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Mathematical Physics

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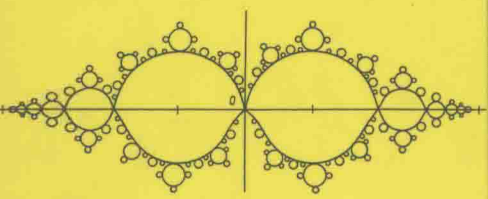
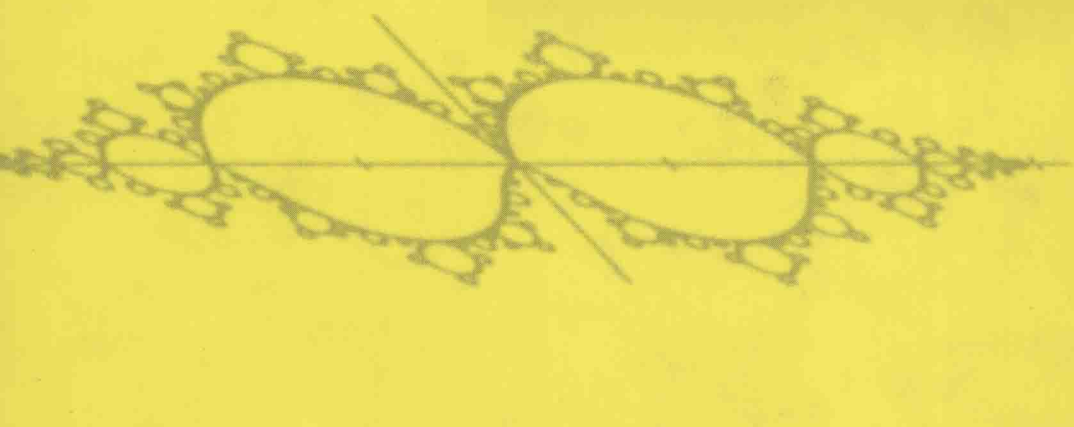


Dynamical Systems, Ergodic Theory and Applications

Second Edition



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Dynamical Systems, Ergodic Theory and Applications

Edited by
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Second, Expanded and Revised Edition
With 25 Figures

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Publisher's Note

While work on this new expanded edition was progressing, Springer-Verlag implemented a new concept for the Encyclopaedia of Mathematical Sciences. Part of this is the new subseries *Mathematical Physics*. A consensus between the editor of this volume, the editors of this new subseries, and Springer-Verlag was quickly established that this volume should become part of the *Mathematical Physics* subseries.

December 1999

Preface to the Second Edition

The first edition of this Encyclopaedia volume was published as Encyclopaedia of Mathematical Sciences Volume 2, "Dynamical Systems II". For this second edition, published as the first volume of the "Mathematical Physics" subseries, two new parts have been added, comprising the contributions by S.G. Dani and J. Smillie.

R.L. Dobrushin and N.B. Maslova, who played a very essential role in the first edition, passed away during the last few years. Their contributions have been left unchanged. The parts by L.A. Bunimovich, M.V. Jakobson and Ya.B. Pesin were essentially revised, updated and extended. In the other contributions of the previous volume some additional references have been added and some stylistic changes have been carried out.

The authors would like to thank J. Mattingly for his critical reading of the manuscript.

December 1999 Ya.G. Sinai

Preface

Each author who took part in the creation of this issue intended, according to the idea of the whole edition, to present his understanding and impressions of the corresponding part of ergodic theory or its applications. Therefore the reader has an opportunity to get both concrete information concerning this quickly developing branch of mathematics and an impression about the variety of styles and tastes of workers in this field.

Ya.G. Sinai

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§ 1. Dynamical Systems with Invariant Measures

Abstract ergodic theory deals with the measurable actions of groups and semigroups of transformations. This means, from the point of view of applications, that the functions defining such transformations need not satisfy any smoothness conditions and should be only measurable.

A pair $(M, \mathscr{M})$ where M is an abstract set and $\mathscr{M}$ is some σ-algebra of subsets of M, is called a measurable space. In the sequel M will be the phase space of a

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§ 1. Dynamical Systems with Invariant Measures

Abstract ergodic theory deals with the measurable actions of groups and semigroups of transformations. This means, from the point of view of applications, that the functions defining such transformations need not satisfy any smoothness conditions and should be only measurable.

A pair $(M, \mathcal{M})$ where M is an abstract set and $\mathcal{M}$ is some σ-algebra of subsets of M, is called a measurable space. In the sequel M will be the phase space of a dynamical system. The choice of $\mathcal{M}$ will always be clear from the context. We shall make use of the notions of the direct product of measurable spaces and of $\mathcal{M}$-measurable functions.

Definition 1.1. A transformation $T: M \to M$ is measurable if $T^{-1}C \in \mathcal{M}$ for any $C \in \mathcal{M}$.

A measurable transformation T is also called *an endomorphism of the measurable space* $(M, \mathcal{M})$. Any endomorphism generates a cyclic semigroup $\{T^n\}$ of endomorphisms ($n = 0, 1, 2, \ldots$).

If T is invertible and T^{-1} (as well as T) is measurable, then T is said to be *an automorphism of the measurable space* $(M, \mathcal{M})$. Any automorphism generates the cyclic group $\{T^n\}$ of automorphisms, $-\infty < n < \infty$.

A natural generalization of the above notions can be achieved by considering an arbitrary countable group or semigroup G and by fixing for each $g \in G$ a measurable transformation T_g such that $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$ for all $g_1, g_2 \in G$, $T_e = id$.

Definition 1.2. The family $\{T_g\}$, $g \in G$, is said to be a *measurable action* of the countable group (semigroup) G.

The simplest example is as follows. Suppose that $(X, \mathcal{X})$ is a measurable space and M is the space of all X-valued functions on G, i.e. any $x \in M$ is a sequence $\{x_g\}$, $x_g \in X$, $g \in G$. For any $g_0 \in G$ define the transformation $T_{g_0} : M \to M$ by the formula $T_{g_0}x = x'$, where $x'_g = x_{g_0 g}$. In this case $\{T_g\}$ is called a group (semigroup) of shifts. In particular,

1) if G is the semigroup $\mathbb{Z}^1_+ = \{n: n \geqslant 0, n \text{ is an integer}\}$, then M is the space of all 1-sided X-valued sequences, i.e. the points $x \in M$ are of the form $x = \{x_n\}$, $x_n \in X$, $n \geqslant 0$, and $T_m x = \{x_{n+m}\}$, $m \in \mathbb{Z}^1_+$. T_1 is called a 1-sided shift.

2) if G is the group $\mathbb{Z}^1 = \{n: -\infty < n < \infty, n \text{ is an integer}\}$, then M is the space of all 2-sided sequences $x = \{x_n\}$, $x_n \in X$, $-\infty < n < \infty$, and $T_m x = \{x_{n+m}\}$, $m \in \mathbb{Z}^1$. T_1 is called a 2-sided shift, or, simply, a shift.

3) if $G = \mathbb{Z}^d = \{(n_1, n_2, \ldots, n_d): n_i \in \mathbb{Z}^1, 1 \leqslant i \leqslant d\}$, $d \geqslant 1$, then M is the space of all sequences x of the form $x = \{x_n\} = \{x_{n_1, \ldots, n_d}\}$, while $T^m x = \{x_{n+m}\}$, $m = \{m_1, \ldots, m_d\} \in \mathbb{Z}^d$.

The above examples arise naturally in probability theory, where the role of M is played by the space of all realizations of d-dimensional random field.

Now suppose G is an arbitrary group or semigroup endowed with the structure of measurable space $(G, \mathcal{G})$ compatible with its group structure, i.e. all transformations $T_{g_0}: g \mapsto g_0 g$ $(g, g_0 \in G)$ are measurable.

Definition 1.3. The family $\{T_g : M \to M\}$, $g \in G$, where G is a measurable group, is called a *measurable action of the group* G (or a G-*flow*) if

1) $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$ for all $g_1, g_2 \in G$;

2) for any $\mathcal{M}$-measurable function $f: M \to \mathbb{R}^1$ the function $f(T_g x)$ considered as a function on the direct product $(M, \mathcal{M}) \times (G, \mathcal{G})$ is also measurable.

Our main example is $G = \mathbb{R}^1$ with the Borel σ-algebra of subsets of $\mathbb{R}^1$ as $\mathcal{G}$. There also exist natural examples with $G = \mathbb{R}^d$, $d > 1$ (cf Chap. 12).

Let now $G = \mathbb{R}^1$. If T^t is the transformation in $\mathbb{R}^1$-flow corresponding to a $t \in \mathbb{R}^1$, then we have $T^{t_1} \cdot T^{t_2} = T^{t_1+t_2}$. We will describe a natural situation in which the actions of $\mathbb{R}^1$ arise.

Suppose M is a smooth compact manifold and α is a smooth vector field on M. Consider the transformation T^t sending each point $x \in M$ to the point $T^t x$ which can be obtained from x by moving x along the trajectory of α for the period of time t (T^t is well defined because of compactness of M). Then $T^{t_1+t_2} = T^{t_1} \cdot T^{t_2}$ and T^t is a measurable action of $\mathbb{R}^1$.

Measurable actions of $\mathbb{R}^1$ are usually called *flows*, and those of $\mathbb{R}^1_+$ —*semiflows*. The cyclic groups and semigroups of measurable transformations are also known as *dynamical systems with discrete time*, while flows and semiflows are known as *dynamical systems with continuous time*.

Now, let $(M, \mathcal{M}, \mu)$ be a measure space (probability space), i.e. $(M, \mathcal{M})$ is a measurable space and μ is a nonnegative normalized ($\mu(M) = 1$) measure on $\mathcal{M}$. Consider a measure ν on $\mathcal{M}$ given by $\nu(C) = \mu(T^{-1}C)$, $C \in \mathcal{M}$. This measure is said to be the image of the measure μ under T (notation: $\nu = T\mu$).

Definition 1.4. A measure μ is *invariant* under a measurable transformation $T: M \to M$ if $T\mu = \mu$.

If μ is invariant under T, then T is called an *endomorphism of the measure space* $(M, \mathcal{M}, \mu)$. If, in addition, T is invertible, it is called an *automorphism* of $(M, \mathcal{M}, \mu)$. If $\{T^t\}$ is a measurable action of $\mathbb{R}^1$ and each T^t, $-\infty < t < \infty$, preserves the measure μ, then $\{T^t\}$ is called a *flow* on the measure space $(M, \mathcal{M}, \mu)$.

Now consider the general case.

Definition 1.5. Let $\{T_g\}$ be a measurable action of a measurable group $(G, \mathcal{G})$ on the space $(M, \mathcal{M})$. A measure μ on $\mathcal{M}$ is called *invariant* under this action if, for any $g \in G$, μ is invariant under T_g.

We now introduce the general notion of metric isomorphism of dynamical systems which allows us to identify systems having similar metric properties.

Definition 1.6. Suppose $(G, \mathcal{G})$ is a measurable group and $\{T_g^{(1)}\}$, $\{T_g^{(2)}\}$ are two G-flows acting on $(M_1, \mathcal{M}_1)$, $(M_2, \mathcal{M}_2)$ respectively and having invariant measures μ_1, μ_2. Such flows are said to be *metrically isomorphic* if there exist G-invariant subsets $M_1' \subset M_1$, $M_2' \subset M_2$, $\mu_1(M_1') = \mu_2(M_2') = 1$, as well as an isomorphism $\varphi: (M_1', \mathcal{M}_1, \mu_1) \to (M_2', \mathcal{M}_2, \mu_2)$ of measure spaces M_1', M_2' such that $T_g^{(2)} \varphi x^{(1)} = \varphi T_g^{(1)} x^{(1)}$ for all $g \in G$, $x^{(1)} \in M_1'$.

Ergodic theory also studies measurable actions of groups on the space $(M, \mathcal{M}, \mu)$ which are not necessarily measure-preserving.

Definition 1.7. Suppose $\{T_g\}$ is a measurable action of a measurable group $(G, \mathcal{G})$ on $(M, \mathcal{M})$. The measure μ on $\mathcal{M}$ is said to be *quasi-invariant* under this action if for any $g \in G$ the measure $\mu_g \stackrel{\text{def}}{=} T_g \mu$, i.e. the image of μ under T_g, is equivalent to μ. In other words, μ and $T_g \mu$ have the same sets of zero measure.