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Global Geometry and Mathematical Physics

Montecatini Terme 1988

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Global Geometry and Mathematical Physics

Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held at Montecatini Terme, Italy, July 4–12, 1988

Editors: M. Francaviglia, F. Gherardelli

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FOREWORD

"Geometry and Physics" is a binomial which has become increasingly more important in the last years and especially in the last two decades. The relations between these two subjects, whose marriage was already hidden in the *"Méchanique Analitique"* of J.-Louis Lagrange (1788), were subtly envisaged as means of understanding the structure of our Universe by the genius of B. Riemann (1854) and fully developed by A. Einstein in his celebrated theory of General Relativity (1916). Since then, although with varying degrees of fortune, the relationship between these two disciplines has grown steadily and robustously, through the contribution of many great scientists (D. Hilbert, T. Levi Civita, E. Cartan, H. Weyl, just to mention a few).

Up to a few years ago the interaction between Geometry and Physics was mainly limited to the domain of Differential Geometry, which plays a fundamental role in the local formulation of the laws of classical and relativistic field theories, and always in a single direction: from the theory to the application. More recently, however, other fundamental branches of Geometry have found their way into Physics, giving an enormous impetus especially to the investigations concerning the global behaviour of physical fields and their quantum properties, the structure of gauge theories, the theory of crystalline defects, the global structure of so-called "completely integrable" dynamical systems, as well as many other domains of application. In particular, the last two decades have seen an increasingly extensive application of Differential Topology and Global Analysis to Field Theory, and an impressive renewed role of Algebraic Geometry in both String Theory and Classical Mechanics. Also the arrow of influence has been somehow reversed, and nice results in pure Mathematics have been sometimes based on ideas originated within the context of Physics (the most striking example is, perhaps, the relatively recent work on 4-manifolds based on the structure of Yang-Mills fields).

* * * *

With exactly this spirit in mind, around 1987 the idea came to one of us (MF) to organize a Summer School on "Global Geometry and Mathematical Physics", to be held under the auspices of the CIME Foundation, with the explicit purpose of offering to both pure mathematicians and theoretical physicists the occasion of an intensive period of interaction on a number of freshly developing fields of common interest.

Also profiting of fruitful conversations with many colleagues (among which A. Cassa, R. Catenacci, M. Cornalba, M. Ferraris and C. Reina), the general structure of the Course was established. It was decided to divide the Course into four modular parts, to be assigned to experienced teachers coming both from pure Geometry and from Mathematical Physics, centering around the main themes of global aspects of field theories (instantons, monopoles, vortices, anomalies, functional integration of gauge fields) and the algebro-geometric methods in Mathematical Physics (KP hierarchies, infinite dimensional Grassmannians, theory of Riemann Surfaces and its applications to String Theory). The CIME Foundation did enthusiastically accept this program, which has been later perfectioned and worked out in detail with the collaboration of the second Editor (FG).

The Course was held in Montecatini Terme, in the beautiful and quiet surroundings of "Villa La Querceta", from July 4 to July 12, 1988. (Incidentally, this was exactly 200 years after the publication of the cited treatise of Lagrange, which laid the foundations of the fruitful interaction between Geometry and Physics and initiated the modern theory of dynamical systems of Lagrangian and Hamiltonian type). It was attended by over 70 participants, belonging to both the mathematical and the physical communities, thus reaching its aim of stimulating further interaction and development. The intensive program was centered around four main Lectures, in which all the major topics were discussed. A number of Seminars was also provided (both by participants and by invited scientists) to cover complements or parts of the subjects which could not be fully exploited during the lectures, and also to discuss some further topics having relation with the Course itself.

These Lecture Notes contain the extended text of three of the four main Lectures, as well as a carefully addressed selection of the Seminars held during the Course. It has to be remarked that all the Seminars were rather interesting and valuable, although only very few of them appear in these Notes. The choice, which has been also worked out together with the four main lecturers, was dictated by a severe limitation of space, so that here only the seminars whose contents are really complementary to the Lectures appear.

* * * *

The first topic discussed in the Course was the application of global methods of Differential Topology to the domain of Field Theory. The Lectures of Nigel HITCHIN (Oxford) on *"The Geometry and Topology of Moduli Spaces"* , were mainly aimed at discussing the various occurrences of the theory of moduli spaces in physical applications. The Lectures begin with a short description of the notion of moduli spaces, to pass immediately to discuss the self-dual Yang-Mills equations and the instantons over the 4-sphere, together with the appropriate moduli space. This opens the way to a short account on Donaldson's work on invariants for four-dimensional manifolds, which are here considered also in the general setting proposed by E. Witten. An interesting section refers to the Riemannian structure of the moduli space of all instantons, which is described in full detail together with its hyper-Kähler structure. Passing to the case of a coupled scalar field, the Yang-Mills-Higgs equations are discussed, in view of the moduli space structure of their "monopole" solutions; also the hyper-Kähler metric of this space is considered. After having discussed the above 3-dimensional and 4-dimensional situations, the Lectures address the gauge-theoretic viewpoint in the 2-dimensional case of a Riemann surface. Various important aspects are touched upon (hyper-Kähler structure, stability, existence of flat connections and related Teichmüller space structures). Finally, vortices and skyrmions, together with the structure of the corresponding moduli spaces are considered.

Some seminars were more or less directly related to the subject of this Lecture. P. HORVATHY (Metz) discussed the *"Dynamic Symmetry of Monopole Scattering"* , giving a short but detailed account about the scattering of Bogomolny-Prasad-Sommerfeld monopoles for SU(2)-gauge theories, the Hamiltonian structure of Taub-Nut geodesics and some remarkable extension to O(4,2)-symmetry. (See Cordani B., Feher L.Gy. and Horvathy P.A., Phys. Lett. **201B**, 481(1988)). M. SEPPÄLA (Helsinki) considered *"Teichmüller spaces and Moduli Spaces of Klein Surfaces"* , presenting a detailed exposition on moduli spaces of real algebraic curves, via a real version of Torelli theorem and suitable Teichmüller spaces (see M.Seppäla and R. Silhol, to appear in Math. Zeitschrift). Another group of seminars addressed the modern investigations of the "supersymmetric extension" of the motion of moduli spaces; these shall be reviewed later in this Introduction, owing to their explicit connections with other main Lectures.

Unfortunately, the Lectures of R. STORA (CERN and Annecy) about *"Differential Algebras in Field Theory"* could not be typed timely and hence could not be collected here. We profit however of a short description of their contents which has been written by the Author: -- The influence of Geometry on Quantum Field Theory has been increasing over the last few years, mainly under the influence of E. Witten, within the Theoretical Physics community, and of M.F. Atiyah, within the Mathematical community. E. Witten has created a new discipline, which one may call "Physical Mathematics", as distinguished from Mathematical Physics. The latter consisted in applied mathematics to well-formulated of some relevance to Physics. The former consist in applying mostly heuristic Field Theory or Quantum Mechanics methods to mathematical problems, mostly in Geometry, often shedding a new light on known theories and pointing to new developments. The methods used in this context, which are mostly non-perturbative, fail to incorporate one of the strongest ingredients of the perturbative methods: locality. The idea was to review consequences of locality in agreement with the geometry of some interesting models based on Lagrangians: (i) locality in perturbation theory, both in Minkowski space and in a compact Riemannian or Euclidean manifold; (ii) 4-dimensional gauge theories and their anomalies: the s-operations connected with gauge fixing; (iii) the quantization of differential forms, s-operations and their relations to the Ray supertorsion; (iv) 2-dimensional conformal models: the free string as an example, the anomaly; (v) topological theories: the topological Yang-Mills theory in four dimensions as an example (in relation with the Jones polynomials). Since then Chern-Simons in 3-dimension has proved to be one of the most interesting ones mathematically, in relation to knot theory, and most tractable models of this sort, although there are still some some subtle quantization problems to be settled --.

The problem of Chern-Simons terms in their supersymmetric version was addressed by G. LANDI (SISSA, Trieste) and U. BRUZZO (Genova) in the seminar *"Geometry of Standard Constraints and Anomalous Supersymmetric Gauge Theories"* which is included in these Proceedings as an integrating part. This was largely based on another seminar by the same authors, on *"Some topics in the Theory of Supervector Bundles"* , were the structure and cohomology of supermanifolds, super vector bundles and super line bundles was addressed, with applications to the existence of connections and characteristic classes on SVB's. Intimately related with the above topics was a seminar by R. CIANCI (Genova) on *"Differential Equations on Supermanifolds"*, which cannot be included here (see Cianci R., Journ. Math. Phys. **29**, 2152 (1988)). Still in the context of "Global Anomalies", although in a rather different perspective, we also mention a beautiful seminar by L. DABROWSKI (SISSA, Trieste), on *"Berry's Phase for Mixed States"* , referring to the rising of an extra phase in the course of various physical processes (quantum optics,...).

* * * *

The Lectures by L. ALVAREZ-GAUMÉ (CERN) form, in a sense, a bridge between the previous aspect of "global geometry applied to field theory" and the algebro-geometric side mentioned earlier in this Introduction. They were in fact concerned with discussing in great detail the mathematical structures connected with (classical, quantum and supersymmetric) string theory, as well as the whole class of so-called "conformal theories". The material presented here is divided in six parts. The Lectures contain a thorough introduction to the methods of conformally invariant theories over Riemann Surfaces of arbitrary genus g , which are considered in the operatorial approach (some of whose aspects have been retaken in the course of appropriate Seminars). In the first two sections it is shown how conserved quantities relate to representations of the "Virasoro algebra" and how Feynman's rule of integration over paths in phase-space leads to integrals over the appropriate moduli space of the relevant Riemann surface which is first supposed to have topology RxS^1. The third lecture extends to cover the case of surfaces having higher genus g and an arbitrary number n of parametrized boundaries. The corresponding moduli space $P(g,n)$ is considered in detail together with quantum states in the appropriate Hilbert space. Lecture 4 addresses the case of an interacting scalar field. In lectures 5 and 6 the author finally considers the problems of generating a connection over $P(g,n)$ out of the Virasoro algebra and of constructing a physically meaningful measure on moduli spaces of Riemann Surfaces with distinguished points.

As we already mentioned above, several specialized Seminars were devoted either to cover in greater detail some of the topics touched upon in these Lectures on Strings and Conformal Fields Theories, or to establish links with the previous two Lectures, as well as with the further Lecture of E. Arbarello. A pedagogical seminar on *"Introduction to Supergravity and Superstrings"* was given by F. GIERES (Berne). An interesting new perspective on strings was addressed in the seminar *"String Field Theory as General Relativity of Loops"* by L. CASTELLANI (Torino and CERN), whereby the dynamics of bosonic and supersymmetric strings was considered in the framework of free differential algebras on group manifolds and using loop representations based on the space Diff(S^1) of diffeomorphisms of the circle (see L. Castellani, R.D'Auria and P.Fre *"Supergravity Theory: a Geometrical Perspective"* , World Sci. (Singapore, 1989)). Strictly related to this group manifold approach and also in deep connection with Stora's Lectures on topological invariant in field theories was a couple of seminars delivered by R. D'AURIA (Padova) and P.FRE (Torino), respectively on *"Superspace Constraints and Chern-Simons Cohomology in D=4 Superstring Effective Theories"* and *"Geometrical Formulation of 4-Dimensional Superstrings"* (see, e.g., P. Fre and F. Gliozzi, Phys. Lett. **B208** , 203 (1988)). S. SHNIDER (Beersheva) gave an interesting seminar on *"Supercommutative Algebra in Higher Dimensions"* , showing in particular, in the algebraic context of Konstant's theory of graded manifolds, that no superconformal algebras exist in dimension strictly greater than six. The operator formalism for string theory in genus g larger than one, which formed the core of Alvarez-Gaumè's Lectures, was discussed in greater detail in the Seminar *"Hamiltonian Formulation of String Theory and Multiloop Amplitudes in the Operator Context"* by A.R. LUGO and J. RUSSO (SISSA, Trieste), which is here included as a complement to the Lectures themselves. On parallel lines M. MATONE (SISSA, Trieste) delivered the seminar *"Conformal Field Theories, Real Weight Differentials and KdV Equation in Higher Genus"* , which is included here; the Seminar was devoted to establish a link between the operator approach of conformal field theories and the algebraic geometric aspects related with Krichever-Novikov algebra on a Riemann surface.

Two seminars addressed, on different perspectives, the important problem of coherently defining the structure of supermoduli spaces of super Riemann surfaces, thus providing mathematically well grounded basis for the discussion of amplitudes in superstring theory: *"Super Riemann Surfaces and Super Moduli Spaces* by M. ROTHSTEIN (Suny at Stony Brook), not included here (see M. Rothstein, Proc. Amer. Math. Soc. **95** , 255-259 (1985)) , and *"Supermoduli and Superstrings"* by G. FALQUI and C. REINA (SISSA, Trieste).

These last seminars bring directly into the core of the applications of "strong" algebraic-geometrical methods in Mathematical Physics, which include nowadays a wide spectrum of techniques and domains of interest. As we already said above, the aim of the Lectures by E. ARBARELLO (Rome) was exactly to make an up-to-date review on some of these relevant topics; the Lectures *"Geometrical Aspects of Kodomchev-Petviashvily Equation"* , written together with C. DE CONCINI (Rome), address in fact all the algebro-geometric machinery involved in the KP generalization of the famous KdV equation. Their first chapter reviews the fundamental concepts from the theory of Riemann surfaces and Abelian varieties (Abel-Jacobi map, Torelli theorem, etc.). Chapter two is devoted to discuss a geometrical criterion to check whether a principal polarized Abelian variety is the Jacobian of a (possibly reducible) algebraic curve. Another criterion, which is based on the so-called "trisecant formula" and which leads naturally to the KP equation, is extensively discussed in Chapter 3. This smoothly introduces to Chapter 4, were the KP equation is used to characterize the Jacobians themselves, and to Chapter 5, where the Hirota bilinear form of the KP hierarchy is discussed. The next Chapters are finally devoted to a through discussion on the infinite dimensional Grassmannian *Gr(H)* and the corresponding t-function on the inverse determinant bundle of the Grassmannian.

A natural complement to these Lectures was a beautiful seminar on *"The Geometrical Construction of W Algebras and their Quantization"* , by D.J. SMIT (Utrecht) whereby various relations between KdV equations, Yang-Baxter equations, quantum groups and bi-Hamiltonian structures for the Virasoro algebra are discussed. We finally mention the nice seminar *"The Hilbert Schmidt Grassmannian is Non-negatively Curved"* , delivered by O. PEKONEN (Palaiseau), which refers to the Kähler structure and sectional curvature of *Gr(H)* (see O. Pekonen, Man. Math. **63**, 21-27 (1989)).

Mauro Francaviglia

LECTURE NOTES IN MATHEMATICS

Edited by A. Dold, B. Eckmann and F. Takens

Some general remarks on the publication of monographs and seminars

In what follows all references to monographs, are applicable also to multiauthorship volumes such as seminar notes.

§1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph.D. theses to be accepted for the Lecture Notes series.

Experience has shown that English language manuscripts achieve a much wider distribution.

§2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg. These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscripts, but a preliminary decision can usually be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information. We generally advise authors not to prepare the final master copy of their manuscript (cf. §4) beforehand.

§3. Final manuscripts should contain at least 100 pages of mathematical text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: this is almost always genuinely helpful for the reader.

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TABLE OF CONTENTS

THE GEOMETRY AND TOPOLOGY OF MODULI SPACES

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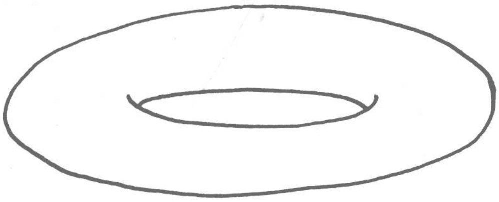
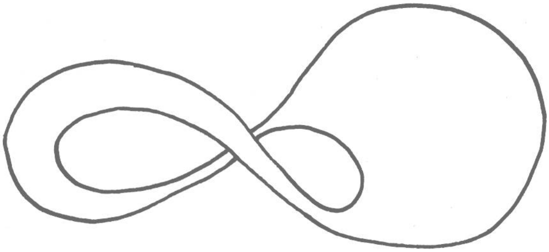
1. What is a moduli space?

To obtain an idea of what a moduli space is, we can go back to the original example of the space of moduli of elliptic curves (or tori). Suppose we ask ourselves the question: "What is the set of all conformal structures on a 2-dimensional torus?" Immediately our mental image is of a torus of revolution:



However, a moment's thought tells us that applying any diffeomorphism to the torus gives us another conformal structure:



This observation forces us to modify the original question to "What is the set of equivalence classes of conformal structures under the notion of equivalence by diffeomorphism?", or alternatively "What are the orbits of the group of diffeomorphisms acting on the space of conformal structures?"

The answer, as is well-known, is that the space of orbits is identical to the orbits of $SL(2;\mathbb{Z})$ acting on the upper half-plane. A point $\tau = x + iy$ in the upper half-plane corresponding to a conformal structure is the modulus of the elliptic curve.

The basic features we encounter here consist of an infinite-dimensional space of geometric structures (here the conformal struc-structures on a torus) and an infinite-dimensional group acting on the space (here the group of diffeomorphisms of the torus), but for which the quotient space is finite-dimensional. This quotient space has an interesting global topology (here generated by $SL(2;\mathbb{Z})$ and its subgroups) and also metric structure (the hyperbolic metric on the upper half-plane is invariant by $SL(2;\mathbb{Z})$). The degenerating behaviour of the conformal structure as we approach the boundary of the upper half-plane can also be analysed and used to produce results about the moduli space as a whole.

Somewhat remarkably, all these features are present in the moduli spaces with which these lectures are principally concerned. These are moduli spaces based on infinite-dimensional spaces of solutions to gauge-theoretic equations acted on by the infinite-dimensional group of gauge transformations. The relevant moduli spaces have on the one hand given a tool, in the hands of Donaldson, for probing the differential topology of four-manifolds and at the other extreme provided a model for the scattering behaviour of non-linear soliton-like objects. They also yield, as we shall see, a new way of approaching the old question of moduli of conformal structures on Riemann surfaces - the same moduli spaces which initiated the whole subject.

§2. The self-dual Yang-Mills equations

The first occasion in which moduli space ideas entered gauge-theory was in the study of "instantons" or self-dual solutions to the Yang-Mills equations on a compact four-manifold. A good reference for this material is the book by Freed and Uhlenbeck [FU].

We begin by taking M to be a compact, oriented, Riemannian manifold and P a principal G-bundle over M where G is a compact Lie group (for example $G = SU(2)$). We denote by $\Omega^p(M;\mathfrak{g})$ the space of exterior differential forms of degree p with values in the vector bundle $P \times_G \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G. Locally, then, $\alpha \in \Omega^p(M;\mathfrak{g})$ is a Lie-algebra valued p-form.

Let us recall the basic objects associated to gauge theories. A connection on P defines a differential operator, the exterior covariant derivative,

$$d_A : \Omega^p(M;\mathfrak{g}) \to \Omega^{p+1}(M;\mathfrak{g})$$

which satisfies

$$d_A(f\alpha) = df \wedge \alpha + f d_A \alpha$$

for any C^∞ function f. The difference of two connections $d_{A_1} - d_{A_2}$ is a zero-order operator defined by $\beta \in \Omega^1(M;\mathfrak{g})$ i.e.

$$d_{A_1}\alpha - d_{A_2}\alpha = [\beta,\alpha] \ .$$

Thus the space of all connections on P is an infinite dimensional affine space $\mathcal{A}$ whose group of translations is $\Omega^1(M;\mathfrak{g})$.

Associated to each connection is its curvature $F_A \in \Omega^2(M;\mathfrak{g})$ which is invariantly defined by

$$F_A = d_A^2 : \Omega^p(M;\mathfrak{g}) \to \Omega^{p+2}(M;\mathfrak{g})$$

Locally, the curvature is a 2-form with values in the Lie algebra.

A gauge transformation is a section of the bundle of groups $P \times_{Ad} G$. It defines an automorphism of the bundle $P \times_G \mathfrak{g}$ and acts on the space of connections by $d_A \to g^{-1} d_A g$. Since d_A is a first order differential operator this is an affine action. The group of all gauge transformations $\mathcal{G}$ therefore acts on the space $\mathcal{A}$ of connections on P.

The Yang-Mills functional is the function defined on the space $\mathcal{A}$ of connections on P by

$$\mathcal{YM}(A) = \int_M |F_A|^2 *1 \ .$$

Here $*1$ denotes the volume form of the metric and the norm $|F_A|$ is obtained by using the Killing form on the Lie algebra and the metric on M. The Yang-Mills functional is invariant under gauge transformations.

The Hodge star-operator is the linear map

$$* : \Omega^p(M) \to \Omega^{4-p}(M)$$

defined by