LECTURE NOTES IN PHYSICS

F. Iachello

Lie Algebras and Applications

Springer

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Preface

In the second part of the 20th century, algebraic methods have emerged as a powerful tool to study theories of physical phenomena, especially those of quantal systems. The framework of Lie algebras, initially introduced by Sophus Lie in the last part of the 19th century, has been considerably expanded to include graded Lie algebras, infinite-dimensional Lie algebras, and other algebraic constructions. Algebras that were originally introduced to describe certain properties of a physical system, in particular behavior under rotations and translations, have now taken center stage in the construction of physical theories.

This book contains a set of notes from lectures given at Yale University and other universities and laboratories in the last 20 years. The notes are intended to provide an introduction to Lie algebras at the level of a one-semester graduate course in physics. Lie algebras have been particularly useful in spectroscopy, where they were introduced by Eugene Wigner and Giulio Racah. Racah's lectures were given at Princeton University in 1951 (Group Theory and Spectroscopy) and they provided the impetus for the initial applications in atomic and nuclear physics. In the intervening years, many other applications have been made. This book contains a brief account of some of these applications to the fields of molecular, atomic, nuclear, and particle physics. The application of Lie algebraic methods in Physics is so wide that often students are overwhelmed by the sheer amount of material to absorb. This book is intended to give a basic introduction to the method and how it is applied in practice, with emphasis on bosonic systems as encountered in molecules (vibron model), and in nuclei (interacting boson model), and to fermionic systems as encountered in atoms (atomic shell model), and nuclei (nuclear shell model), and hadrons (quark model). Exactly solvable problems in quantum mechanics are also discussed.

Associated with a Lie algebra there is a Lie group. Although the emphasis of these lecture notes is on Lie algebras, a chapter is devoted to Lie groups and to the relation between Lie algebras and Lie groups.

Many exhaustive books exist on the subject of Lie algebras and Lie groups. Reference to these books is made throughout, so that the interested student can study the subject in depth. A selected number of other books, not explicitly mentioned in the text, are also included in the reference list, to serve as additional introductory material and for cross-reference.

In the early stages of preparing the notes, I benefited from many conversations with Morton Hamermesh, Brian Wybourne, Asim Barut, and Jin-Quan Chen, who wrote books on the subject, but are no longer with us. This book is dedicated to their memory. I also benefited from many conversations with Robert Gilmore, who has written a book on the subject, and with Phil Elliott, Igal Talmi, Akito Arima, Bruno Gruber, Arno Böhm, Yuval Ne'eman, Marcos Moshinsky, and Yuri Smirnov, David Rowe, who have made major contributions to this field.

I am very much indebted to Mark Caprio for a critical reading of the manuscript, and for his invaluable help in preparing the final version of these lecture notes.

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May 2006

Francesco Iachello

List of Symbols

$\mathcal{A}$	Abelian algebra
$[,]_+$	anticommutator
$\{,\}$	anticommutator
$\mathcal{B}$	basis (bosons)
$\mathcal{F}$	basis (fermions)
$\in$	belongs to
b	boson annihilation operator
$b^\dagger$	boson creation operator
$\langle\vert$	bra
$C_i(g)$	Casimir operator of g
$[,]$	commutator
C	complex
$\supset$	contains
$\subset$	contained in
$\equiv$	defined as
Der	derivation
dim	dimension of the representation
$\oplus$	direct sum
DS	dynamic symmetry
X_α	element of an algebra
$\square$	entry in Young tableau
$\doteqdot$	equal to
$\langle\rangle$	expectation value
$\downarrow \exp$	exponential map
a	fermion annihilation operator
$a^\dagger$	fermion creation operator
$\backslash/$	Gel'fand pattern
$\approx$	homomorphic groups
$\cap$	intersect
IRB	irreducible basis
$\sim$	isomorphic algebras
$\langle\vert\rangle$	isoscalar factors
$\vert\rangle$	ket
$\downarrow$	labels of the representation

∇^2	Laplace operator
$\mathcal{G}$	Lie algebra
g	Lie algebra
G	Lie group
$\mathcal{V}$	linear vector space
$\circ$	long root
$\mathcal{F}$	number field
O	octonion
$\alpha^{(+)}$	positive root
Q	quaternion
$\langle\|\|\|\rangle$	reduced matrix elements
$\cdot$	scalar product
$\oplus_s$	semidirect sum
$\bullet$	short root
SGA	spectrum generating algebra
$\otimes$	tensor product
$\oplus$	tensor sum
$\|0\rangle$	vacuum
$\mathcal{L}$	vector space

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Contents

1 Basic Concepts

1.1 Definitions

The key notion in the definition of Lie algebras is that of the commutator (or bracket), denoted by $[,]$. The commutator of X and Y is defined as

$$[X,Y] = XY - YX\ . \tag{1.1}$$

It satisfies the relations

$$[X,X] = 0; \qquad [X,Y] = -[Y,X]\ . \tag{1.2}$$

Another key notion is that of number field, $\mathcal{F}$. The number fields of interest are: Real, R, Complex, C, Quaternion, Q, and Octonion, O. Since these lecture notes are intended primarily for applications to quantal systems, where the basic commutation relations between coordinates and momenta are

$$\left[x, \frac{1}{i}\frac{\partial}{\partial x}\right] = i\ , \tag{1.3}$$

only real and complex fields will be considered. Although formulations of quantum mechanics in terms of quaternions and octonions have been suggested, Lie algebras over the quaternion and octonion number fields will not be discussed here.

1.2 Lie Algebras

Lie algebras are named after the Norwegian mathematician Sophus Lie (1842-1899). Most of what we know about the original formulation comes from Lie's lecture notes in Leipzig, as collected by Scheffers. [S. Lie and G. Scheffers, *Vorlesungen über Kontinuerliche Gruppen*, Leipzig, 1893].

A set of elements $X_\alpha (\alpha = 1, \ldots, r)$ is said to form a Lie algebra $\mathcal{G}$, written as $X_\alpha \in \mathcal{G}$, if the following axioms are satisfied:

Axiom 1 *The commutator of any two elements is a linear combination of the elements in the Lie algebra*

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