Advances in Analysis

Problems of Integration

Mark Burgin
Editor

Mathematics Research Developments

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MATHEMATICS RESEARCH DEVELOPMENTS

ADVANCES IN ANALYSIS

PROBLEMS OF INTEGRATION

MARK BURGIN
EDITOR

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MANY FACES OF INTEGRATION AND THEIR MATHEMATICAL REFLECTIONS

Mark Burgin

To integrate, as suggests the origin of this term from the Latin word *integer*, means to make something into a whole or to unify. Consequently, integration is an extremely widespread operation used in a huge diversity of areas.

In mathematics, the term *integration* is one of the most important concepts and has an exact and rather restricted, but not unique, meaning. First, integration means corresponding to a function and a subset of the function domain some value. This value is a number for numerical functions, a vector for vector valued functions, and operator for operator valued functions. Second, integration means an inverse operation to differentiation, corresponding to a function another function, or more exactly, a family of functions. From a general point of view, solving differential equations is also a kind of integration. This gives us a standard view on integration in mathematics and reflects the scope of the mathematical theory of integration.

There are concepts that, in spite of their simplicity, are exceptionally useful in mathematics. They constitute the actual core of mathematics. The development of mathematics brings with it a branching and separation of mathematical fields. At the same time, the same core concept emerges in different fields and settings. As a result, this concept provides a basis for unification and interconnectedness in mathematics, supplying effective means for solving various problems in pure and applied mathematics. One such concept is that of integration.

Integration emerged in mathematics from problems of finding lengths, areas, and volumes. According to Wikipedia, integration can be traced as far back as ancient Egypt, *circa* 1800 B.C.E. However, the majority of historical sources imply that the first essential contributions to integration were made by ancient Greek mathematicians, whose works display the beginnings of integration aimed at finding areas of curved regions and volumes of curved solids. Such mathematicians as Antiphon the Sophist (circa 430 B.C.E.), Eudoxus of Cnidus (circa 360 B.C.E.), and the great Archimedes (circa 287- 212 B.C.E.) were at the origin of integration as a mathematical field.

The major breakthrough in the area of mathematical integration was made by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) when they created, or it is better to say, completed creation of the integral calculus. This outstanding achievement inspired many mathematicians to further develop this area and to apply integration to a variety of practical and theoretical problems. The theory of integration has become an active and permanently developing field of modern mathematics. Mathematicians have elaborated many kinds and forms of integrals and integration in mathematics: definite and indefinite integrals, curve/line, contour, surface and volume integration, improper integrals, multiple integrals, Cauchy integral, Riemann integral, Lebesgue integral, Henstock-Kurzweil (gauge) integral, Feynman (path) integral, Wiener integral, non-commutative integration, p-adic (nonarchimedean) integration, integration of differential forms, stochastic integrals, McShane integral, Bochner integral, Denjoy-Bochner integral, Kolmogorov integral, Denjoy integral, Perron integral, Darboux integral, Stieltjes integral, Stieltjes-Lebesgue integral, and so on.

Now we can see a Renaissance in the field of mathematical integration. New constructions appear. Numerous new results are obtained. Many new books and papers on integration are published.

In addition to the development of the general theory of integration, researchers are engaged in active elaboration of concrete integration techniques and methods. For instance, numerical integration forms one of the most important fields in numerical analysis and computational science. Numerical integration consists of a broad family of algorithms and methods for calculating the numerical value of a definite integral, or in some cases, for finding numerical solutions of differential equations. By tradition, numerical integration of one-variable functions is also called quadrature and numerical integration of two-variable functions (or even of functions with more variables) is called cubature.

Integration in mathematics is the inverse operation and thus, opposition to differentiation. By the fundamental Newton-Leibniz Theorem, if we apply integration to a function f and then apply differentiation, we come to the same function f. By the same token, if we apply differentiation to a function f and then apply integration, we come to the same function f (possibly changed by some additive constant). All mathematicians know this but this journal is not only for mathematicians. All researchers who study problems of integration in their area and use mathematical models can publish their papers in the journal INTEGRATION.

This connection between integration and differentiation is similar to the connection (opposition) between synthesis and analysis. Synthesis, originating from the Greek words συνθεσιζ, συν (with) and θεσιζ (placing), combines elements together to form a new whole system or to create a new structure or pattern that has not previously existed. Thus, synthesis is, in some sense, integration and these words are used interchangeably in many contexts, while as its first step, analysis usually includes differentiation of a system into subsystems, components or elements.

In addition to integration in the pure and applied mathematics, we can observe integration processes everywhere. For example, the main goal of science is discovery of laws of nature. The main tools are experiments and observations. Laws come as generalization and unification of observational data. Consequently, it is possible to treat scientific cognition as integration of observational data into laws of nature. It is possible to find many examples that demonstrate importance of integration in physics. An important achievement of quantum field theory was Feynman's discovery that virtually all quantum problems could be formulated in terms of integrals and path integrals that presuppose a summation over configurations of classical states. As a result, heuristic Feynman-type integrals have become an essential tool both in theoretical and applied areas, including optics, study of macromolecules, quantum mechanics, quantum field theory, gauge theory, quantum gravity and string theory. Such integrals have influence in biology, electrical engineering, and finance.

Distributions, also called generalized functions, are one of the most used in contemporary physics mathematical structures. The standard representation of a distribution is a functional. In turn, functionals are an advanced development of definite integrals, which are special cases of functionals. Thus, one more pivotal for physics mathematical structure, distribution, is based on integration.

It is possible to find examples that demonstrate importance of integration in any branch and field of science. As a result, integration forms a base for many mathematical models in a huge diversity of areas.

Consequently, the meaning of the word *integration* comprises a variety of interpretations. For example, according to Wikipedia, integration is a process of combining or accumulating. It can also mean: *racial integration*, which refers to social and cultural behavior; *political integration*, which refers to the centralization of power within a polity; *economic integration*; *business integration*, which seeks to harmonize company resources, including people, processes, suppliers, customers, and technology, by creating a cohesive and accessible it infrastructure to support the needs of business users; *market integration*; *regional integration*; *horizontal integration* and *vertical integration*, in microeconomics and strategic management, refer to a style of ownership and control; *integration clause*, in a contract, a term used to declare the contract the final and complete understanding of the parties; and so on. International migration and the *integration of immigrants and ethnic minorities* into the fabric of city life are issues of global significance today. There is *functional integration* in mathematics, which extends the domain of integration from a region of a finite dimensional linear space to a space of functions. There is *functional integration* in sociology, which means the interdependence among the parts of a social system. There is also *functional integration* in neurobiology, which means the hypothesis that the integration within and among specialized areas of the brain is mediated by effective connectivity. Only in information technology and computer science, several kinds of integration are utilized and studied: *digital integration* allows data from one device or software to be read or manipulated by another, resulting in ease of use; *enterprise application integration*, also known as *systems integration*, is the use of software and computer systems to bring together a set of enterprise computer applications; *database integration* is pulling together and reconciling dispersed in multiple, heterogeneous systems data for analytic purposes; *data integration* is the extraction, transformation and loading of data from disparate systems into a single data store for the purposes of manipulation and evaluation (reporting); *schema*, or *view*, *integration* is used in large databases with many expected users and applications; *binary ladder integration* and *N-ary integration* are strategies for view integration; *knowledge integration* is the process of integrating knowledge from distinct environments and resolving ontological alignment differences; *integration testing* exposes defects in the interfaces and interaction between integrated components; *system integration testing* verifies that a system is integrated to any external or third party system defined in the system requirements. *circuit integration*; *network integration*; *numerical integration* aimed at computing values of integrals of functions, etc. *Continuous integration* is a useful programming practice. Each of these diverse meanings corresponds to some area of people's activity where there problems that is possible to treat mathematically. Consequently, papers where problems of integration in any area are studied using mathematical structures and technique are welcome to the journal INTEGRATION.

Many processes in society are built on integration. Coming to a new school, a girl or a boy has to integrate into her or his new class. Coming to a new country, an emigrant has to integrate into society of this country. Technology of multicomponent systems is impossible without integration of components into the whole system. To make Europe stronger and more competitive in the modern world, European powers decided to integrate major European countries into the European Union.

Synthesis in chemistry is a kind of integration as chemical synthesis is the formation of a compound from simpler compounds or elements. It is also the production of a substance, e.g., DNA synthesis, or protein synthesis, by the union of chemical elements and groups of elements. In music, synthesis is the artificial generation of a sound by electronic devices and it is also a kind of integration. Thus, mathematical problems of chemical synthesis and synthesis in music are included in the scope of the journal INTEGRATION.

It is possible to find an evidence of popularity of integration with search engines on the Internet. For instance, a hit for the word integration on Google gave 165,000,000 results in 0.25 sec.

All this demonstrates that, on the one hand, if we analyze what people are doing and what they are writing and speaking about, we can see that a very popular kind of activity and topic of discourse is integration.

On the other hand, integration is one of the two main operations in calculus and analysis, while calculus, analysis and related disciplines form the major body of mathematics. Thus, the pivotal topic for the Journal will be integration as an area of mathematics and its various applications. However, traditionally solving differential equations has been also called integration of differential equations. That is why problems of differential equations will occupy an important position in *INTEGRATION*'s publications.

However, integration has a much more extended meaning in mathematics, not speaking about its external understanding. Thus, one of the first (if not the first) mathematical operation, counting, is a kind of integration when a set of objects is integrated into a single number. Taking a limit of a sequence is a kind of integration because different sequence elements are unified by its limit. The whole development of mathematics is based on generalization and abstraction. Both of these processes are forms of integration. Indeed, generalization means integration of some collection of structures into a generalized structure, which reflects common properties of initial structures. It is possible to make abstraction from a single structure, but as a rule abstraction is based on a set of initial structures and its result represents in an integral form many different structures. This gives us a broad understanding of integration in mathematics.

Thus, we can see that the term *integration* has a very broad meaning both in mathematics and beyond. Many researchers from diverse fields work in the area of integration. They publish many papers and books. More than ten journals on different issues of integration have been published: *Journal of Integration Processes and Energy Saving*, *The VLSI Journal of Integration*, *Business Integration Journal*, *Journal of Psychotherapy Integration*, *Journal of Economic Integration*, *Electronic Journal for the Integration of Technology in Education*, *Journal of Systems Integration*, *Journal of European Integration* (Revue d'Intégration Européenne), *Journal of European Integration History*, *Journal of International Migration* *and Integration*, *Journal of Microelectronic System Integration*, *The Life Integration Journal*, etc. However, neither specifically covers mathematical problems of integration.

At the same time, integration, as the main operation in the integral calculus, which, in turn, is included in analysis, is among the most basic tools of mathematics. There are several journals on analysis and its applications. However, now integration is a big area even inside analysis, while there are other kinds of integration in mathematics.

In addition, mathematics with its models and techniques is becoming pivotal for integration outside mathematics.

Thus, this huge area of mathematical aspects of integration in a variety of realms needs a journal to collect in one place those papers that are aimed at mathematical aspects of integration and to represent the community of mathematicians and scientists who work in this area or whose works are related to these problems.

However, no scholarly journal is currently devoted to the provision of publication opportunities for research results specifically in the theory of mathematical integration and its applications, not speaking about mathematical aspects of integration in a diversity areas in technology, society and nature. The journal *INTEG RATION*: *Mathema tical Th eory a nd Applications* fills this void and permits high level exchanges between mathematicians, scientists and other researchers who apply integration in their studies or study integration in their area by mathematical tools. The journal aims to provide a forum for the communication and development of integration and mathematical models that involve integration.

The scope covered by the journal *INTEGRATION* includes all topics related to mathematical problems of integration in any area (in mathematics, computer technology, knowledge and databases, education, society, etc.), as well as all applications of mathematical integration. Mathematical integration is understood in very broad sense. For example, historically solving differential equations is called integration. Thus, papers on problems of differential equations are welcome to the journal *INTEGRATION* . Moreover, any classification or factorization in the mathematical sense is also a kind of integration. Indeed, classes of equivalent elements are integrated in their representatives in the process of factorization or classes of elements are integrated by their names in a classification. When some characteristics is corresponded to some mathematical system, it is integration of properties of the elements from this system. Homotopy classes of functions, cohomology classes of topological spaces and cardinalities of sets are examples of integrals (integral characteristics). Quantification in mathematical logic is also a kind of integration of truth values. Summation is a kind of mathematical integration, namely, it is a discrete integration. Leibniz, who was very inventive with developing consistent and useful notation and concepts, introduced the $\int$ notation for integration that are used now in mathematics. The symbol $\int$ actually is a stylization of the letter S. Thus, integration as a mathematical operation was considered a generalized summation by Leibniz.

As integration is a very general and ever-evolving notion, mathematically oriented articles on other related topics are also invited. Mathematical problems are studied and mathematical technique is used in such area as computer technology (integrated circuits), data- and knowledge bases (knowledge integration).

Any mathematical results that use integration can be published in the Journal. For instance, Laplace transforms are defined in terms of integrals. Thus, all research that uses or studies Laplace transforms is accepted to *INTEGRATION*. Actually integration is used in the majority of fields in mathematics and physics: in number theory, stochastic analysis, dynamical systems, ergodic theory, calculus, differential equations, harmonic analysis, Fourier analysis, optimization, algebra, geometry, numerical analysis, statistics, probability theory, potential theory, quantum physics, classical mechanics, thermodynamics, and astrophysics.

INTEGRATION publishes research papers, notes and short communications, reviews of books, expository and survey articles. The emphasis is going to be on special issues on a focused topic where the aim is to give a picture of recent developments in a particular area.

The Journal will be oriented at publication of special issues each of which will represent mathematical problems of integration or integration applications in some area, e.g., the Feynman integral, gauge integral, integration in financial and economical problems, mathematical problems of circuit integration, mathematical problems of knowledge integration, etc. Researchers are welcome to send proposals for such issues.

Mark Burgin
Chief Editor of the international journal
"Integration: Mathematical Theory and Applications"

PREFACE

In mathematics, the term integration is one of the most important concepts and has an exact and rather restricted. This book examines the development of integration as a mathematical field, as well as mathematical models in different areas that involve integration. (Imprint: Nova Press)

In Chapter 1, a stochastic transfer principle, that relates multiple Itô-type integrals with respect to a general Gaussian random field X, to multiple stochastic integrals defined with respect to a Volterra-type field Y with respect to X of the form

$$Y_{t_1,\ldots,t_d} = \int_0^{t_1} \cdots \int_0^{t_d} \prod_{i=1}^{d} K_i(t_i, s_i) dX_{s_1,\ldots,s_d},$$

is established. As an illustration, the result is applied to deduce the product formula for $I_n^Y(f) I_m^X(g)$ and the independence criterion for $I_n^Y(f)$, $I_m^X(g)$; in the special case when Y is a Volterra-type field with respect to a persistent fractional Brownian sheet, the continuity (for the topology of total variation on the space of signed measures) of the law of integrals $I_n^Y(f)$ with respect to the kernel f is also deduced.

The Feynman path integral, being very popular in physics, has not yet found a concise unified mathematical representation. In Chapter 2, a new approach to the path integral is developed. It is based on hyperintegration, which extends the path integral to the path hyperintegral. The theory of hyperfunctionals and generalized distributions, as a part of hyperanalysis that includes hyperintegration, is a novel approach in functional analysis that provides flexible means for analysis in infinite dimensional spaces. Although, the new theory resembles nonstandard analysis, there are several distinctions between these theories. For example, nonstandard analysis changes spaces of real and complex numbers by injecting infinitely small numbers and other nonstandard entities. In contrast to this, the theory of hyperfunctionals and generalized distributions does not change the inner structure of spaces of real and complex numbers, but adds to them infinitely big and oscillating numbers as external objects. This better correlates with the situation in contemporary physics, which often encounters infinitely big numbers in a form of divergent series and integrals, while there are no infinitely small number in physics. Here our main goal is to find conditions when path hyperintegral has finite value or its value is a real or complex number, i.e., when path hyperintegration coincedes with path integration.

In Chapter 3, for a very general conditioning function, we obtain a simple formula for expressing conditional function space integrals in terms of ordinary (i.e., nonconditional) function space integrals. We then use this simple formula to establish several conditional function space integration formulas.

In Chapter 4, Schrödinger equations in *functional derivatives* are solved via quantized Galerkin limit of antinormal functional Feynman integrals for Schrödinger equations in *partial derivatives*.

In Chapter 5 we introduce the time slicing approximation ([26], [13]) of Feynman path integrals to Gaussian processes as an analysis which has functional integrals and smooth functional derivatives. More precisely, we give a fairly general class of functionals so that the path integrals for Gaussian processes have a mathematically rigorous meaning. For any functional belonging to our class, the time slicing approximation of the path integral converges uniformly on compact subsets of the configuration space. Our class is closed under addition, multiplication, translation, real linear transformation and functional differentiation. The invariance under translation and orthogonal transformation, the interchange of the order with Riemann-Stieltjes integrals and limits, the integration by parts and the Taylor expansion formula with respect to functional differentiation, and the fundamental theorem of calculus hold in the path integrals.

In Chapter 6, let $L^{(\Phi)}[0,1]$ be the Orlicz function space on the Lebesgue unit interval defined by an N-function Φ. If $L^{(\Phi)}[0,1]$ is reflexive, then the Jung constant $JC(L^{(\Phi)}[0,1])$ always satisfies the inequality

$$\max\left(\beta_\Phi, \frac{1}{2\alpha_\Phi^*}\right) \leqslant JC(L^{(\Phi)}[0,1]) \,,$$

where

$$\beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,, \quad \alpha_\Phi^* = \inf_{u\geqslant 1} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,.$$

In Chapter 7, we first construct a new class of separable Banach spaces which contains all Henstock–Kurzweil integrable functions (in particular, the Feynman kernel and the Dirac measure) as norm bounded elements. We then construct infinite tensor product Banach spaces along the lines used by von Neumann to construct infinite tensor product Hilbert spaces. These spaces are used to extend our constructive representation theory for the Feynman operator calculus to the Banach space setting. In order to show the power of our approach, we generalized a few of the important theorems of semigroup theory to the time-ordered setting, including the important Hille–Yosida theorem. It is thus clear that all of semigroup theory can be extended to the time-ordered setting. This means that the formulation of physical theory using our approach is a natural extension of basic operator theory to the time ordered setting. This also means that the problematic disentanglement method used by Feynman to justify his theory by relating it to standard methods is not required when our approach is used. As an application, we unify and extend the theory of time-dependent parabolic and hyperbolic evolution equations. We then develop a general perturbation theory and use it to prove that all theories generated by semigroups are asympotic in an extended sense. We extend the Dyson expansion and provide a general theory for the interaction representation of relativistic quantum theory. We also show that our theory can be reformulated as a physically motivated sum over paths, and use this version to extend the Feynman path integral to a very general setting, providing a generalized Feynman-Kac theorem which is independent of the space of continuous paths.

In Chapter 8 we consider Feynman's operational calculus for two bounded linear operators on Hilbert space with Lebesgue measure as one time-ordering measure and Brownian motion replacing the second time-ordering measure. A representation for solutions of linear stochastic differential equations in Hilbert space is obtained.

The physicist Richard Feynman found a method for forming functions of noncommuting operators. When applied to appropriate functions (such as exponentials of sums of suitable noncommuting operators) the resulting formulas gave the evolution of quantum systems. Feynman interpreted this method as a generalization of his previously defined path integral.

The work in Chapter 9 rests primarily on a family of mathematically rigorous operational calculi developed recently by the first author and Brian Jefferies and inspired by earlier work of the present authors.

We introduce two noncommutative auxiliary operations and study the related algebra and analysis. The disentangling maps are the central objects of Feynman's operational calculi; they put products of operators in their proper order. We explore the connections between the auxiliary operations and the disentangling maps. Also, we use the auxiliary operations to show that the product of two disentangled operators is itself a disentangled operator. Finally, we give some examples of the many formulas that can be established via the auxiliary operations.

In Chapter 10 we develop an integral equation satisfied by Feynman's operational calculi in formalism of B. Jefferies and G. W. Johnson. In particular a "reduced" disentangling is derived and an evolution equation of DeFacio, Johnson, and Lapidus is used to obtain the integral equation. After the integral equation is presented, we show that solutions to the heat and Schrodinger's equation can be obtained from the reduced disentangling and its integral equation. We also make connections between the Jefferies and Johnson development of the operational calculi and the analytic Feynman integral.

In Chapter 11 we consider the numerical approximation of quantum mechanical wave packets by a finite difference method for the Schrödinger equation. We discuss known results for the solutions of the equations for N coupled harmonic oscillators and separation of variables solutions of finite difference equations for the heat and wave equations. We find separation of variables solutions of the Schrödinger finite difference equations with the same spatial part as the above solutions. We compare approximating a stationary state of the Schrödinger equation via four approaches: full finite difference equations for the Schrödinger and wave equations, and individual separation of variables solutions for each set of difference equations. We approximate the Fourier integral representation of a Gaussian wave packet by a Riemann sum, thus writing the wave packet as a finite superposition of plane waves. We then approximate the wave packet using the Schrödinger and wave finite difference equations.

In Chapter 12, te functional integration scheme for Feynman path integrals advanced by

Cartier and DeWitt-Morette is extended to the case of fields. The extended scheme is then applied to quantum field theory. Several aspects of the construction are discussed.

In Chapter 13 we explain the notion of stochastic backward differential equations and its relationship with classical (backward) parabolic differential equations of second order. The paper contains a combination of stochastic processes like Markov processes and martingale theory and semi-linear partial differential equations of parabolic type. Emphasis is put on the fact that the solutions to BSDE's obtained by stochastic methods to BSDE's are often viscosity solutions.

In Chapter 14 we consider the linear integral equations of Fredholm and Volterra

$$x(t) - \int_a^b \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b],$$

and

$$x(t) - \int_a^t \alpha(t,s)\, x(s)\, dg(s) = f(t), \quad t \in [a,b],$$

in the frame of the Henstock-Kurzweil integral and we prove results on the existence and uniqueness of solutions. More precisely, we consider the above equations in the sense of Henstock-Kurzweil and we state a Fredholm Alternative theorem for the first equation and an existence and uniqueness result for the second equation for which the solution is given explicitly.

The refinement integral already surfaced in implicit form in Darboux's reformulation of the Riemann(-Stieltjes) integral, although the recognition that the upper and lower integrals are of refinement type depended on the subsequent convergence theory of E.H. Moore, who also observed that the Lebesgue integral could be construed as refinement. For a sub- (or super-) additive subset integrand, the refinement integral exists as the sup (or inf) over the approximating sums; it can sometimes be evaluated as a standard integral (e.g. for BV functions of quotients of measures by replacing the quotient with the derivative and integrating this point function). Shannon's definitions of information theory notions for continuous distributions can be corrected, and the statisticians' formal integrals made sense of, as refinement integrals. A substantial study of the refinement integral was carried out by Kolmogorov in Math. Ann. 103 (1930) 654–696—he integrates real-valued integrands of a subset argument on a domain of subsets closed only for intersection. In Chapter 15, some extension and application of this basic source will be presented.

In Chapter 16, we define the product of a distribution and a Boehmian and investigate its properties. We also provide an example of a singular distribution and a Boehmian which can be multiplied with each other.

In Chapter 17 we show that every Henstock-Kurzweil integrable mapping taking values in a locally convex space X satisfies the Saks-Henstock lemma if and only if every unconditionally convergent series in X is absolutely convergent.

In Chapter 18, using modern integration theory based on the generalized Riemann integration of Henstock and Kurzweil, stochastic processes such as those of financial theory can be modelled by elementary methods not involving the Itô calculus. To demonstrate this, the following form of the Black-Scholes partial differential equation is established:

$$\frac{\partial E^{\mu}(f)}{\partial t} + \mu\xi \frac{\partial E^{\mu}(f)}{\partial \xi} + \frac{1}{2}\sigma^2\xi^2 \frac{\partial^2 E^{\mu}(f)}{\partial \xi^2} = rE^{\mu}(f).$$

With the underlying asset price process x, with t as the initial time and ξ the initial value ($x(t) = \xi$), and with r an arbitrary discounting parameter, the expectation value $E^{\mu}(f)$ is the expected initial value of a discounted derivative asset price process f relative to the probabilities which are induced in the sample space by assuming an arbitrary growth rate μ in the underlying process x. This new approach also separates those mathematical features which depend on the martingale property from those which depend on the assumption of geometric Brownian motion.

CONTENTS

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INTRODUCTION TO INTEGRATION: INTEGRATION AS A BASIC OPERATION

M. Burgin
Department of Mathematics
University of California, Los Angeles
Los Angeles, CA, U. S.

To integrate, as suggests the origin of this term from the Latin word *integer*, means to make something into a whole or to unify. It is possible to find an evidence of popularity of integration with search engines on the Internet. For instance, a hit for the word integration on Google gave 165,000,000 results in 0.25 sec. Many papers, books and more than ten journals on different issues of integration are published:

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- Electronic Journal for the Integration of Technology in Education;
- Journal of Systems Integration, Journal of European Integration (Revue d'Intégration Européenne);
- Journal of European Integration History;
- Journal of International Migration and Integration;
- Journal of Microelectronic System Integration;
- The Life Integration Journal, and so on.