Graduate Texts in Mathematics

Anders Björner
Francesco Brenti

Combinatorics of Coxeter Groups

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With 81 Illustrations

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To Annamaria and Christine

Foreword

Coxeter groups arise in a multitude of ways in several areas of mathematics. They are studied in algebra, geometry, and combinatorics, and certain aspects are of importance also in other fields of mathematics. The theory of Coxeter groups has been exposited from algebraic and geometric points of view in several places, also in book form. The purpose of this work is to present its core combinatorial aspects.

By "combinatorics of Coxeter groups" we have in mind the mathematics that has to do with reduced expressions, partial order of group elements, enumeration, associated graphs and combinatorial cell complexes, and connections with combinatorial representation theory. There are some other topics that could also be included under this general heading (e.g., combinatorial properties of reflection hyperplane arrangements on the geometric side and deeper connections with root systems and representation theory on the algebraic side). However, with the stated aim, there is already more than plenty of material to fill one volume, so with this "disclaimer" we limit ourselves to the chosen core topics.

It is often the case that phenomena of Coxeter groups can be understood in several ways, using either an algebraic, a geometric, or a combinatorial approach. The interplay between these aspects provides the theory with much of its richness and depth. When alternate approaches are possible, we usually choose a combinatorial one, since it is our task to tell this side of the story. For a more complete understanding of the subject, the reader is urged to study also its algebraic and geometric aspects. The notes at the end of each chapter provide references and hints for further study.

The book is divided into two parts. The first part, comprising Chapters 1 – 4, gives a self-contained introduction to combinatorial Coxeter group theory. We treat the combinatorics of reduced decompositions, Bruhat order, weak order, and some aspects of root systems. The second part consists of four independent chapters dealing with certain more advanced topics. In Chapters 5 – 7, some external references are necessary, but we have tried to minimize reliance on other sources. Chapter 8, which is elementary, discusses permutation representations of the most important finite and affine Coxeter groups.

Exercises are provided to all chapters — both easier exercises, meant to test understanding of the material, and more difficult ones representing results from the research literature. Open problems are marked with an asterisk. Thus, the book is meant to have a dual character as both graduate textbook (particularly Part I) and as research monograph (particularly Part II).

Acknowledgments: Work on this book has taken place at highly irregular intervals during the years 1993–2004. An essentially complete and final version was ready in 1999, but publication was delayed due to unfortunate circumstances. During the time of writing we have enjoyed the support of the Volkswagen-Stiftung (RiP-program at Oberwolfach), of the Fondazione San Michele, and of EC grants Nos. CHRX-CT93-0400 and HPRN-CT-2001-00272 (Algebraic Combinatorics in Europe).

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Stockholm and Rome, September 2004

Anders Björner and Francesco Brenti

Notation

We collect here some notation that is adhered to throughout the book.

$\mathbb{Z}$	the integers
$\mathbb{N}$	the non-negative integers
$\mathbb{P}$	the positive integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	the rational, real, and complex numbers
$[n]$	the set $\{1, 2, \ldots, n\}$ $(n \in \mathbb{N})$, in particular $[0] = \emptyset$
$[a, b]$	the set $\{n \in \mathbb{Z} : \ a \leq n \leq b\}$ $(a, b \in \mathbb{Z})$
$[\pm n]$	the set $[-n, n] \setminus \{0\}$
$\{a_1, \ldots, a_n\}_<$	the set $\{a_1, \ldots, a_n\}$ with total order $a_1 < \cdots < a_n$
$\lfloor a \rfloor$	the largest integer $\leq a$ $(a \in \mathbb{R})$
$\lceil a \rceil$	the smallest integer $\geq a$ $(a \in \mathbb{R})$
$\operatorname{sgn}(a)$	the sign of a real number: $\operatorname{sgn}(a) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -1, & \text{if } a < 0. \end{cases}$
δ_{ij} or $\delta(i, j)$	the Kronecker delta: $\delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$
$\lvert A \rvert$, $\# A$, or $\operatorname{card}(A)$	the cardinality of a set A
$A \uplus B$	the union of two disjoint sets
$A \Delta B$	the symmetric difference $A \cup B \setminus (A \cap B)$
2^A	the family of all subsets of a finite set A
$\binom{A}{k}$	the family of all k-element subsets of a finite set A
A^*	the set of all words with letters from an alphabet A

Each result (theorem, corollary, proposition, or lemma) is numbered consecutively within sections. So, for example, Theorem 2.3.3 is the third result in the third section of Chapter 2 (i.e., in Section 2.3). The symbol □ denotes the end of a proof or an example. A □ appearing at the end of the statement of a result signifies that the result should be obvious at that stage of reading, or else that a reference to a proof is given.

Contents

1
The basics

Coxeter groups are defined in a simple way by generators and relations. A key example is the symmetric group S_n, which can be realized as permutations (combinatorics), as symmetries of a regular $(n-1)$-dimensional simplex (geometry), or as the Weyl group of the type A_{n-1} root system or of the general linear group (algebra). The general theory of Coxeter groups expands and interweaves the many mathematical themes and aspects suggested by this example.

In this chapter, we give the basic definitions, present some examples, and derive the most elementary combinatorial facts underlying the rest of the book. Readers who already know the fundamentals of the theory can skim or skip this chapter.

1.1 Coxeter systems

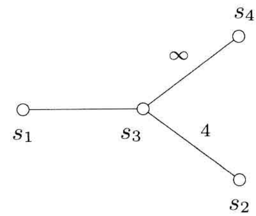
Let S be a set. A matrix $m : S \times S \to \{1, 2, \ldots, \infty\}$ is called a *Coxeter matrix* if it satisfies

$$m(s, s') = m(s', s)\,; \tag{1.1}$$

$$m(s, s') = 1 \Leftrightarrow s = s'. \tag{1.2}$$

Equivalently, m can be represented by a *Coxeter graph* (or *Coxeter diagram*) whose node set is S and whose edges are the unordered pairs $\{s, s'\}$ such that $m(s, s') \geq 3$. The edges with $m(s, s') \geq 4$ are labeled by that number. For instance,

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix} \longleftrightarrow$$



Let $S^2_{\text{fin}} = \{(s,s') \in S^2 : m(s,s') \neq \infty\}$. A Coxeter matrix m determines a group W with the presentation

$$\begin{cases} \text{Generators: } S \\ \text{Relations: } (ss')^{m(s,s')} = e, \quad \text{for all } (s,s') \in S^2_{\text{fin}}. \end{cases} \tag{1.3}$$

Here, and in the sequel, "e" denotes the identity element of any group under consideration. Since $m(s,s) = 1$, we have that

$$s^2 = e, \qquad \text{for all } s \in S, \tag{1.4}$$

which, in turn, shows that the relation $(ss')^{m(s,s')} = e$ is equivalent to

$$\underbrace{s\,s'\,s\,s'\,s\ldots}_{m(s,s')} = \underbrace{s'\,s\,s'\,s\,s'\ldots}_{m(s,s')}. \tag{1.5}$$

In particular, $m(s,s') = 2$ (i.e., two distinct nodes s and s' are not neighbors in the Coxeter graph) if and only if s and s' commute.

For instance, the group determined by the above Coxeter diagram is generated by s_1, s_2, s_3, and s_4 subject to the relations

$$\begin{cases} s_1^2 = s_2^2 = s_3^2 = s_4^2 = e \\ s_1 s_2 = s_2 s_1 \\ s_1 s_3 s_1 = s_3 s_1 s_3 \\ s_1 s_4 = s_4 s_1 \\ s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2 \\ s_2 s_4 = s_4 s_2. \end{cases}$$

If a group W has a presentation such as (1.3), then the pair (W,S) is called a *Coxeter system.* The group W is the *Coxeter group* and S is the set of *Coxeter generators.* The cardinality of S is called the *rank* of (W,S). Most groups of interest will be of finite rank. The system is *irreducible* if its Coxeter graph is connected.

When referring to an abstract group as a Coxeter group, one usually has in mind not only W but the pair (W,S), with a specific generating set S tacitly understood. Some caution is necessary in such cases, since the isomorphism type of (W,S) is not determined by the group W alone; see Exercise 2.

The following three statements are equivalent and make explicit what it means for W to be determined by m via the presentation (1.3):

1. (Universality Property) If G is a group and $f : S \to G$ is a mapping such that
$$(f(s)f(s'))^{m(s,s')} = e$$
for all $(s, s') \in S^2_{\text{fin}}$, then there is a unique extension of f to a group homomorphism $f : W \to G$.

2. $W \cong F/N$, where F is the free group generated by S and N is the normal subgroup generated by $\{(ss')^{m(s,s')} : (s, s') \in S^2_{\text{fin}}\}$.

3. Let S^* be the free monoid generated by S (i.e., the set of words in the alphabet S with concatenation as product). Let $\equiv$ be the equivalence relation generated by allowing insertion or deletion of any word of the form
$$(ss')^{m(s,s')} = \underbrace{s\, s'\, s\, s'\, s \ldots s'\, s\, s'}_{2m(s,s')}$$
for $(s, s') \in S^2_{\text{fin}}$. Then, $S^*/\equiv$ forms a group isomorphic to W.

It might seem that to be precise we should use different symbols for the elements of S and for their images in $W \cong S^*/\equiv$ under the surjection
$$\varphi : S^* \to W. \tag{1.6}$$
However, this is needlessly pedantic since, in practice, the possibility of confusion is negligible. It will be shown (Proposition 1.1.1) that $s \neq s'$ in S implies $\varphi(s) \neq \varphi(s')$ in W and (Corollary 1.4.8) that S is a minimal generating system for W.

Let (W, S) be a Coxeter system. Definition (1.3) leaves some uncertainty about the orders of pairwise products ss' as elements of W $(s, s' \in S)$. All that immediately follows is that the order of ss' divides $m(s, s')$ if $m(s, s')$ is finite. This leaves open the possibility that distinct Coxeter graphs might determine isomorphic Coxeter systems. However, this is not the case.

Proposition 1.1.1 *Let (W, S) be the Coxeter system determined by a Coxeter matrix m. Let s and s' be distinct elements of S. Then, the following hold:*

(i) (The classes of) s and s' are distinct in W.

(ii) The order of ss' in W is $m(s, s')$.

The proof is postponed to Section 4.1, where it is obtained for free as a by-product of some other material. Section 4.1 makes no use of (or even mention of) any material in the intermediate sections, so it is possible for a systematic reader, who wants to see a proof for Proposition 1.1.1 at this stage of reading, to go directly from here to Section 4.1.

It is a consequence of Proposition 1.1.1 that the Coxeter matrix $(m(s, s'))_{s,s' \in S}$ can be fully reconstructed from the group W and the generating set S. This leads to an important conclusion.

Theorem 1.1.2 *Up to isomorphism there is a one-to-one correspondence between Coxeter matrices and Coxeter systems.* □

The finite irreducible Coxeter systems, as well as certain classes of infinite ones, have been classified. See Appendix A1 for the classification of the finite and so-called affine groups and [306] for additional information. From now on, we will every now and then refer to these Coxeter groups by their conventional names mentioned in Appendix A1, but the classification as such will not play any significant role in the book. There is no essential restriction in confining attention to the irreducible case, since reducible Coxeter groups decompose uniquely as a product of irreducible ones (see Exercise 2.3).

The finite Coxeter groups for which $m(s, s') \in \{2, 3, 4, 6\}$ for all $(s, s') \in S^2, s \neq s'$ are called *Weyl groups*, a name motivated by Lie theory (see Example 1.2.10). The Coxeter groups for which $m(s, s') \in \{2, 3\}$ for all $(s, s') \in S^2, s \neq s'$ are called *simply-laced.*

1.2 Examples

Let us now look at a few examples. The following list is not intended to be systematic — the aim is merely to acquaint the reader with some of the groups that play an important role in the combinatorial theory of Coxeter groups and to exemplify some of the diverse ways in which Coxeter groups arise. More examples can be found in Chapter 8.

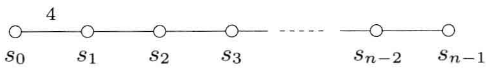
Example 1.2.1 The graph with n isolated vertices (no edges) is the Coxeter graph of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of order 2^n. □

Example 1.2.2 The *universal Coxeter group* U_n of rank n is defined by the complete graph with all $\binom{n}{2}$ edges marked by "∞." Equivalently, it is the group having n generators of order 2 and no other relations. Each group element can be uniquely expressed as a word in the alphabet of generators, and these words are precisely the ones where no adjacent letters are equal. □

Example 1.2.3 The path

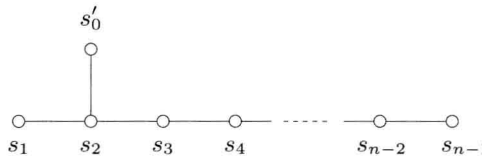
is the Coxeter graph of the symmetric group S_n with respect to the generating system of adjacent transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$. This is proved in Proposition 1.5.4. An understanding of this particular example is very valuable, both because of the importance of the symmetric group as such and its role as the most accessible nontrivial example of a Coxeter group. We will frequently return to S_n in order to concretely illustrate various general concepts and constructions. □

Example 1.2.4 The graph



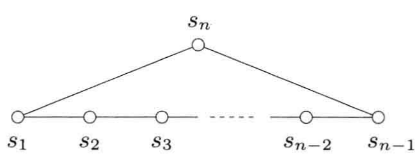
is the Coxeter graph of the group S_n^B of all signed permutations of the set $[n] = \{1, 2, \ldots, n\}$. See Section 8.1 for a detailed description of this group. It can be thought of in terms of the following combinatorial model. Suppose that we have a deck of n cards, such that the j-th card has "$+j$" written on one side and "$-j$" on the other. The elements of S_n^B can then be identified with the possible rearrangements of stacks of cards; that is, a group element is a permutation of $[n] = \{1, 2, \ldots, n\}$ (the order of the cards in the stack) together with the sign information $[n] \to \{+, -\}$ (telling which side of each card is up). The Coxeter generators s_i, $1 \leq i \leq n-1$, interchange the card in position i with that in position $i+1$ in the stack (preserving orientation), and s_0 flips card 1 (the top card).

The group S_n^B has a subgroup, denoted S_n^D, with Coxeter graph



Here, $s'_0 = s_0 s_1 s_0$. In terms of the card model this group consists of the stacks with an even number of turned-over cards (i.e., with minus side up). The generators s_i, $1 \leq i \leq n-1$, are adjacent interchanges as before, and s_0 flips cards 1 and 2 together (as a package). See Section 8.2 for more about this group. □

Example 1.2.5 The circuit



is the Coxeter graph of the group $\widetilde{S}_n$ of *affine permutations* of the integers. This is the group of all permutations x of the set $\mathbb{Z}$ such that

$$x(j+n) = x(j) + n, \quad \text{for all } j \in \mathbb{Z},$$

and

$$\sum_{i=1}^{n} x(i) = \binom{n+1}{2},$$

with composition as group operation. The Coxeter generators are the periodic adjacent transpositions $\widetilde{s}_i = \prod_{j \in \mathbb{Z}} (i + jn, i + 1 + jn)$ for $i = 1, \ldots, n$. See Section 8.3 for more about these infinite permutation groups. □