (positrons). The appearance of these functions in the expansion of the wave packet expresses the (in general) inevitable production of electron–positron pairs in the process of measuring the coordinates of an electron. This formation of new particles in a way which cannot be detected by the process itself clearly renders meaningless the measurement of the electron coordinates.

In the rest frame of the electron, the least possible error in the measurement of its coordinates is

$$\Delta q \sim \hbar/mc. \tag{1.3}$$

This value (which purely dimensional arguments show to be the only possible one) corresponds to a momentum uncertainty $\Delta p \sim mc$, which in turn corresponds to the threshold energy for pair production.

In a frame of reference in which the electron is moving with energy ε, (1.3) becomes

$$\Delta q \sim \hbar c/\varepsilon. \tag{1.4}$$

In particular, in the limiting ultra-relativistic case the energy is related to the momentum

by $\varepsilon \approx cp$, and

$$\Delta q \sim \hbar/p, \tag{1.5}$$

i.e. the error Δq is the same as the de Broglie wavelength of the particle.

For photons, the ultra-relativistic case always applies, and the expression (1.5) is therefore valid. This means that the coordinates of a photon are meaningful only in cases where the characteristic dimensions of the problem are large in comparison with the wavelength. This is just the "classical" limit, corresponding to geometrical optics, in which radiation can be said to be propagated along definite paths or rays. In the quantum case, however, where the wavelength cannot be regarded as small, the concept of coordinates of the photon has no meaning. We shall see later (§4) that, in the mathematical formalism of the theory, the fact that the photon coordinates cannot be measured is evident because the photon wave function cannot be used to construct a quantity which might serve as a probability density satisfying the necessary conditions of relativistic invariance.

It is clear from the foregoing that, in a consistent relativistic quantum mechanics, the coordinates of particles cannot act as dynamical variables, since these must by their nature have a precise significance. Nor can the particle momentum retain its former meaning. Since an accurate measurement of the momentum requires a long interval of time, there is no possibility of following its change in the process.

Having regard to the discussion at the beginning of this section, we reach the conclusion that the entire formalism of non-relativistic quantum mechanics becomes insufficient in the relativistic case. The wave functions $\psi(q)$, in their original sense as the carriers of unobservable information, cannot appear in the formalism of a consistent relativistic theory.

The momentum can figure in a consistent theory only for free particles; for these it is conserved, and can therefore be measured with any desired accuracy. This indicates that the theory will not consider the time dependence of particle interaction processes. It will show that in these processes there are no characteristics precisely definable (even within the usual limitations of quantum mechanics); the description of such a process as occurring in the course of time is therefore just as unreal as the classical paths are in non-relativistic quantum mechanics. The only observable quantities are the properties (momenta, polarisations) of free particles: the initial particles which come into interaction, and the final particles which result from the process (L. D. Landau and R. E. Peierls, 1930).

A typical problem as formulated in relativistic quantum theory is to determine the probability amplitudes of transitions between specified initial and final states ($t \to \mp\infty$) of a system of particles. The set of such amplitudes between all possible states constitutes the *scattering matrix* or *S-matrix*. This matrix will embody all the information about particle interaction processes that has an observable physical meaning (W. Heisenberg, 1938).

In such a theory, moreover, the concepts of "elementary" and "composite" particles lose their earlier significance; the problem of "what consists of what" cannot be formulated without considering the process of interaction between particles, and if this is not done the whole problem becomes meaningless. All particles which occur as initial or final particles in any physical collision phenomenon must appear in the theory on an equal footing. In this sense the difference between those particles usually said to be "composite" and those said to be "elementary" is only a quantitative one, and amounts to the value of the mass defect with respect to decay into specified "component parts". For example, the statement

that the deuteron is complex (its binding energy with respect to disintegration into a proton and a neutron being fairly small) differs only quantitatively from the statement that the neutron "consists of" a proton and a pion.

There is as yet no logically consistent and complete relativistic quantum theory. We shall see that the existing theory introduces new physical features into the nature of the description of particle states, which acquires some of the features of field theory (see §10). The theory is, however, largely constructed on the pattern of ordinary quantum mechanics and makes use of the latter's concepts. This structure of the theory has yielded good results in quantum electrodynamics. The lack of complete logical consistency in this theory is shown by the occurrence of divergent expressions when the mathematical formalism is directly applied, although there are quite well-defined ways of eliminating these divergences. Nevertheless, such methods remain, to a considerable extent, semiempirical rules, and our confidence in the correctness of the results is ultimately based only on their excellent agreement with experiment, not on the internal consistency or logical ordering of the fundamental principles of the theory.

A quite different situation occurs in the theory of effects depending on the strong interactions of particles (nuclear forces). Here, attempts to construct a theory by similar methods have led to no significant results bearing on physical reality. The construction of a complete theory embracing strong interactions will probably call for the application of fundamentally new physical ideas.

CHAPTER I

PHOTONS

§2. Quantisation of the free electromagnetic field

WITH the purpose of treating the electromagnetic field as a quantum object, it is convenient to begin from a classical description of the field in which it is represented by an infinite but discrete set of variables. This description permits the immediate application of the customary formalism of quantum mechanics. The representation of the field by means of potentials specified at every point in space is essentially a description by means of a continuous set of variables.

Let $A(r, t)$ be the vector potential of the free electromagnetic field, which satisfies the "transversality condition"

$$\operatorname{div} A = 0. \tag{2.1}$$

The scalar potential $\Phi = 0$, and the fields E and H are

$$E = -\dot{A}, \qquad H = \operatorname{curl} A. \tag{2.2}$$

Maxwell's equations reduce to the wave equation for A:

$$\Delta A - \partial^2 A/\partial t^2 = 0. \tag{2.3}$$

In classical electrodynamics (see *Fields*, §52) the change to the description by means of a discrete set of variables is brought about by considering the field in a large but finite volume V.† The following is a brief *résumé* of the argument.

The field in a finite volume can be expanded in terms of travelling plane waves, and its potential is then represented by a series

$$A = \sum_k (a_k e^{ik.r} + a_k^* e^{-ik.r}), \tag{2.4}$$

where the coefficients a_k are functions of the time such that

$$a_k \sim e^{-i\omega t}, \qquad \omega = |k|. \tag{2.5}$$

The condition (2.1) shows that the complex vectors a_k are orthogonal to the corresponding wave vectors: $a_k . k = 0$.

The summation in (2.4) is taken over an infinite discrete set of values of the wave vector (i.e. of its components k_x, k_y, k_z). The change to an integral over a continuous distribution may be made by means of the expression $d^3k/(2\pi)^3$ for the number of possible values of k belonging to the volume element $d^3k = dk_x\, dk_y\, dk_z$ in k-space.

† We shall take $V = 1$, in order to reduce the number of factors in the formulae.

If the vectors $\boldsymbol{a}_k$ are specified, the field in the volume considered is completely determined. Thus these quantities may be regarded as a discrete set of classical "field variables". In order to explain the transition to the quantum theory, however, a further transformation of these variables is needed, whereby the field equations take a form analogous to the canonical equations (Hamilton's equations) of classical mechanics. The canonical field variables are defined by

$$\begin{aligned} \boldsymbol{Q}_k &= \frac{1}{\sqrt{(4\pi)}}(\boldsymbol{a}_k + \boldsymbol{a}_k^*), \\ \boldsymbol{P}_k &= \frac{-i\omega}{\sqrt{(4\pi)}}(\boldsymbol{a}_k - \boldsymbol{a}_k^*) = \dot{\boldsymbol{Q}}_k, \end{aligned} \tag{2.6}$$

and are evidently real. The vector potential is expressed in terms of the canonical variables by

$$\boldsymbol{A} = \sqrt{(4\pi)} \sum_{\boldsymbol{k}} \left(\boldsymbol{Q}_k \cos \boldsymbol{k}.\boldsymbol{r} - \frac{1}{\omega} \boldsymbol{P}_k \sin \boldsymbol{k}.\boldsymbol{r} \right). \tag{2.7}$$

To find the Hamiltonian H, we must calculate the total energy of the field,

$$\frac{1}{8\pi} \int (\boldsymbol{E}^2 + \boldsymbol{H}^2)\, d^3x,$$

and express it in terms of the $\boldsymbol{Q}_k$ and $\boldsymbol{P}_k$. When $\boldsymbol{A}$ is written as the expansion (2.7), and $\boldsymbol{E}$ and $\boldsymbol{H}$ are found from (2.2), the result of the integration is

$$H = \tfrac{1}{2} \sum_{\boldsymbol{k}} (\boldsymbol{P}_k^2 + \omega^2 \boldsymbol{Q}_k^2).$$

Each of the vectors $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ is perpendicular to the wave vector $\boldsymbol{k}$, and therefore has two independent components. The direction of these vectors determines the direction of polarisation of the corresponding wave. Denoting the two components of the vectors $\boldsymbol{Q}_k$ and $\boldsymbol{P}_k$ (in the plane perpendicular to $\boldsymbol{k}$) by $\boldsymbol{Q}_{k\alpha}, \boldsymbol{P}_{k\alpha}$ ($\alpha = 1, 2$), we can write the Hamiltonian as

$$H = \sum_{\boldsymbol{k},\alpha} \tfrac{1}{2}(P_{k\alpha}^2 + \omega^2 Q_{k\alpha}^2). \tag{2.8}$$

Thus the Hamiltonian is the sum of independent terms, each of which contains only one pair of quantities $Q_{k\alpha}, P_{k\alpha}$. Each such term corresponds to a travelling wave with a definite wave vector and polarisation, and has the form of the Hamiltonian for a one-dimensional harmonic oscillator. This expansion is therefore often referred to as an *oscillator expansion* of the field.

Let us now consider the quantisation of the free electromagnetic field. The classical description of the field given above makes the manner of transition to the quantum theory obvious. We have now to use canonical variables (generalised coordinates $Q_{k\alpha}$ and generalised momenta $P_{k\alpha}$) as operators, with the commutation rule

$$\mathrm{P}_{k\alpha}\mathrm{Q}_{k\alpha} - \mathrm{Q}_{k\alpha}\mathrm{P}_{k\alpha} = -i; \tag{2.9}$$

operators with different values of $\boldsymbol{k}$ and α always commute. The potential $\mathbf{A}$ and, according to (2.2), the fields $\mathbf{E}$ and $\mathbf{H}$ likewise become operators.

The consistent determination of the Hamiltonian requires the calculation of the integral

$$\mathrm{H} = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{H}^2)\, d^3x, \tag{2.10}$$