Probability Theory in Finance

A Mathematical Guide to the Black-Scholes Formula

Seán Dineen

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Dedicated to the memory of
LEOPOLDO NACHBIN (1922–1993)
Teacher and Friend

Preface

To doubt all or believe all are two equally convenient solutions, in that both dispense with thinking.

Henri Poincaré, 1854-1912

Mathematics occupies a unique place in modern society and education. It cannot be ignored and almost everyone has an opinion on its place and relevance. This has led to problems and questions that will never be solved or answered in a definitive fashion. At third level we have the perennial debate on the mathematics that is suitable for non-mathematics majors and the degree of abstraction with which it should be delivered. We mathematicians are still trusted with this task and our response has varied. Some institutions offer generic mathematics courses to all and sundry, and faculties, such as engineering and business, respond by directing their students to the courses they consider appropriate. In other institutions departments design specific courses for students who are not majoring in mathematics. The response of many departments lies somewhere in between. This can lead to tension between the professional mathematicians' attitude to mathematics and the client faculties' expectations. In the first case non-mathematics majors may find themselves obliged to accept without explanation an approach that is, in their experience, excessively abstract. In the second, a recipe-driven approach often produces students with skills they have difficulty using outside a limited number of well-defined settings. Some students, however, do arrive, by sheer endurance, at an intuitive feeling for mathematics. Clearly both extremes are unsatisfactory and it is natural to ask if an alternative approach is possible.

It is, and the difficulties to be overcome are not mathematical. The understanding of mathematics that we mathematicians have grown to appreciate and accept, often slowly and unconsciously, is not always shared by non-mathematicians, be they students or colleagues, and the benefits of abstract mathematics are not always obvious to academics from other disciplines. This is not their fault. They have, for the most part, been conditioned to think differently. They accept that mathematics is useful and for this reason are willing to submit their students to our courses. We can—and it is in our own hands, since we teach the courses—show that it is possible to *combine* abstract mathematics and good technical skills. It is not easy, it is labor intensive, and the benefits are usually not apparent in the short term. It requires patience and some unconditional support that we need to earn from our students and colleagues.

Although this book is appearing as a graduate text in mathematics, it is based on a one-semester undergraduate course given to economics and finance students at University College Dublin. It is the result of an opportunity given to the author to follow an alternative approach by mixing the abstract and the practical. We feel that all students benefited, but some were not convinced that this was indeed the case.

The students had the usual mathematical background, an acquaintance with the *techniques* of one variable differential and integral calculus and linear algebra. The aim of the course was to provide a mathematical foundation for further studies in financial mathematics, a discipline that has made enormous advances in the last twenty-five years and has been the surprise catalyst in the introduction of certain high-level mathematics courses for non-mathematics majors at universities in recent years. Even though the eventual applications are concrete the mathematics involved is quite abstract, and as a result business students, who specialize in finance, are today exposed to more demanding mathematics than their fellow students in engineering and science. The students' motivation, background, aspirations and future plans were the constraints under which we operated, and these determined the balance between the choice of topics, the degree of abstraction and pace of the presentation.

In view of its overall importance there was no difficulty in choosing the *Black-Scholes formula* for pricing a call option as our ultimate goal. This provided a focus for the students' motivation. As the students were *not* mathematics majors but the majority would have one or two further years of mathematically oriented courses, it seemed appropriate to aim for an understanding that would strengthen their overall mathematical background. This meant it was necessary to initiate the students into what has unfortunately become for many an alien and mysterious subject, *modern abstract mathematics.* For this approach to take root the security associated with recipe-driven and technique-oriented mathematics has to be replaced by a more mature and intrinsic confidence which accepts a degree of intellectual uncertainty as part of the thinking process. Even with highly motivated students, this requires a gradual introduction to mathematical abstraction, and at the same time it is necessary to remain, for reasons of motivation, in contact with the financial situation.

Probability theory, Lebesgue integration and the Itô calculus are the main ingredients in the *Black-Scholes formula*, and these rely on set theory, analysis and an axiomatic approach to mathematics. We take, on the financial side, a first principles approach and include only the minimum necessary to justify the introduction of mathematical concepts and place in context mathematical developments. We move slowly initially and provide elementary examples at an early stage. Hopefully, this makes the apparently more difficult mathematics in later chapters more intuitive and obvious. This cultural change explains why we felt it necessary on occasion to digress into non-technical, and even psychological, matters and why we attempted to present mathematics as a living culture with a history and a future. In particular, we tried to explain the importance of properly understanding questions and recognizing situations which required justification. This helped motivate, and place in perspective, the need for clear definitions and proofs. For example, in considering the concept of a convergent sequence of real numbers, on which all stochastic notions of convergence and all theories of integration rely, we begin by assuming an intuitive concept of limit in Chapter 1; in Chapter 3 we define the limit of a bounded increasing sequence of real numbers; in Chapter 4 we define the limit of a sequence of real numbers; in Chapter 6 we use upper and lower limits to characterize limits; in Chapter 9 we use Doob's upcrossing approach to limits; and in Chapter 11 we employ subsequences to obtain an equivalent definition of limit. In all cases the different ways of considering limits of sequences of real numbers are used as an introduction to similar but more advanced concepts in probability theory.

The introduction of peripheral material, the emphasis on simple examples, the repetition of basic principles, and attention to the students' motivation all take time. The real benefits only become apparent later, both to the students and their non-mathematical academic advisors, when they, the students, proceed to mix with other students in mathematically demanding courses.

The main mathematical topics covered in this book, for which we assume no background, are all essentially within probability theory. These are *measure theory, expected values, conditional expectation, martingales, stochastic processes, Wiener processes* and the *Itô integral.* We do not claim to give a fully comprehensive treatment, and we presented, even though otherwise tempted, certain results without proof. Readers who have worked their way through this book should be quite capable of following the standard proofs in the literature of The

Central Limit Theorem, The Radon-Nikodým Theorem, etc., and we hope they will be motivated to do so. Our self-imposed attempt at self-sufficiency sometimes led to awkward proofs. Although probability theory was the initial focus for our studies, we found as we progressed that more and more analysis was required. Having introduced sequences and continuous functions and proved a number of their basic properties, it did not require much effort to complete the process and present with complete proofs the fundamental properties of continuous and convex functions in Sections 7.2 and 7.6 respectively.

Different groups may benefit from reading this book. Students of financial mathematics at an early, but not too early, stage in their studies could follow, as our students did, Chapters 1-5; Sections 6.1, 6.2, 6.3 and 7.3; the statements of the main results in Sections 6.3, 6.4, and 7.5; and Chapters 8-10. Students of mathematics and statistics interested in analysis and probability theory could follow Chapters 3-7 with the option of two additional topics: the combination of Section 8.2, Chapter 9 and Section 10.3 forming one topic and Chapter 11 the other. Students of mathematics could follow Chapters 3-6 as an introduction to measure theory, while Chapter 11 is, modulo a modest background in probability theory, a self-contained introduction to stochastic integration and the Itô integral. Finally anyone beginning their university studies in mathematics or merely interested in modern mathematics, from a philosophical or aesthetic point of view, will find Chapters 1-5 accessible, challenging and rewarding.

The exercises played an important role in the course, on which we based this book. Some are easy, others difficult; many are included to clarify simple points; some introduce new ideas and techniques; a few contain deep results; and there is a high probability that some of our solutions are incorrect. However, *an hour or two attempting a problem is never a waste of time*, and *to make sure that this happened* these exercises were the focus of our small-group weekly workshops. This is a secret that we mathematicians all too often keep to ourselves. Mathematics is an active discipline, progress *cannot* be achieved by passive participation, and with sustained active participation progress *will* be achieved.

It is pleasure to see this book, written for undergraduate non-mathematics majors, appearing in a series devoted to graduate studies in mathematics. I greatly appreciate the support and encouragement that I received from the editorial staff of the *American Mathematical Society*. In particular, I would like to thank Sergei Gelfand, for being so positive and helpful, and Deborah Smith, for her suggestions and impressive proof-reading.

Dan Golden, from the Department of Banking and Finance at University College Dublin was the main inspiration behind this book. He set up the degree programme in economics and finance, and his constant advice, insight and encouragement were an enormous help over the last five years. It is a pleasure to thank Shane Whelan for numerous conversations on all matters connected with this book, Maciej Klimek for his interesting and constructive suggestions, Michael Mackey for his mathematical insight and excellent diagrams, Milne Anderson for encouragement and perspective over many years and Chris Boyd for his suggestions. Maria Meehan, Louis Murray, Cora Stack, Silvia Lassalle and David Horan helped more than they think. I would especially like to thank the students of economics and finance at University College Dublin, who were subjected to many pre-book versions of this material and who, by their questions and answers, left a lasting impression on the final shape of this book. Finally I would like to thank Carol, Deirdre, and Stephen for displaying great patience while the endless revisions of this book were taking place.

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Contents

Chapter 1

Money and Markets

There are very few things which we know, which are not capable of being reduc'd to a Mathematical Reasoning: and when they cannot, it's a sign our knowledge of them is very small and confus'd; and where a mathematical reasoning can be had, it's as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle standing by you.

John Arbuthnot, 1692,
Preface, Of the Laws of Chance

Summary

We give an extremely basic introduction to the financial markets and use some simple mathematics to examine interest rates.

1.1. Introduction

In this book we lay the mathematical foundations necessary to model certain transactions in the world of finance. Our goal is to provide a complete self-contained mathematical background to the Black-Scholes formula for pricing a call option. This involves two cultures, mathematics and finance, each having its own internal intuitions, concepts, rules and language. In finance, we confine ourselves to the minimal background necessary to achieve our purpose. This involves concepts such as interest rates, present worth or value, discounted value, hedging, risk, bonds, stocks, shares, options, expected return and arbitrage. In the first two chapters we explore these concepts and begin the process of interpreting them mathematically. To illustrate certain points we use examples, artificial from a finance perspective, but as we progress we make them more realistic.

We suppose the reader has some acquaintance with the techniques of one variable differential and integral calculus. All other mathematics required, for example, set theory, integration theory and probability theory, are developed *ab initio* as we proceed. History shows that intuition generally precedes rigor in mathematics, and, guided by this principle, we adopt an intuitive approach in the first two chapters. Afterwards we introduce the necessary rigorous mathematical definitions and provide proofs. The mathematical examples given are often elementary and are provided to improve our *understanding* of basic concepts. Complicated mathematical formulae and equations often turn out to be nothing more than clever combinations of simple well-known mathematical facts.

1.2. Money

In ancient times trade was conducted by exchanging goods, a system known as *bartering.* To simplify this process a fixed amount of a single commodity, often silver or gold, was chosen as a *unit of value* and goods were valued in units of this standard. We call this standard *money.*[1] Silver and gold are maintenance free and easily divided and thus suitable choices. Life would have been more complicated if the unit chosen had been a live chicken. Money's original role as a *medium of exchange* led to the separation of the acts of buying and selling, and it assumed a further role as a *store of value* as people realized its potential to be used *when* it suited them. Thus began the relationship between *money* and *time.*

When prices are stable, those with money feel financially secure. However, prices do change depending on *supply* and *demand.* The *rate of change over time* in the price of a commodity or a number of commodities is called *inflation.* If product A cost \$10 this time last year while today it costs \$12, then the percentage increase in price over the year is $\frac{12-10}{10} \times 100\% = 20\%$ and product A has a 20% annual rate of inflation. The inflation rate for a country is obtained by taking the weighted average of a basket of goods in the overall economy. If we call the *real,* in contrast to the *nominal,* value of money what it is capable of buying, then the presence of inflation means that the real value of money is a *function of time.*

[1] Similar to the way we have developed standard units of measurement for distance, temperature, land, etc.

Inflation is a problem for those with money. In its absence they can estimate their financial obligations and requirements. The presence of inflation reduces their financial security and forces them to confront an intrinsic problem: *how to maintain the future real value of money?* Money securely locked away is safe but may be losing value. On the other hand there are others who need money to buy houses, to set up businesses, etc. To cater to these needs, renting money became a business, and successful moneylenders prospered and became respectable bankers. Those with money and no immediate need of it rented it to the bank, and those who needed money rented it from the bank. The price of renting money is called *interest.*[2] Money deposited in a savings account grows at the prevailing rate of interest,[3] and as most deposits are insured and often guaranteed by governments, they are, for all practical purposes, a risk-free way of maintaining *some growth.* Any other way, such as investing in a business venture, involves *risk.* Interest rates and inflation rates are distinct processes, one increasing the nominal value of money, the other reducing its real value. However, it is often observed in economies that interest rates tend to be slightly higher than inflation rates. It seems savers generally demand a positive real interest rate and borrowers generally are willing to pay it. We can also identify two groups with different approaches to the management of money. *Hedgers* are those who wish to eliminate risk as much as possible, while *speculators* are willing to take risks in the expectation of higher profits.

1.3. Interest Rates

We now discuss interest rates and at the same time review some important results from one variable calculus. Interest rates are presented in various forms: simple interest, compound interest, continuously compounded interest, effective rate of interest, etc., with charges usually given as annual percentage rates, say $5\%, 10\%$. Since all involve the same basic concept they are comparable. We show how to compare them and having done so, settle on one and use it more or less exclusively afterwards. We let t denote the time variable, $t = 0$ will denote the present, while $t = 10$ will be 10 units of time, usually measured in years, into the future. Interest rates vary with time, but initially we assume they are constant.

[2] Nowadays we think of interest in this way, but essentially interest is the price of renting any object or service. Interest has been around for over *five thousand years* and for two thousand years before coins were introduced. Early Irish law, *The Brehon Law*, operated from around 200 BC to 1600 AD and relied heavily on the use of pledges to ensure that legal obligations were carried out. A *pledge* was an object of value delivered into the custody of another for a fixed period. A person who gave a pledge on behalf of another was entitled to *interest* while the pledged object was out of his possession. For example, if a lord supplied a goblet as a pledge, he was entitled to receive interest of 2 ounces of silver every three days until nine days were up and afterwards the rate of interest increased.

[3] The method of setting bank interest rates is complicated and involves central banks, governments, supply and demand, etc.

We begin with the simplest case, simple interest. Ten percent *simple interest* on a loan of \$1,000 for five years means that 10% of the amount borrowed, the *principal*, is charged for each year of the loan. Thus the interest charged is $\$\frac{10}{100} \times 5 \times 1,000 = \$500$. The general formula for calculating simple interest is straightforward: if an amount A is borrowed or saved for T years at a rate[4] r of simple interest, then the repayment due at time T is

$$A + ArT = A(1 + rT).$$

Simple interest is rarely used by banks, and it is easy to see why. If \$1,000 is deposited for 2 years at a rate of 10% simple interest, then the amount accumulated at the end of two years, the maturity date, would be \$1,200. If, however, at the end of year one the amount accumulated at that time, \$1,100, is withdrawn and immediately deposited for a further year at the same rate of simple interest, then, at maturity, the amount accumulated would be \$1,210, a gain of \$10 on the previous amount. If simple interest was the norm, people would be in and out of banks regularly withdrawing and immediately re-depositing their savings. For this reason a different method of calculating interest is normally used. This is called *compound interest*[5] and is based on applying simple interest over regular preassigned periods of the savings or loan to the amount accumulated at the beginning of each period. If a savings account offers 5% interest per annum compounded every six months, then the amount accumulated by \$2,000 deposited for two years is calculated as follows. The simple interest rule applied to the first six months' period shows that the amount will earn \$50 interest, and the amount deposited will have increased to \$2,050 at the end of six months. During the second six months, the \$2,050 will grow to $\$2,050(1 + \frac{5}{100} \times \frac{1}{2}) = \$2,101.25$, during the next period the amount will reach \$2,153.78, and in the final six months' period the amount will reach \$2,207.63.

Interest can, of course, be compounded at various other intervals of time, and the more frequent the compounding the greater the interest earned. Suppose an amount A is borrowed for T years at a rate r per annum compounded at n equally spaced intervals of time per year. Each interval of time $\frac{1}{n}$ has a simple interest rate of $\frac{r}{n}$. Thus after the first time interval the amount due has grown to $A(1 + \frac{r}{n})$, after two intervals it becomes $A(1 + \frac{r}{n})(1 + \frac{r}{n}) = A(1 + \frac{r}{n})^2$ and so on. Since there are a total of nT intervals of time, the total repayment at the end of T years will be $A(1 + \frac{r}{n})^{nT}$.

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[4] That is at a $100r$ percentage rate.

[5] The word *compound* comes from the Latin words *com* (together) and *ponere* (to put) and is used because compound interest is a putting together of simple interest. The words used in mathematics are taken from our everyday language and given precise mathematical meanings. They are usually chosen because one of their common usages approximates their meaning within mathematics. By simply consulting a dictionary, one can sometimes gain helpful mathematical insights.

We compare different interest rates by finding their *effective rate of interest.* This is the rate of simple interest which would give the same return over *one year.* One thousand dollars borrowed for one year at a rate of 10% per annum compounded every six months would result in a repayment of \$1, 102.50 at the end of the year. If the same amount is borrowed for one year at 10.25% simple interest, then the amount due would also be \$1, 102.50. Thus we say that the rate 10% per annum compounded every six months has a 10.25% effective rate of interest. It is clear that the more frequent the compounding, the higher the effective rate of interest.

Example 1.1. By comparing effective rates of interest we find which of the following gives the highest and lowest return:

(a) 6% compounded once a year

(b) 5.8% compounded quarterly

(c) 5.9% compounded quarterly

(d) 5.8% compounded monthly

(e) 5.6% compounded daily.

In practical cases such as this it is not advisable to rush in and blindly apply a mathematical formula but to pause and examine the situation from a common sense point of view. Since (a) is compounded only once a year, its effective rate of interest is 6%. Since (b) and (c) are compounded at the same time, but (b) has a lower rate of interest, it follows that (b) will have a lower effective rate of interest. Comparing (b) and (d) we see that they have the same rate of interest but (d) is compounded more frequently and thus will have a higher effective rate of interest.

Interest rates are independent of the amount borrowed or saved, so we compare them by considering \$1 borrowed for one year. For (b) the amount to be repaid is $\$1(1+\frac{.058}{4})^4 = \$1.0593$ and thus its effective rate of interest is 5.93%. For (c) we have $\$1(1+\frac{.059}{4})^4 = \$1.0603$, and its effective rate of interest is 6.03%. Similarly for (d), $\$1(1+\frac{.058}{12})^{12} = \$1.0596$, and its effective rate of interest is 5.96%; and for (e), $\$1(1+\frac{.056}{365})^{365} = \$1.0576$, and its effective rate of interest is 5.63%. Hence for the borrower (e) offers the cheapest rate, while (c) is the most expensive.

Example 1.2. A bank is offering 4% interest per annum compounded monthly to savers, and a customer wishes to save a fixed amount each month in order to accumulate a lump sum of \$10, 000 at the end of five years. We wish to determine how much should be saved each month.

As the customer is saving each month and also gaining interest, the amount deposited over the five years must be less than the lump sum \$10, 000. Since