LIE GROUPS FOR PHYSICISTS

Robert Hermann

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LIE GROUPS FOR PHYSICISTS

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LIE GROUPS
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CONTENTS

CHAPTER 1

INTRODUCTION

In these lectures I shall give some of the qualitative, geometric background of Lie group theory that is not readily found in a direct manner in the standard treatises, e.g., Chevalley (1946), Helgason (1962), Jacobson (1962), or Pontrjagin (1939). I shall assume that the readers are physicists interested in the possible applications of Lie group theory to elementary-particle theory, and that they have already made some sort of beginning toward studying the subject. Most proofs will be omitted or sketched. In general, there is a great need for expositions of modern mathematics for physicists and engineers which only present the most important ideas. This book was written in that spirit.

I shall emphasize the theory of homogeneous spaces of Lie groups (particularly the theory of symmetric spaces), since apparently this is the side of the theory that is least known to physicists, and there are many ideas here that might be useful if they were better known. It is strongly recommended that Helgason's book, *Differential Geometry and Symmetric Spaces*, be consulted for the details that I have omitted. The book by Auslander and Mackenzie, *Introduction to Differentiable Manifolds*, is recommended as useful background for the more elementary general background on manifolds and Lie groups. In addition, much of the "fine structure" of Lie group theory, particularly that which involves topology and the classification of semisimple Lie algebras, will be omitted.

On the other hand, I have refrained from trying to make this report more *immediately* accessible to physicists, by using the primitive, but ingenious, notations that now seem to be standard in the physics literature. As long as the only Lie groups that appeared in physics were of the very simple type [e.g., $SO(3)$, $U(2)$], mathematicians could not really complain too strongly; however, it has been found in mathematics that the more abstract coordinate and basis-free methods developed in the last twenty years are very powerful when dealing with the more complicated Lie groups that seem to be creeping into physics [e.g., $SU(3)$ and $SU(6)$], and physicists who want to push on in these directions will find themselves needlessly wasting much time and effort if they do not learn the "modern" tricks.

I shall concentrate then on those general principles of "geometric" Lie group theory that I believe to be relevant to physics, rather than attempting to duplicate the material that is now traditional in expositions of Lie groups for physicists. For example, the expositions by Behrends et al. (1962), Boerner (1963), Dynkin (1950), Hamermesh (1962), Mathews and Walker (1964), Racah (1951), Salam (1963), and Wightman (1960) can be recommended to the physicist reader as background. As a geometer, I have a strong preference for coordinate-free methods. While the technicalities of manifold theory will be used as little as possible, the reader should at least be familiar with the theory of linear vector spaces as it is used in recent mathematical literature, i.e., in a way independent of bases, and with emphasis on "mapping" ideas. It will be assumed that the reader is familiar with the tensor product, and with such basic ideas as "invariant subgroup," "Lie algebra," etc. For example, the book by Kastler (1961) will be useful to physicists as a bridge between the physicist's and mathematician's versions of linear algebras.

It is a great pleasure to express my thanks to the many people who have helped me in this interdisciplinary endeavor. My conversations with L. Brown, N. Burgoyne, J. Cook, Y. Dothan, M. Gell-Mann, M. Hamermesh, W. McGlinn, L. Michel, J. Moyal, Y. Ne'eman, and B. Sakita have been particularly helpful. I am indebted to W. Givens, W. Miller, and R. Sachs, who arranged on short notice my stay in the uniquely stimulating environment of Argonne National Laboratory. Doris Haight has been a great help as typist. Some of the preliminary work on this material was done under grant from the Mathematics Division of the Air Force Office of Scientific Research. The work was performed in part under the auspices of the U.S. Atomic Energy Commission.

CHAPTER 2

LIE GROUPS AS TRANSFORMATION GROUPS

Groups first arose as transformation groups on spaces. If M is a space, a transformation of a space, denoted, say, by g, is a one-to-one map of M onto itself. If p is a point of M, gp will denote the transform of p by g. Two such maps, say g_1 and g_2, can be composed, to obtain the product g_1g_2:

$$(g_1g_2)p = g_1(g_2p).$$

This product, together with the inverse g^{-1} (i.e., $g^{-1}p$ is that point q such that $gq = p$), give a group structure to the set of transformations. Of course, groups can also be considered as abstract objects, denoted by such letters as G, H, L, etc. Exhibiting such a group as a set of transformations of M, with the "abstract" group operations agreeing with the "geometric" operations on transformations, defines G as a *transformation* group on M. Another way of putting this is to say that exhibiting G on a transformation group on M amounts to exhibiting a map $G \times M \to M$ satisfying an obvious set of conditions. [The image of $(g, p) \in G \times M$ is just gp, the transform of p by g.]

We shall be interested in the cases where G is a Lie group, M is a differentiable manifold, and the action of G on M is given in local coordinates by differentiable functions. (To define a Lie group one must impose on an abstract group the condition that it also is a differentiable manifold, and that the group operations are differentiable in the local coordinates.) There are a series of basic definitions associated with such a group action on M.

1. Let p be a point of M. The set of $g \in G$ that leave p fixed, i.e., satisfy $gp = p$, forms a subgroup of G, called the *isotropy group* of G at p, denoted by G^p.
2. The set of points of M that can be reached by applying elements of G to a single point $p \in M$ is called the *orbit* of G at p, denoted by Gp.

G acts transitively on M (or M is a *homogeneous space* of G) if $Gp = M$ for at least one $p \in M$. (It follows then that this is so for every point of M.) Clearly, G acts transitively on every orbit.

If G acts transitively on M, M can essentially be reconstructed knowing G and G^p, for one point p in M. In general, if H is a subgroup of a group G, we can construct the *coset* space G/H. An *element* of G/H is a subset of G of the form gH, for *one* $g \in G$. G can be made to act as a group of transformations on G/H: g_0 applied to the coset gH is by definition the coset g_0gH. Note now that $M = Gp$ can be identified with G/G^p: Since $gG^pp = gp$, the map $G(p) \to M$ "passes to the quotient" to define a map $G/G^p \to M$ that, it is readily verified, is one-to-one and onto.

Hence the study of homogeneous spaces can, in principle, be reduced to the study of coset spaces G/H, where H is a subgroup of G, hence to the study of pairs (G, H) consisting of a Lie group G and subgroup H. In turn, many of the properties of such pairs can be deduced from algebraic properties of pairs (**G**, **H**) consisting of a Lie algebra **G** and subalgebra **H**. Finally, note that the identification of M with G/G^p really does not depend on the point $p \in M$ chosen: If q is another point of M, and if G acts transitively, there is a $g \in G$ with $gp = q$. Then $G^q = gG^pg^{-1}$; i.e., G is a conjugate subgroup of G. It is clear that the coset spaces corresponding to conjugate subgroups are basically the same.

Example 1

M = Minkowski space, $= G/H$, where G = the Poincaré group (i.e., the inhomogeneous Lorentz group); H = the homogeneous Lorentz group.

Recall the notation for the "classical groups." $GL(n, R)$ and $GL(n, C)$ are the groups of invertible $n \times n$ real and complex matrices. $SL(n, R)$ and $SL(n, C)$ are those (invariant) subgroups consisting of elements of $GL(n, R)$ and $SL(n, C)$ which have determinant 1. The Lie algebra (see Chapter 3 for its definition) of $GL(n, R)$ and $GL(n, C)$ consists of *all* $n \times n$ real and complex matrices. The Jacobi bracket relation for matrices, say α and β, is the commutator $[\alpha, \beta] = \alpha\beta - \beta\alpha$. $O(n, R)$ is the subgroup of $GL(n, R)$ consisting of orthogonal matrices. $O(n, C)$ is a subgroup of $GL(n, C)$ in a similar way. $SO(n, R) = SL(n, R) \cap O(n, R)$, the *rotation group*. $SO(n, C) = SL(n, C) \cap O(n, C)$. $Sp(n, C)$ is the subgroup of $SL(2n, C)$ that leaves invariant a given nondegenerate skew-symmetric form. $U(n)$ is the unitary subgroup of $GL(n, C)$. $SU(n) = U(n) \cap SL(n, C)$. All these groups, except $O(n, R)$, are connected. All are semi-simple, except $GL(n, R)$ and $GL(n, C)$, and $U(n)$.

Example 2

$P_n(R)$ and $P_n(C)$, real and complex projective spaces of n real and complex (respectively) dimensions. (n complex dimensions means $2n$ real dimensions.) For example, $P_n(C)$ is constructed as follows: Start with

C^{n+1}, the space of $(n+1)$ complex variables. A point of C^{n+1} is then an $(n+1)$ tuple $z = (z_1, \ldots, z_{n+1})$ of complex numbers.

Set up an equivalence relation on the nonzero elements of C^{n+1} in the following way. Two vectors z and z' are equivalent if there is a nonzero scalar λ such that $z = \lambda z'$. A "point" of $P_n(C)$ is then an equivalence class of such vectors. [If C^{n+1} is regarded more abstractly as a complex vector space, a "point" of $P_n(C)$ can be regarded as a one-dimensional linear subspace of C^{n+1}.] A function $z \to f(z)$ (not necessarily holomorphic, of course) on C^{n+1} can then be regarded as a function on $P_n(C)$ if it is homogeneous of zeroth degree, i.e., if $f(\lambda z) = f(z)$. The coordinate functions $z_1, \ldots, z_{n+1}$ are not, of course; hence they are not actually functions on $P_n(C)$. In classical language, they are "homogeneous coordinates" for $P_n(C)$. However, genuine functions on $P_n(C)$ can be constructed by taking rational functions in $z_1, \ldots, z_{n+1}$. For example,

$$y_i = \frac{z_i}{z_1} \qquad (2 \leqslant i \leqslant n+1).$$

These are "inhomogeneous coordinates" for $P_n(C)$. Note that they are not defined everywhere on $P_n(C)$, but just on the set of points arising from vectors z whose first component is nonzero (in classical language, on the complement of the "hyperplane at infinity" defined by $z_1 = 0$). There is a topological reason for this: $P_n(C)$ is a compact topological space, hence does not have a coordinate system that is defined everywhere. On the other hand, functions such as

$$f_{ij}(z) = \frac{z_i \bar{z}_j}{z_k \bar{z}_k}$$

are defined everywhere on $P_n(C)$. (Let indices $i, j, k, \ldots$ have the range from 1 to $n+1$, and adopt the summation convention.) Note however that they are not holomorphic ("complex analytic") functions of z.

Now we can exhibit $P_n(C)$ as a homogeneous space of $GL(n+1, C)$. Consider an element α of $GL(n+1, C)$ as an $(n+1 \times n+1)$ matrix (α_{ij}). Let α act on C^{n+1} by the rule

$$(\alpha z)_i = \alpha_{ij} z_j.$$

Note that if $z' = \lambda z$, $\alpha z' = \lambda \alpha z$ also; hence α "passes to the quotient" to define a transformation on $P_n(C)$. We leave it to the reader to show that the action of $GL(n+1, C)$ is transitive on $P_n(C)$; i.e., any pair of one-dimensional subspaces of C^{n+1} can be mapped into each other by a suitably chosen linear transformation.

Let us determine the isotropy subgroup H of $GL(n+1, C)$ at a point

$P_n(C)$, for example, the point determined by the vector $z^0 = (1, 0, \ldots, 0)$. The matrix α leaves this point fixed if there is a scalar λ with

$$\alpha z^0 = \lambda z^0;$$

i.e.,

$$\alpha_{ij} z_j^0 = \lambda z_i^0,$$

or

$$\begin{aligned} \alpha_{i1} &= 0 \qquad \text{for } i > 1, \\ \alpha_{11} &= \lambda. \end{aligned}$$

Thus, H is the subgroup of $GL(n+1, C)$ determined by the condition

$$\alpha_{i1} = 0 \qquad \text{for } i > 1.$$

$GL(n+1, C)$ does not act "effectively" on $P_n(C)$. Let us pause to explain what this means in general. Consider a group G that acts as a transformation group on a space M. The set of $g \in G$ which acts as the identity transformation, i.e., such that

$$gp = p \qquad \text{for all } p \in M,$$

forms a subgroup L of G. In fact, it is an invariant subgroup of G. For any other $g_0 \in G$,

$$(g_0 g g_0^{-1})(p) = g_0 g g_0^{-1} p = g_0 g_0^{-1} p = p;$$

i.e., $g_0 g g_0^{-1} \in L$. The quotient group G/L can then be formed. Since L acts as the identity on M, the action of G "passes to the quotient" to define an action of G/L on M. G/L acts *effectively* on M; i.e., each element of G/L that is not the identity element of the abstract group does not act as the identity on M.

Return now to the case $M = P_n(C)$. $\alpha \in GL(n+1, C)$ acts as the identity if

$$\alpha_{ij} z_j = \lambda(z) z_i \qquad \text{for all } z \in C^{n+1}.$$

This forces

$$\alpha_{ij} = \lambda \delta_{ij},$$

i.e., α is a diagonal matrix. Another way of looking at it is to note that the diagonal matrices form the center of $GL(n+1, C)$; i.e.,

> The quotient of $GL(n+1, C)$ by its center acts on $P_n(C)$. The quotient group is sometimes called the projective or collineation group, since it acts effectively on $P_n(C)$.

We can also consider $SL(n+1, C)$, the group of matrices of determinant 1. It, too, acts transitively on $P_n(C)$. Its center is now discrete, in fact, forms the multiples $\lambda\delta_{ij}$ of the identity matrix, with $\lambda^{n+1} = 1$. Thus, the center is the cyclic group with $n+1$ elements, sometimes denoted by Z_{n+1}. The quotient $SL(n+1, C)/Z_{n+1}$ can then also be identified with the projective group which acts effectively on $P_n(C)$. The subgroup $SU(n+1)$ also acts transitively on $P_n(C)$. The center of $SL(n+1, C)$, namely Z_{n+1}, belongs to $SU(n+1)$. For example, it is an interesting fact that it is precisely the representations of $SU(3)/Z_3$ that occur as symmetries of the strongly interacting particles. The subgroup of $SU(n+1)$ that leaves a point $P_n(C)$ fixed can readily be identified with $U(n)$, so that $SU(n+1)/U(n) = P_n(C)$.

$P_n(R)$ can be dealt with in a similar way by using the real instead of the complex numbers in these constructions.

Example 3

S_n, the n-sphere, $= SO(n+1, R)/SO(n, R)$. Here we start off with R^{n+1}, the space of $(n+1)$-triples of real numbers $x = (x_1, \ldots, x_{n+1})$. S_n is the set of x's with $x_1^2 + \cdots + x_{n+1}^2 = 1$. The matrix groups $SO(n+1, R)$ and $O(n+1, R)$ both act transitively on S_n. The computation of the isotropy subgroup at one point is left to the reader.

CHAPTER 3

LIE ALGEBRAS AND THE CORRESPONDENCE BETWEEN SUBGROUPS AND SUBALGEBRAS

Let G be a Lie group. A *one-parameter subgroup* of G is a mapping $t \to g(t)$ of the real numbers in G that is a homomorphism between the additive group of the real numbers and G; i.e., it satisfies

$$g(t_1 + t_2) = g(t_1)g(t_2).$$

Let ρ be a linear representation of G by linear automorphisms of a (finite-dimensional) vector space V. Thus, for each $g \in G$, $\rho(g)$ is an invertible linear transformation $V \to V$; the rules to be satisfied can be summed up by saying that the map $G \times V \to V$, defined by $(g, v) \to \rho(g)(v)$, defines G as a transformation group on V. Alternatively, of course, one may say that ρ is a homomorphism from G to the group of linear automorphisms of V. To each one-parameter group $t \to g(t)$, we have a one-parameter group $t \to \rho(g(t))$ of linear transformations of V, for which one can find an "infinitesimal generator" α, which is also a linear transformation: $V \to V$ (not necessarily invertible, however).

On the one hand, α can be obtained from the one-parameter group: For $v \in V$,

$$\alpha(v) = \frac{d}{dt}\rho(g(t))(v)\Big|_{t=0}.$$

On the other hand, the one-parameter group can be reconstructed from α: For $v \in V$, $\rho(g(t))(v)$ is the solution of $dv/dt = \alpha v$, with $v(0) = v$, or, explicitly,

$$\rho(g(t))(v) = \exp(t\alpha)v = \sum_{j\to 0}^{\infty} \frac{(t\alpha)^j}{j!}(v).$$

Thus, we have set up a correspondence between the set of one-parameter subgroups of G and certain linear transformations of a vector space. The linear transformations of a vector space V form a vector space; α_1 and α_2 can be added:

$$(\alpha_1 + \alpha_2)(v) = \alpha_1(v) + \alpha_2(v) \qquad \text{for } v \in V.$$