Groups, Graphs and Trees

An Introduction to the Geometry of Infinite Groups

JOHN MEIER

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For my driver, Piotr

Preface

Groups are algebraic objects, consisting of a set with a binary operation that satisfies a short list of required properties: the binary operation must be associative; there is an identity element; and every element has an inverse. Presenting groups in this formal, abstract algebraic manner is both useful and powerful. Yet it avoids a wonderful geometric perspective on group theory that is also useful and powerful, particularly in the study of infinite groups. This perspective is hinted at in the combinatorial approach to finite groups that is often seen in a first course in abstract algebra. It is my intention to bring the geometric perspective forward, to establish some elementary results that indicate the utility of this perspective, and to highlight some interesting examples of particular infinite groups along the way. My own bias is that these groups are just as interesting as the theorems.

The topics covered in this book fit inside of "geometric group theory," a field that sits in the impressively large intersection of abstract algebra, geometry, topology, formal language theory, and many other fields. I hope that this book will provide an introduction to geometric group theory at a broadly accessible level, requiring nothing more than a single-semester exposure to groups and a naive familiarity with the combinatorial theory of graphs.

The chapters alternate between those devoted to general techniques and theorems (odd numbers) and brief chapters introducing some of the standard examples of infinite groups (even numbers). Chapter 2 presents a few groups generated by reflections; Chapter 4 presents the Baumslag–Solitar group $\mathrm{BS}(1,2)$ in terms of linear functions; Chapter 6 is the Gupta–Sidki variant of Grigorchuk's group; the Lamplighter group is discussed in Chapter 8; and Thompson's group F is the subject of Chapter 10. When I taught this material at Lafayette College I referred to the material in these even numbered chapters as "field trips to the Zoo of Infinite Groups."

The first chapter should be relatively easy to work through, as it reviews material on groups (mainly finite groups), group actions, and the combinatorial theory of graphs. It establishes quite a bit of notation and introduces the construction of Cayley graphs. While some material in this chapter may be new to the reader, most of it should seem to be a repackaging of ideas that she or he has previously encountered.

Chapter 3 is an introduction to free groups and free products of groups. Chapters 5 and 7 are devoted to connections between finitely generated groups and formal language theory. Chapters 9 and 11 deal with the geometry of infinite groups, with Chapter 9 focusing on what might be called the "fine geometry" of Cayley graphs, while Chapter 11 treats what is called the large-scale geometry of groups.

While no background beyond elementary group theory is necessary for this book, a broader undergraduate exposure to mathematics is certainly helpful. My experience in the classroom indicates that the material in Chapter 7 is demanding for people who have not previously encountered formal language theory. Similarly Chapter 11 is easier for people who have had a course in real analysis. Because hyperbolic geometry is not a standard undergraduate topic, Gromov's theory of hyperbolic groups does not appear in this book. Similarly, because algebraic topology is not a standard undergraduate topic, I have avoided fundamental groups and covering spaces.

There are two forms of exercises in this book. A few exercises are embedded within the chapters. These should be done, at least at the level of the reader convincing themselves they know how to do them, while reading through the material. There are also end-of-chapter exercises that are arranged in the order that material is presented in the chapter. Some of these end-of-chapter exercises are challenging but most are reasonably accessible.

Groups, Graphs and Trees was developed from notes used in two undergraduate course offerings at Lafayette College, and it can certainly serve as a primary text for an advanced undergraduate course. It should also be useful as a text for a reading course and as a gentle introduction to geometric group theory for mathematicians with a broader background than this. An undergraduate course that attempted to cover this text, omitting no details, in one semester, would have to move at a rather brisk pace. The critical background information is contained in the first five chapters and those should not be trimmed. With a bit of forethought an instructor can cover much of the rest of this book, if for example the material in Chapter 7 or Chapter 11 is presented more as a colloquium than as course material. My own hope is that various classes will find the space in their semester to pursue tangents of interest to them, and then let me know the results of their exploration.

I have many people to thank. My wife Trisha and son Robert were unreasonably supportive of this project. Many students provided important feedback as I fumbled through the process of presenting this bit of advanced mathematics at an elementary level: George Armagh, Kari Barkley, Jenna Bratz, Jacob Carson, Joellen Cope, Joe Dudek, Josh Goldstein, Ekaterina Jager, Brian Kronenthal, Rob McEwen, and Zachary Reiter. I also benefited from extensive feedback given by my colleagues Ethan Berkove and Jon McCammond. Finally, a number of anonymous referees provided comments on various draft chapters. I was impressed by the fact that there was no intersection between the comments provided by students, the comments provided by colleagues, and the anonymous referees!

Contents

1

Cayley's Theorems

As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

–Arthur Cayley

An introduction to group theory often begins with a number of examples of finite groups (symmetric, alternating, dihedral, ...) and constructions for combining groups into larger groups (direct products, for example). Then one encounters Cayley's Theorem, claiming that every finite group can be viewed as a subgroup of a symmetric group. This chapter begins by recalling Cayley's Theorem, then establishes notation, terminology, and background material, and concludes with the construction and elementary exploration of Cayley graphs. This is the foundation we use throughout the rest of the text where we present a series of variations on Cayley's original insight that are particularly appropriate for the study of infinite groups.

Relative to the rest of the text, this chapter is gentle, and should contain material that is somewhat familiar to the reader. A reader who has not previously studied groups and encountered graphs will find the treatment presented here "brisk."

1.1 Cayley's Basic Theorem

You probably already have good intuition for what it means for a group to act on a set or geometric object. For example:

- The cyclic group of order n – denoted $\mathbb{Z}_n$ – acts by rotations on a regular n-sided polygon.

- The dihedral group of order $2n$ – denoted D_n – also acts on the regular n-sided polygon, where the elements either rotate or reflect the polygon.
- We use SYM_n to denote the symmetric group of all permutations of $[n] = \{1, 2, \ldots, n\}$. (More common notations are S_n and Σ_n.) By its definition, SYM_n acts on this set of numbers, as does its index 2 subgroup, the alternating group A_n, consisting of the even permutations.
- Matrix groups, such $\mathrm{GL}_n(\mathbb{R})$ (the group of invertible n-by-n matrices with real number entries), act on vector spaces.

Because the general theme of this book is to study groups via actions, we need a bit of notation and a formal definition.

Convention 1.1. If X is a mathematical object (such as a regular polygon or a set of numbers), then we use $\mathrm{SYM}(X)$ to denote all bijections from X to X that preserve the indicated mathematical structure. For example, if X is a set, then $\mathrm{SYM}(X)$ is simply the group of permutations of the elements of X. In fact, if $n = |X|$ then $\mathrm{SYM}(X) \approx \mathrm{SYM}_n$. Moreover, if X and X' have the same cardinality, then $\mathrm{SYM}(X) \approx \mathrm{SYM}(X')$. If X is a regular polygon, then angles and lengths are important, and $\mathrm{SYM}(X)$ will be composed of rotations and reflections (and it will in fact be a dihedral group). Similarly, if X is a vector space, then $\mathrm{SYM}(X)$ will consist of bijective linear transformations.

What we are referring to as "$\mathrm{SYM}(X)$" does have a number of different names in different contexts within mathematics. For example, if G is a group, then the collection of its symmetries is referred to as $\mathrm{AUT}(G)$, the group of automorphisms. If we are working with the Euclidean plane, $\mathbb{R}^2$, and are considering functions that preserve the distance between points, then we are looking at $\mathrm{ISOM}(\mathbb{R}^2)$, the group of isometries of the plane.

It is quite useful to have individual names for these groups, as their names highlight what mathematical structures are being preserved. Our convention of lumping these various groups all together under the name "SYM" is vague, but we believe that in context it will be clear what is intended, and we like the fact that this uniform terminology emphasizes that these various situations where groups arise are not all that different.[1] One egregious example, which highlights the need to be care-

[1] In his book, *Symmetry*, Hermann Weyl wrote: "[W]hat has indeed become a guiding principle in modern mathematics is this lesson: *Whenever you have to deal with a structure-endowed entity* Σ *try to determine its group of automorphisms*, the group of those element-wise transformations which leave all structural relations

ful in using our convention, comes from the integers. If the integers are thought of as simply a set, containing infinitely many elements, then $\text{SYM}(\mathbb{Z})$ is an infinite permutation group, which contains SYM_n for any n. On the other hand, if $\mathbb{Z}$ denotes the group of integers under addition, then $\text{SYM}(\mathbb{Z}) \approx \mathbb{Z}_2$. (The only non-trivial automorphism of the group of integers sends n to $-n$ for all $n \in \mathbb{Z}$.)

Definition 1.2. An *action* of a group G on a mathematical object X is a group homomorphism from G to $\text{SYM}(X)$. Equivalently, it is a map from $G \times X \to X$ such that

1. $e \cdot x = x$, for all $x \in X$; and
2. $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$ and $x \in X$.

We denote "G acts on X" by $G \curvearrowright X$.

If one has a group action $G \curvearrowright X$, then the associated homomorphism is a *representation* of G. The representation is *faithful* if the map is injective. In other words, it is faithful if, given any non-identity element $g \in G$, there is some $x \in X$ such that $g \cdot x \neq x$.

Example 1.3. The dihedral group D_n is the symmetry group of a regular n-gon. As such, it also permutes the vertices of the n-gon, hence there is a representation $D_n \to \text{SYM}_n$. As every non-identity element of D_n moves at least $(n-2)$ vertices, this representation is faithful.

Remark 1.4 (left vs. right)**.** In terms of avoiding confusion, this is perhaps the most important remark in this book. Because not all groups are abelian, it is very important to keep left and right straight. All of our actions will be *left* actions (as described above). We have chosen to work with left actions since it matches function notation and because left actions are standard in geometric group theory and topology.

Groups arise in a number of different contexts, most commonly as symmetries of any one of a number of possible mathematical objects X. In these situations, one can often understand the group directly from our understanding of X. The dihedral and symmetric groups are two examples of this. However, groups are abstract objects, being merely a set with a binary operation that satisfies a certain minimal list of requirements. Cayley's Theorem shows that the abstract notion of a group and the notion of a group of permutations are one and the same.

undisturbed. You can expect to gain a deep insight into the constitution of Σ in this way." Our use of $\text{SYM}(\Sigma)$ instead of $\text{AUT}(\Sigma)$ is a small notational deviation from Weyl's recommendation.

Theorem 1.5 (Cayley's Basic Theorem). *Every group can be faithfully represented as a group of permutations.*

Proof. The objects that G permutes are the elements of G. In this proof we use "SYM_G" to denote $\text{SYM}(G)$, to emphasize that "G" denotes the underlying *set* of elements, not the group. The permutation associated to $g \in G$ is defined by left multiplication by g. That is, $g \mapsto \pi_g \in \text{SYM}_G$ where $\pi_g(h) = g \cdot h$ for all $h \in G$. This is a permutation of the elements of G, since if $g \cdot h = g \cdot h'$, then by left cancellation, $h = h'$. Denote the map taking the element g to the permutation π_g by $\pi : G \to \text{SYM}_G$.

To check that π is a group homomorphism we need to verify that $\pi(gh) = \pi(g) \cdot \pi(h)$. In other words, we need to show that $\pi_{gh} = \pi_g \cdot \pi_h$. We do this by evaluating what each side does to an arbitrary element of G. We denote the arbitrary element by "x", thinking of it as a variable. The permutation π_{gh} takes $x \mapsto (gh) \cdot x$, and successively applying π_h then π_g sends $x \mapsto h \cdot x \mapsto g \cdot (h \cdot x)$. Thus checking that ϕ is a homomorphism amounts to verifying the associative law: $(gh) \cdot x = g \cdot (h \cdot x)$. As this is part of the definition of a group, the equation holds.

In order to see that the map is faithful it suffices to show that no non-identity element is mapped to the trivial permutation. One can do this by simply noting that if $g \in G \setminus \{e\}$, then $g \cdot e = g$, hence $\pi_g(e) = g$, and so π_g is not the identity (or trivial) permutation. □

The proof of Cayley's Basic Theorem constructs a representation of G as a group of permutations of itself. Before moving on we should examine what these permutations look like in some concrete situations. We first consider SYM_3, the group of all permutations of three objects.

Notation 1.6 (cycle notation). In describing elements of SYM_n we use cycle notation, and multiply (that is, compose permutations) right to left. This matches with our intuition from functions where $f \circ g(x)$ means that you first apply g then apply f, and it is consistent with our use of left actions. Here is a concrete example: $(12)(35) \in \text{SYM}_5$ is the element that transposes 1 and 2, as well as 3 and 5; the element (234) sends 2 to 3, 3 to 4 and 4 to 2; the product $(12)(35) \cdot (234) = (12534)$. (The product is not (13542), which is the result of multiplying left to right.)

Example 1.7. The group SYM_3 has six elements, shown as disjoint vertices in Figure 1.1. The permutations described by Cayley's Basic Theorem – for the elements (12) and (123) – are also shown.

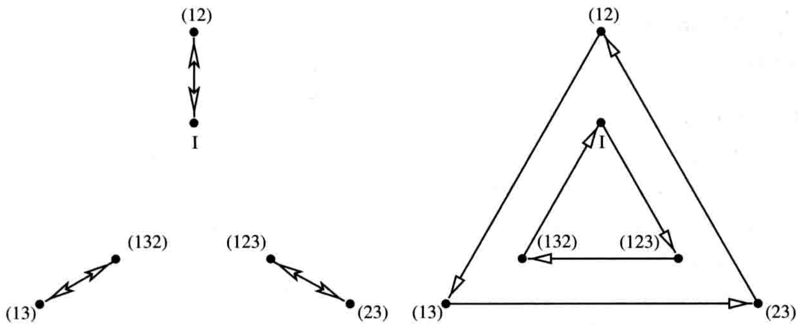

Fig. 1.1. The permutation of SYM_3 induced by (12) is shown on the left, and the permutation induced by (123) is shown on the right.

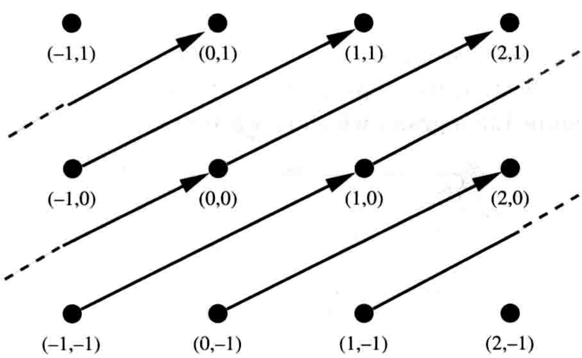

Fig. 1.2. The action of $(2,1)$ on $\mathbb{Z} \oplus \mathbb{Z}$.

Example 1.8. In most introductions to group theory, Cayley's Basic Theorem is stated for *finite* groups. But we made no such assumption in our statement and the same proof as is given for finite groups works for infinite groups. Consider for example the direct product of two copies of the group of integers, $G = \mathbb{Z} \oplus \mathbb{Z}$. Here elements are represented by pairs of integers, and the binary operation is coordinatewise addition: $(a, b) + (c, d) = (a + c, b + d)$. In Figure 1.2 we have arranged the vertices corresponding to elements of G as the integral lattice in the plane. The arrows indicate the permutation of the elements of $\mathbb{Z} \oplus \mathbb{Z}$ induced by the element $(2, 1)$.

1.2 Graphs

One of the key insights into the study of groups is that they can be viewed as symmetry groups of graphs. We refer to this as "Cayley's Better Theorem," which we prove in Section 1.5.2. In this section we establish some terminology from graph theory, and in the following section we discuss groups acting on graphs.

Definition 1.9. A *graph* Γ consists of a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of *edges*, each edge being associated to an unordered pair of vertices by a function "ENDS": $\text{ENDS}(e) = \{v, w\}$ where $v, w \in V$. In this case we call v and w the *ends* of the edge e and we also say v and w are *adjacent*.

We allow the possibility that there are multiple edges with the same associated pair of vertices. Thus for two distinct edges e and e' it can be the case that $\text{ENDS}(e) = \text{ENDS}(e')$. We also allow loops, that is, edges whose associated vertices are the same. Graphs without loops or multiple edges are *simple* graphs.

Graphs are often visualized by making the vertices points on paper and edges arcs connecting the appropriate vertices. Two simple graphs are shown in Figure 1.3; a graph which is not simple is shown in Figure 1.4.

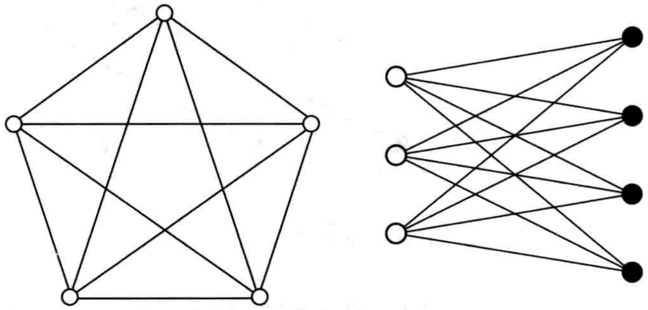

Fig. 1.3. The complete graph on five vertices, K_5, and the complete bipartite graph $K_{3,4}$.

There are a number of families of graphs that arise in mathematics. The *complete graph* on n vertices has exactly one edge joining each pair of distinct vertices, and is denoted K_n. At the opposite extreme are the *null graphs*, which have no edges.

A graph is *bipartite* if its vertices can be partitioned into two subsets – by convention these subsets are referred to as the "black" and "white" vertices – such that, for every $e \in E(\Gamma)$, $\textsc{Ends}(e)$ contains one black vertex and one white vertex. The *complete bipartite* graphs are simple graphs whose vertex sets have been partitioned into two collections, $V_\circ$ and $V_\bullet$, with edges joining each vertex in $V_\circ$ with each vertex in $V_\bullet$. If $|V_\circ| = n$ and $|V_\bullet| = m$ then the corresponding complete bipartite graph is denoted $K_{n,m}$.

The *valence* or *degree* of a vertex is the number of edges that contain it. For example, the valence of any vertex in K_n is $n-1$. If a vertex v is the vertex for a loop, that is an edge e where $\textsc{Ends}(e) = \{v, v\}$, then this loop contributes twice to the computation of the valence of v. For example, the valence of the leftmost vertex in the graph shown in Figure 1.4 is six.

A graph is *locally finite* if each vertex is contained in a finite number of edges, that is, if the valence of every vertex is finite.

An *edge path*, or more simply a *path*, in a graph consists of an alternating sequence of vertices and edges, $\{v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n\}$ where $\textsc{Ends}(e_i) = \{v_{i-1}, v_i\}$ (for each i). A graph is *connected* if any two vertices can be joined by an edge path. In Figure 1.4 we have indicated an

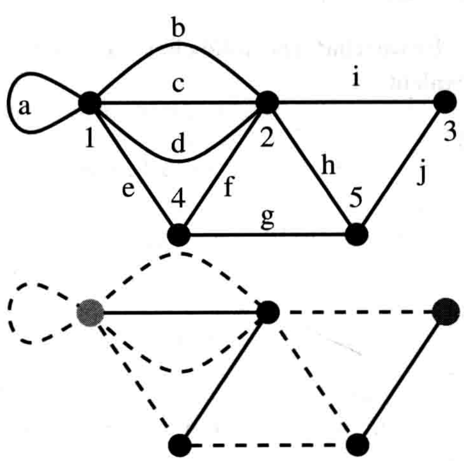

Fig. 1.4. On top is a graph which is not simple, with its vertices labelled by numbers and its edges labelled by letters. Below is the set of edges traversed in an edge path, joining the vertex labelled 1 to the vertex labelled 3, is indicated.

edge path from the leftmost vertex to the rightmost vertex. If v_i is the vertex labelled i and e_α is the edge labelled α, then this path is:

$$\{v_1, e_a, v_1, e_e, v_4, e_g, v_5, e_h, v_2, e_b, v_1, e_d, v_2, e_i, v_3\}$$