THE SCHUR COMPLEMENT AND ITS APPLICATIONS

NUMERICAL ALGORITHMS

Edited by
FUZHEN ZHANG

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Claude Brezinski
Université des Sciences et Technologies de Lille, France

To our families, friends, and the matrix community

Issai Schur (1875-1941)

This portrait of Issai Schur was apparently made by the "Atelieir Hanni Schwarz, N. W. Dorotheenstraße 73" in Berlin, *c.* 1917, and appears in *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis: Gewidmet dem großen Mathematiker Issai Schur (1875–1941)* edited by Bernd Fritzsche & Bernd Kirstein, pub. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1991.

Emilie Virginia Haynsworth (1916-1985)

This portrait of Emilie Virginia Haynsworth is on the Auburn University Web site www.auburn.edu/~fitzpjd/ben/images/Emilie.gif and in the book *The Education of a Mathematician* by Philip J. Davis, pub. A K Peters, Natick, Mass., 2000.

Preface

What's in a name? To paraphrase Shakespeare's Juliet, that which Emilie Haynsworth called the *Schur complement*, by any other name would be just as beautiful. Nevertheless, her 1968 naming decision in honor of Issai Schur (1875–1941) has gained lasting acceptance by the mathematical community. The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications.

Our goal is to expose the Schur complement as a rich and basic tool in mathematical research and applications and to discuss many significant results that illustrate its power and fertility. Although our book was originally conceived as a research reference, it will also be useful for graduate and upper division undergraduate courses in mathematics, applied mathematics, and statistics. The contributing authors have developed an exposition that makes the material accessible to readers with a sound foundation in linear algebra.

The eight chapters of the book (Chapters 0–7) cover themes and variations on the Schur complement, including its historical development, basic properties, eigenvalue and singular value inequalities, matrix inequalities in both finite and infinite dimensional settings, closure properties, and applications in statistics, probability, and numerical analysis. The chapters need not be read in the order presented, and the reader should feel at leisure to browse freely through topics of interest.

It was a great pleasure for me, as editor, to work with a wonderful group of distinguished mathematicians who agreed to become chapter contributors: T. Ando (Hokkaido University, Japan), C. Brezinski (Université des Sciences et Technologies de Lille, France), R. A. Horn (University of Utah, Salt Lake City, USA), C. R. Johnson (College of William and Mary, Williamsburg, USA), J.-Z. Liu (Xiangtang University, China), S. Puntanen (University of Tampere, Finland), R. L. Smith (University of Tennessee, Chattanooga, USA), and G. P. H. Styan (McGill University, Canada).

I am particularly thankful to George Styan for his great enthusiasm in compiling the master bibliography for the book. We would also like to acknowledge the help we received from Gülhan Alpargu, Masoud Asgharian, M. I. Beg, Adi Ben-Israel, Abraham Berman, Torsten Bernhardt, Eva Brune, John S. Chipman, Ka Lok Chu, R. William Farebrother, Bernd Fritsche, Daniel Hershkowitz, Jarkko Isotalo, Bernd Kirstein, André Klein, Jarmo Niemelä, Geva Maimon Reid, Timo Mäkeläinen, Lindsey E. McQuade, Aliza K. Miller, Ingram Olkin, Emily E. Rochette, Vera Rosta, Eugénie Roudaia, Burkhard Schaffrin, Hans Schneider, Shayle R. Searle, Daniel N. Selan, Samara F. Strauber, Evelyn M. Styan, J. C. Szamosi, Garry J. Tee, Götz Trenkler, Frank Uhlig, and Jürgen Weiß. We are also very grateful to the librarians in the McGill University Interlibrary Loan and Document Delivery Department for their help in obtaining the source materials for many of our references. The research of George P. H. Styan was supported in part by the Natural Sciences and Engineering Research Council of Canada.

Finally, I thank my wife Cheng, my children Sunny, Andrew, and Alan, and my mother-in-law Yun-Jiao for their understanding, support, and love.

Fuzhen Zhang
September 1, 2004
Fort Lauderdale, Florida

Contents

Chapter 0

Historical Introduction: Issai Schur and the Early Development of the Schur Complement

0.0 Introduction and mise-en-scène

In this introductory chapter we comment on the history of the Schur complement from 1812 through 1968 when it was so named and given a notation. As Chandler & Magnus [113, p. 192] point out, "The coining of new technical terms is an absolute necessity for the evolution of mathematics." And so we begin in 1968 when the mathematician Emilie Virginia Haynsworth (1916–1985) introduced a name and a notation for the Schur complement of a square nonsingular (or invertible) submatrix in a partitioned (two-way block) matrix [210, 211].

We then go back fifty-one years and examine the seminal lemma by the famous mathematician Issai Schur (1875–1941) published in 1917 [404, pp. 215–216], in which the *Schur determinant formula* (0.3.2) was introduced. We also comment on earlier implicit manifestations of the Schur complement due to Pierre Simon Laplace, later Marquis de Laplace (1749–1827), first published in 1812, and to James Joseph Sylvester (1814–1897), first published in 1851.

Following some biographical remarks about Issai Schur, we present the *Banachiewicz inversion formula* for the inverse of a nonsingular partitioned matrix which was introduced in 1937 [29] by the astronomer Tadeusz Banachiewicz (1882–1954). We note, however, that closely related results were obtained earlier in 1933 by Ralf Lohan [290], following results in the book [66] published in 1923 by the geodesist Hans Boltz (1883–1947).

We continue with comments on material in the book *Elementary Matrices and Some Applications to Dynamics and Differential Equations* [171], a classic by the three aeronautical engineers Robert Alexander Frazer (1891–1959), William Jolly Duncan (1894–1960), and Arthur Roderick Collar (1908–1986), first published in 1938, and in the book *Determinants and Matrices* [4] by the mathematician and statistician Alexander Craig Aitken (1895–1967), another classic, and first published in 1939.

We introduce the *Duncan inversion formula* (0.8.3) for the sum of two matrices, and the very useful *Aitken block-diagonalization formula* (0.9.1), from which easily follow the *Guttman rank additivity formula* (0.9.2) due to the social scientist Louis Guttman (1916–1987) and the *Haynsworth inertia additivity formula* (0.10.1) due to Emilie Haynsworth.

We conclude this chapter with some biographical remarks on Emilie Haynsworth and note that her thesis adviser was Alfred Theodor Brauer (1894–1985), who completed his Ph.D. degree under Schur in 1928.

This chapter builds on the extensive surveys of the Schur complement published (in English) by Brezinski [73], Carlson [105], Cottle [128, 129], Ouellette [345], and Styan [432], and (in Turkish) by Alpargu [8]. In addition, the role of the Schur complement in matrix inversion has been surveyed by Zielke [472] and by Henderson & Searle [219], with special emphasis on inverting the sum of two matrices, and by Hager [200], with emphasis on the inverse of a matrix after a small-rank perturbation.

0.1 The Schur complement: the name and the notation

The term *Schur complement* for the matrix

$$S - RP^{-1}Q, \tag{0.1.1}$$

where the nonsingular matrix P is the leading submatrix of the complex partitioned matrix

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{0.1.2}$$

was introduced in 1968 in two papers [210, 211] by Emilie Haynsworth published, respectively, in the *Basel Mathematical Notes* and in *Linear Algebra and its Applications.*

The notation

$$(M/P) = S - RP^{-1}Q \tag{0.1.3}$$

for the Schur complement of P in $M = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ was apparently first used in 1968 by Haynsworth, in the *Basel Mathematical Notes* [210] but not in *Linear Algebra and its Applications* [211], where its first appearance seems to be in the 1970 paper by Haynsworth [212]. This notation does appear, however, in the 1969 paper [131] by Haynsworth with Douglas E. Crabtree in the *Proceedings of the American Mathematical Society* and is still in use today, see e.g., the papers by Brezinski & Redivo Zaglia [88] and N'Guessan [334] both published in 2003; the notation (0.1.3) is also used in the six surveys [8, 73, 128, 129, 345, 432].

The notation $(M|P)$, with a vertical line separator rather than a slash, was introduced in 1971 by Markham [295] and is used in the book by Prasolov [354, p. 17]; see also [296, 332, 343] published in 1972–1980. The notation $M|P$ without the parentheses was used in 1976 by Markham [297].

In this book we will use the original notation (0.1.3) but without the parentheses,

$$M/P = S - RP^{-1}Q, \tag{0.1.4}$$

for the Schur complement of the nonsingular matrix P in the partitioned matrix $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. This notation (0.1.4) without the parentheses was introduced in 1974 by Carlson, Haynsworth & Markham [106] and seems to be very popular today, see, e.g., the recent books by Ben-Israel & Greville [45, p. 30], Berman & Shaked-Monderer [48, p. 24], and by C. R. Rao & M. B. Rao [378, p. 139], and the recent papers [160, 287, 471].

0.2 Some implicit manifestations in the 1800s

According to David Carlson in his 1986 survey article [105] entitled "What are Schur complements, anyway?":

> The idea of the Schur complement matrix goes back to the 1851 paper [436] by James Joseph Sylvester. It is well known that the entry a_{ij} of [the Schur complement matrix] A, $i = 1, \ldots, m - k$, $j = 1, \ldots, n - k$, is the minor of [the partitioned matrix] M determined by rows $1, \ldots, k, k+i$ and columns $1, \ldots, k, k+j$, a property which was used by Sylvester as his definition. For a discussion of this and other appearances of the Schur complement matrix in the 1800s, see the paper by Brualdi & Schneider [99].

Farebrother [162, pp. 116–117] discusses work by Pierre Simon Laplace, later Marquis de Laplace, and observes that Laplace [273, livre II, §21 (1812); *Œuvres*, vol. 7, p. 334 (1886)] obtained a ratio that we now recognize as the ratio of two successive leading principal minors of a symmetric positive definite matrix. Then the ratio $\det(M)/\det(M_1)$ is the determinant of what we now know as the Schur complement of M_1 in M, see the

Schur determinant formula (0.3.2) below. Laplace [273, §3 (1816); *Œuvres*, vol. 7, pp. 512–513 (1886)] evaluates the ratio $\det(M)/\det(M_1)$ with $n = 3$.

0.3 The lemma and the Schur determinant formula

The adjectival noun "Schur" in "Schur complement" was chosen by Haynsworth because of the lemma (Hilfssatz) in the paper [404] by Issai Schur published in 1917 in the *Journal für die reine und angewandte Mathematik*, founded in Berlin by August Leopold Crelle (1780–1855) in 1826 and edited by him until his death. Often called Crelle's *Journal* this is apparently the oldest mathematics periodical still in existence today [103]; Frei [174] summarizes the long history of the *Journal* in volume 500 (1998).

The picture of Issai Schur facing the opening page of this chapter appeared in the 1991 book *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis: Gewidmet dem großen Mathematiker Issai Schur (1875–1941)* [177, p. 20]; on the facing page [177, p. 21] is a copy of the title page of volume 147 (1917) of the *Journal für die reine und angewandte Mathematik* in which the Schur determinant lemma [404] was published.

This paper [404] is concerned with conditions for power series to be bounded inside the unit circle; indeed a polynomial with roots within the unit disk in the complex plane is now known as a *Schur polynomial*, see e.g., Lakshmikantham & Trigiante [271, p. 49].

The lemma appears in [404, pp. 215–216], see also [71, pp. 148–149], [177, pp. 33–34]. Our English translation, see also [183, pp. 33–34], follows. The Schur complement $S - RP^{-1}Q$ is used in the proof but the lemma holds even if the square matrix P is singular. We refer to this lemma as the *Schur determinant lemma*.

LEMMA. *Let P, Q, R, S denote four $n \times n$ matrices and suppose that P and R commute. Then the determinant* $\det(M)$ *of the $2n \times 2n$ matrix*

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

is equal to the determinant of the matrix $PS - RQ$.

Proof. We assume that the determinant of P is not zero. Then, with I denoting the $n \times n$ identity matrix,

$$\begin{pmatrix} P^{-1} & 0 \\ -RP^{-1} & I \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I & P^{-1}Q \\ 0 & S - RP^{-1}Q \end{pmatrix}.$$

Taking determinants yields $\det(P^{-1}) \cdot \det(M) = \det(S - RP^{-1}Q)$ and so