

ENCYCLOPAEDIA OF MATHEMATICAL SCIENCES

Volume 100

Mathematical  
Physics

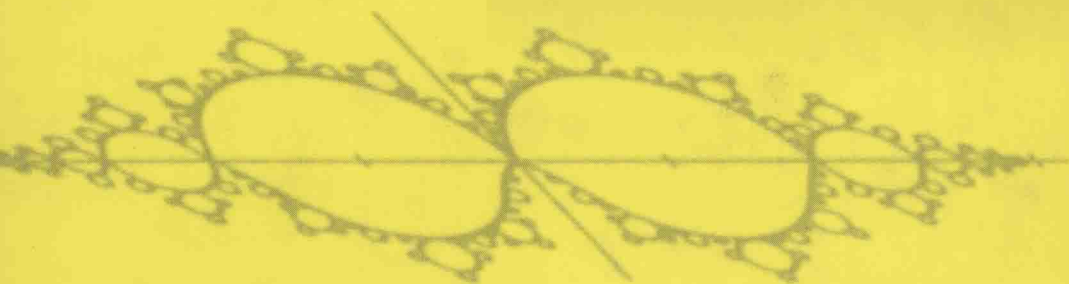
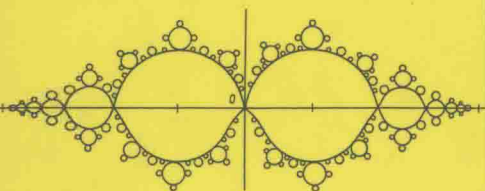
I

J. FRÖHLICH  
S. P. NOVIKOV  
D. RUELLE  
Subseries Editors

L. A. BUNIMOVICH  
S. G. DANI  
R. L. DOBRUSHIN  
M. V. JAKOBSON  
I. P. KORNFELD  
N. B. MASLOVA  
YA. B. PESIN  
YA. G. SINAI  
J. SMILLIE  
YU. M. SUKHOV  
A. M. VERSHIK

# Dynamical Systems, Ergodic Theory and Applications

Second Edition



Springer

L. A. Bunimovich S. G. Dani R. L. Dobrushin  
M. V. Jakobson I. P. Kornfeld N. B. Maslova  
Ya. B. Pesin Ya. G. Sinai  
J. Smillie Yu. M. Sukhov A. M. Vershik

# Dynamical Systems, Ergodic Theory and Applications

Edited by  
Ya. G. Sinai

Second, Expanded and Revised Edition  
With 25 Figures



Springer

*Subseries Editors*

Prof. Dr. J. Fröhlich  
Theoretische Physik  
Dept. Physik (D-PHYS)  
HPZ G 17  
ETH Hönggerberg  
8093 Zürich, Switzerland  
e-mail: juerg.froehlich@itp.phys.ethz.ch

Prof. S. P. Novikov  
Department of Mathematics  
University of Maryland at College Park-IPST  
College Park, MD 20742-2431, USA  
e-mail: novikov@ipst.umd.edu

Prof. D. Ruelle  
IHES, Le Bois-Marie  
35, Route de Chartres  
91440 Bures-Sur-Yvette, France  
e-mail: ruelle@ihes.fr

Founding editor of the Encyclopaedia of Mathematical Sciences:  
R. V. Gamkrelidze

The first edition of this book was published as  
Dynamical Systems II, Volume 2 of the Encyclopaedia of Mathematical Sciences

ISSN 0938-0396

ISBN 3-540-66316-9 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

Springer-Verlag is a company in the BertelsmannSpringer publishing Group.

© Springer-Verlag Berlin Heidelberg 2000

Printed in Germany

Typesetting: Camera ready by authors

Production: PRO EDIT GmbH, 69126 Heidelberg, Germany

Cover Design: E. Kirchner, Heidelberg, Germany

Printed on acid-free paper SPIN: 10654592 46/3143 hs 5 4 3 2 1 0

Encyclopaedia of Mathematical Sciences  
Volume 100

---

*Mathematical Physics I*

Subseries Editors:

J. Fröhlich S. P. Novikov D. Ruelle

**Springer**

*Berlin  
Heidelberg  
New York  
Barcelona  
Hong Kong  
London  
Milan  
Paris  
Singapore  
Tokyo*

## Publisher's Note

While work on this new expanded edition was progressing, Springer-Verlag implemented a new concept for the Encyclopaedia of Mathematical Sciences. Part of this is the new subseries *Mathematical Physics*. A consensus between the editor of this volume, the editors of this new subseries, and Springer-Verlag was quickly established that this volume should become part of the *Mathematical Physics* subseries.

December 1999

## Preface to the Second Edition

The first edition of this Encyclopaedia volume was published as Encyclopaedia of Mathematical Sciences Volume 2, "Dynamical Systems II". For this second edition, published as the first volume of the "Mathematical Physics" subseries, two new parts have been added, comprising the contributions by S.G. Dani and J. Smillie.

R.L. Dobrushin and N.B. Maslova, who played a very essential role in the first edition, passed away during the last few years. Their contributions have been left unchanged. The parts by L.A. Bunimovich, M.V. Jakobson and Ya.B. Pesin were essentially revised, updated and extended. In the other contributions of the previous volume some additional references have been added and some stylistic changes have been carried out.

The authors would like to thank J. Mattingly for his critical reading of the manuscript.

December 1999

Ya.G. Sinai

## Preface

Each author who took part in the creation of this issue intended, according to the idea of the whole edition, to present his understanding and impressions of the corresponding part of ergodic theory or its applications. Therefore the reader has an opportunity to get both concrete information concerning this quickly developing branch of mathematics and an impression about the variety of styles and tastes of workers in this field.

Ya.G. Sinai



# List of Editor, Authors and Translators

## *Consulting Editor*

Ya.G. Sinai, Princeton University, Dept. of Mathematics, 708 Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA;  
e-mail: [sinai@math.princeton.edu](mailto:sinai@math.princeton.edu);  
Russian Academy of Sciences, L.D. Landau Institute for Theoretical Physics, ul. Kosygina 2, 117940 Moscow V-334, GSP-1, Russia

## *Authors*

L.A. Bunimovich, Georgia Institute of Technology, School of Mathematics, Atlanta, GA 30332-016, USA; e-mail: [bunimov@math.gatech.edu](mailto:bunimov@math.gatech.edu)  
S.G. Dani, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Bombay 400005, India;  
e-mail: [dani@math.tifr.res.in](mailto:dani@math.tifr.res.in)

R.L. Dobrushin †

M.V. Jakobson, Department of Mathematics, University of Maryland, College Park, MD 20742, USA; e-mail: [mvy@math.umd.edu](mailto:mvy@math.umd.edu)

I.P. Kornfeld, North Dakota State University, College of Science and Mathematics, Minard Hall 302C, Fargo, ND 58105, USA;  
e-mail: [kornfeld@plains.nodak.edu](mailto:kornfeld@plains.nodak.edu)

N.B. Maslova †

Ya.B. Pesin, Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; e-mail: [pesin@math.psu.edu](mailto:pesin@math.psu.edu)

Ya.G. Sinai, Princeton University, Dept. of Mathematics, 708 Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA;  
e-mail: [sinai@math.princeton.edu](mailto:sinai@math.princeton.edu);

Russian Academy of Sciences, L.D. Landau Institute for Theoretical Physics, ul. Kosygina 2, 117940 Moscow V-334, GSP-1, Russia

J. Smillie, Department of Mathematics, Cornell University, Malott Hall, Ithaca, NY 14853, USA; e-mail: [smillie@math.cornell.edu](mailto:smillie@math.cornell.edu)

Y.M. Sukhov, Statistical Laboratory, DPMMS, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, UK;  
e-mail: [y.m.suhov@statslab.cam.ac.uk](mailto:y.m.suhov@statslab.cam.ac.uk)

A.M. Vershik, Institute of Mathematics of the Russian Academy of Sciences, Fontanka 27, 191011 St. Petersburg, Russia; e-mail: [vershik@pdmi.ras.ru](mailto:vershik@pdmi.ras.ru)

*Translators*

- L.A. Bunimovich, Georgia Institute of Technology, School of Mathematics,  
Atlanta, GA 30332-016, USA; e-mail: bunimov@math.gatech.edu
- M.V. Jakobson, Department of Mathematics, University of Maryland, College  
Park, MD 20742, USA; e-mail: mvy@math.umd.edu
- I.P. Kornfeld, North Dakota State University, College of Science and Math-  
ematics, Minard Hall 302C, Fargo, ND 58105, USA;  
e-mail: kornfeld@plains.nodak.edu
- Y.M. Sukhov, Statistical Laboratory, DPMMS, University of Cambridge, 16  
Mill Lane, Cambridge CB2 1SB, UK;  
e-mail: y.m.suhov@statslab.cam.ac.uk

# Contents

## **I. General Ergodic Theory of Groups of Measure Preserving Transformations**

1

## **II. Ergodic Theory of Smooth Dynamical Systems**

103

## **III. Dynamical Systems on Homogeneous Spaces**

264

## **IV. The Dynamics of Billiard Flows in Rational Polygons**

360

## **V. Dynamical Systems of Statistical Mechanics and Kinetic Equations**

383

## **Subject Index**

455

# I. General Ergodic Theory of Groups of Measure Preserving Transformations

## Contents

Chapter 1. Basic Notions of Ergodic Theory and Examples of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	2
§ 1. Dynamical Systems with Invariant Measures	2
§ 2. First Corollaries of the Existence of Invariant Measures. Ergodic Theorems	11
§ 3. Ergodicity. Decomposition into Ergodic Components. Various Mixing Conditions	18
§ 4. General Constructions	23
4.1. Direct Products of Dynamical Systems	23
4.2. Skew Products of Dynamical Systems	24
4.3. Factor-Systems	25
4.4. Integral and Induced Automorphisms	25
4.5. Special Flows and Special Representations of Flows	26
4.6. Natural Extensions of Endomorphisms	28
Chapter 2. Spectral Theory of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	30
§ 1. Groups of Unitary Operators and Semigroups of Isometric Operators Adjoint to Dynamical Systems	30
§ 2. The Structure of the Dynamical Systems with Pure Point and Quasidiscrete Spectra	33
§ 3. Examples of Spectral Analysis of Dynamical Systems	35
§ 4. Spectral Analysis of Gauss Dynamical Systems	36
Chapter 3. Entropy Theory of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	38
§ 1. Entropy and Conditional Entropy of a Partition	39
§ 2. Entropy of a Dynamical System	40
§ 3. The Structure of Dynamical Systems of Positive Entropy	43
§ 4. The Isomorphy Problem for Bernoulli Automorphisms and $K$ -Systems	45

§ 5. Equivalence of Dynamical Systems in the Sense of Kakutani . . .	53
§ 6. Shifts in the Spaces of Sequences and Gibbs Measures . . . . .	57
Chapter 4. Periodic Approximations and Their Applications.	
Ergodic Theorems, Spectral and Entropy Theory for the General	
Group Actions ( <i>I.P. Kornfeld, A.M. Vershik</i> ) . . . . .	61
§ 1. Approximation Theory of Dynamical Systems by Periodic Ones.	
Flows on the Two-Dimensional Torus . . . . .	61
§ 2. Flows on the Surfaces of Genus $p \geq 1$ and Interval Exchange	
Transformations . . . . .	66
§ 3. General Group Actions . . . . .	69
3.1. Introduction . . . . .	69
3.2. General Definition of the Actions of Locally Compact	
Groups on Lebesgue Spaces . . . . .	70
3.3. Ergodic Theorems . . . . .	71
3.4. Spectral Theory . . . . .	74
§ 4. Entropy Theory for the Actions of General Groups . . . . .	76
Chapter 5. Trajectory Theory ( <i>A.M. Vershik</i> ) . . . . .	80
§ 1. Statements of Main Results . . . . .	80
§ 2. Sketch of the Proof. Tame Partitions . . . . .	84
§ 3. Trajectory Theory for Amenable Groups . . . . .	89
§ 4. Trajectory Theory for Non-Amenable Groups. Rigidity . . . . .	91
§ 5. Concluding Remarks. Relationship Between Trajectory Theory	
and Operator Algebras . . . . .	94
Bibliography . . . . .	95
Additional Bibliography . . . . .	101

## Chapter 1

### Basic Notions of Ergodic Theory and Examples of Dynamical Systems

I.P. Kornfeld, Ya.G. Sinai

#### § 1. Dynamical Systems with Invariant Measures

Abstract ergodic theory deals with the measurable actions of groups and semigroups of transformations. This means, from the point of view of applications, that the functions defining such transformations need not satisfy any smoothness conditions and should be only measurable.

A pair  $(M, \mathcal{M})$  where  $M$  is an abstract set and  $\mathcal{M}$  is some  $\sigma$ -algebra of subsets of  $M$ , is called a measurable space. In the sequel  $M$  will be the phase space of a

# I. General Ergodic Theory of Groups of Measure Preserving Transformations

## Contents

Chapter 1. Basic Notions of Ergodic Theory and Examples of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	2
§ 1. Dynamical Systems with Invariant Measures	2
§ 2. First Corollaries of the Existence of Invariant Measures. Ergodic Theorems	11
§ 3. Ergodicity. Decomposition into Ergodic Components. Various Mixing Conditions	18
§ 4. General Constructions	23
4.1. Direct Products of Dynamical Systems	23
4.2. Skew Products of Dynamical Systems	24
4.3. Factor-Systems	25
4.4. Integral and Induced Automorphisms	25
4.5. Special Flows and Special Representations of Flows	26
4.6. Natural Extensions of Endomorphisms	28
Chapter 2. Spectral Theory of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	30
§ 1. Groups of Unitary Operators and Semigroups of Isometric Operators Adjoint to Dynamical Systems	30
§ 2. The Structure of the Dynamical Systems with Pure Point and Quasidiscrete Spectra	33
§ 3. Examples of Spectral Analysis of Dynamical Systems	35
§ 4. Spectral Analysis of Gauss Dynamical Systems	36
Chapter 3. Entropy Theory of Dynamical Systems ( <i>I.P. Kornfeld, Ya.G. Sinai</i> )	38
§ 1. Entropy and Conditional Entropy of a Partition	39
§ 2. Entropy of a Dynamical System	40
§ 3. The Structure of Dynamical Systems of Positive Entropy	43
§ 4. The Isomorphy Problem for Bernoulli Automorphisms and $K$ -Systems	45

§ 5. Equivalence of Dynamical Systems in the Sense of Kakutani . . .	53
§ 6. Shifts in the Spaces of Sequences and Gibbs Measures . . . . .	57
Chapter 4. Periodic Approximations and Their Applications. Ergodic Theorems, Spectral and Entropy Theory for the General Group Actions ( <i>I.P. Kornfeld, A.M. Vershik</i> ) . . . . .	61
§ 1. Approximation Theory of Dynamical Systems by Periodic Ones. Flows on the Two-Dimensional Torus . . . . .	61
§ 2. Flows on the Surfaces of Genus $p \geq 1$ and Interval Exchange Transformations . . . . .	66
§ 3. General Group Actions . . . . .	69
3.1. Introduction . . . . .	69
3.2. General Definition of the Actions of Locally Compact Groups on Lebesgue Spaces . . . . .	70
3.3. Ergodic Theorems . . . . .	71
3.4. Spectral Theory . . . . .	74
§ 4. Entropy Theory for the Actions of General Groups . . . . .	76
Chapter 5. Trajectory Theory ( <i>A.M. Vershik</i> ) . . . . .	80
§ 1. Statements of Main Results . . . . .	80
§ 2. Sketch of the Proof. Tame Partitions . . . . .	84
§ 3. Trajectory Theory for Amenable Groups . . . . .	89
§ 4. Trajectory Theory for Non-Amenable Groups. Rigidity . . . . .	91
§ 5. Concluding Remarks. Relationship Between Trajectory Theory and Operator Algebras . . . . .	94
Bibliography . . . . .	95
Additional Bibliography . . . . .	101

## Chapter 1

### Basic Notions of Ergodic Theory and Examples of Dynamical Systems

I.P. Kornfeld, Ya.G. Sinai

#### § 1. Dynamical Systems with Invariant Measures

Abstract ergodic theory deals with the measurable actions of groups and semigroups of transformations. This means, from the point of view of applications, that the functions defining such transformations need not satisfy any smoothness conditions and should be only measurable.

A pair  $(M, \mathcal{M})$  where  $M$  is an abstract set and  $\mathcal{M}$  is some  $\sigma$ -algebra of subsets of  $M$ , is called a measurable space. In the sequel  $M$  will be the phase space of a

dynamical system. The choice of  $\mathcal{M}$  will always be clear from the context. We shall make use of the notions of the direct product of measurable spaces and of  $\mathcal{M}$ -measurable functions.

**Definition 1.1.** A transformation  $T: M \rightarrow M$  is measurable if  $T^{-1}C \in \mathcal{M}$  for any  $C \in \mathcal{M}$ .

A measurable transformation  $T$  is also called an *endomorphism of the measurable space*  $(M, \mathcal{M})$ . Any endomorphism generates a cyclic semigroup  $\{T^n\}$  of endomorphisms ( $n = 0, 1, 2, \dots$ ).

If  $T$  is invertible and  $T^{-1}$  (as well as  $T$ ) is measurable, then  $T$  is said to be an *automorphism of the measurable space*  $(M, \mathcal{M})$ . Any automorphism generates the cyclic group  $\{T^n\}$  of automorphisms,  $-\infty < n < \infty$ .

A natural generalization of the above notions can be achieved by considering an arbitrary countable group or semigroup  $G$  and by fixing for each  $g \in G$  a measurable transformation  $T_g$  such that  $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$  for all  $g_1, g_2 \in G$ ,  $T_e = id$ .

**Definition 1.2.** The family  $\{T_g\}$ ,  $g \in G$ , is said to be a *measurable action* of the countable group (semigroup)  $G$ .

The simplest example is as follows. Suppose that  $(X, \mathcal{X})$  is a measurable space and  $M$  is the space of all  $X$ -valued functions on  $G$ , i.e. any  $x \in M$  is a sequence  $\{x_g\}$ ,  $x_g \in X$ ,  $g \in G$ . For any  $g_0 \in G$  define the transformation  $T_{g_0}: M \rightarrow M$  by the formula  $T_{g_0}x = x'$ , where  $x'_g = x_{g_0 g}$ . In this case  $\{T_g\}$  is called a group (semigroup) of shifts. In particular,

1) if  $G$  is the semigroup  $\mathbb{Z}_+^1 = \{n: n \geq 0, n \text{ is an integer}\}$ , then  $M$  is the space of all 1-sided  $X$ -valued sequences, i.e. the points  $x \in M$  are of the form  $x = \{x_n\}$ ,  $x_n \in X$ ,  $n \geq 0$ , and  $T_m x = \{x_{n+m}\}$ ,  $m \in \mathbb{Z}_+^1$ .  $T_1$  is called a 1-sided shift.

2) if  $G$  is the group  $\mathbb{Z}^1 = \{n: -\infty < n < \infty, n \text{ is an integer}\}$ , then  $M$  is the space of all 2-sided sequences  $x = \{x_n\}$ ,  $x_n \in X$ ,  $-\infty < n < \infty$ , and  $T_m x = \{x_{n+m}\}$ ,  $m \in \mathbb{Z}^1$ .  $T_1$  is called a 2-sided shift, or, simply, a shift.

3) if  $G = \mathbb{Z}^d = \{(n_1, n_2, \dots, n_d): n_i \in \mathbb{Z}^1, 1 \leq i \leq d\}$ ,  $d \geq 1$ , then  $M$  is the space of all sequences  $x$  of the form  $x = \{x_n\} = \{x_{n_1, \dots, n_d}\}$ , while  $T^m x = \{x_{n+m}\}$ ,  $m = \{m_1, \dots, m_d\} \in \mathbb{Z}^d$ .

The above examples arise naturally in probability theory, where the role of  $M$  is played by the space of all realizations of  $d$ -dimensional random field.

Now suppose  $G$  is an arbitrary group or semigroup endowed with the structure of measurable space  $(G, \mathcal{G})$  compatible with its group structure, i.e. all transformations  $T_{g_0}: g \mapsto g_0 g$  ( $g, g_0 \in G$ ) are measurable.

**Definition 1.3.** The family  $\{T_g: M \rightarrow M\}$ ,  $g \in G$ , where  $G$  is a measurable group, is called a *measurable action of the group*  $G$  (or a  $G$ -flow) if

- 1)  $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$  for all  $g_1, g_2 \in G$ ;
- 2) for any  $\mathcal{M}$ -measurable function  $f: M \rightarrow \mathbb{R}^1$  the function  $f(T_g x)$  considered as a function on the direct product  $(M, \mathcal{M}) \times (G, \mathcal{G})$  is also measurable.

Our main example is  $G = \mathbb{R}^1$  with the Borel  $\sigma$ -algebra of subsets of  $\mathbb{R}^1$  as  $\mathcal{G}$ . There also exist natural examples with  $G = \mathbb{R}^d$ ,  $d > 1$  (cf Chap. 12).



Let now  $G = \mathbb{R}^1$ . If  $T^t$  is the transformation in  $\mathbb{R}^1$ -flow corresponding to a  $t \in \mathbb{R}^1$ , then we have  $T^{t_1} \cdot T^{t_2} = T^{t_1+t_2}$ . We will describe a natural situation in which the actions of  $\mathbb{R}^1$  arise.

Suppose  $M$  is a smooth compact manifold and  $\alpha$  is a smooth vector field on  $M$ . Consider the transformation  $T^t$  sending each point  $x \in M$  to the point  $T^t x$  which can be obtained from  $x$  by moving  $x$  along the trajectory of  $\alpha$  for the period of time  $t$  ( $T^t$  is well defined because of compactness of  $M$ ). Then  $T^{t_1+t_2} = T^{t_1} \cdot T^{t_2}$  and  $T^t$  is a measurable action of  $\mathbb{R}^1$ .

Measurable actions of  $\mathbb{R}^1$  are usually called *flows*, and those of  $\mathbb{R}_+^1$  — *semiflows*. The cyclic groups and semigroups of measurable transformations are also known as *dynamical systems with discrete time*, while flows and semiflows are known as *dynamical systems with continuous time*.

Now, let  $(M, \mathcal{M}, \mu)$  be a measure space (probability space), i.e.  $(M, \mathcal{M})$  is a measurable space and  $\mu$  is a nonnegative normalized ( $\mu(M) = 1$ ) measure on  $\mathcal{M}$ . Consider a measure  $\nu$  on  $\mathcal{M}$  given by  $\nu(C) = \mu(T^{-1}C)$ ,  $C \in \mathcal{M}$ . This measure is said to be the image of the measure  $\mu$  under  $T$  (notation:  $\nu = T\mu$ ).

**Definition 1.4.** A measure  $\mu$  is *invariant* under a measurable transformation  $T: M \rightarrow M$  if  $T\mu = \mu$ .

If  $\mu$  is invariant under  $T$ , then  $T$  is called an *endomorphism of the measure space*  $(M, \mathcal{M}, \mu)$ . If, in addition,  $T$  is invertible, it is called an *automorphism* of  $(M, \mathcal{M}, \mu)$ . If  $\{T^t\}$  is a measurable action of  $\mathbb{R}^1$  and each  $T^t$ ,  $-\infty < t < \infty$ , preserves the measure  $\mu$ , then  $\{T^t\}$  is called a *flow* on the measure space  $(M, \mathcal{M}, \mu)$ .

Now consider the general case.

**Definition 1.5.** Let  $\{T_g\}$  be a measurable action of a measurable group  $(G, \mathcal{G})$  on the space  $(M, \mathcal{M})$ . A measure  $\mu$  on  $\mathcal{M}$  is called *invariant* under this action if, for any  $g \in G$ ,  $\mu$  is invariant under  $T_g$ .

We now introduce the general notion of metric isomorphism of dynamical systems which allows us to identify systems having similar metric properties.

**Definition 1.6.** Suppose  $(G, \mathcal{G})$  is a measurable group and  $\{T_g^{(1)}\}, \{T_g^{(2)}\}$  are two  $G$ -flows acting on  $(M_1, \mathcal{M}_1), (M_2, \mathcal{M}_2)$  respectively and having invariant measures  $\mu_1, \mu_2$ . Such flows are said to be *metrically isomorphic* if there exist  $G$ -invariant subsets  $M'_1 \subset M_1, M'_2 \subset M_2$ ,  $\mu_1(M'_1) = \mu_2(M'_2) = 1$ , as well as an isomorphism  $\varphi: (M'_1, \mathcal{M}_1, \mu_1) \rightarrow (M'_2, \mathcal{M}_2, \mu_2)$  of measure spaces  $M'_1, M'_2$  such that  $T_g^{(2)}\varphi x^{(1)} = \varphi T_g^{(1)}x^{(1)}$  for all  $g \in G, x^{(1)} \in M'_1$ .

Ergodic theory also studies measurable actions of groups on the space  $(M, \mathcal{M}, \mu)$  which are not necessarily measure-preserving.

**Definition 1.7.** Suppose  $\{T_g\}$  is a measurable action of a measurable group  $(G, \mathcal{G})$  on  $(M, \mathcal{M})$ . The measure  $\mu$  on  $\mathcal{M}$  is said to be *quasi-invariant* under this action if for any  $g \in G$  the measure  $\mu_g \stackrel{\text{def}}{=} T_g\mu$ , i.e. the image of  $\mu$  under  $T_g$ , is equivalent to  $\mu$ . In other words,  $\mu$  and  $T_g\mu$  have the same sets of zero measure.