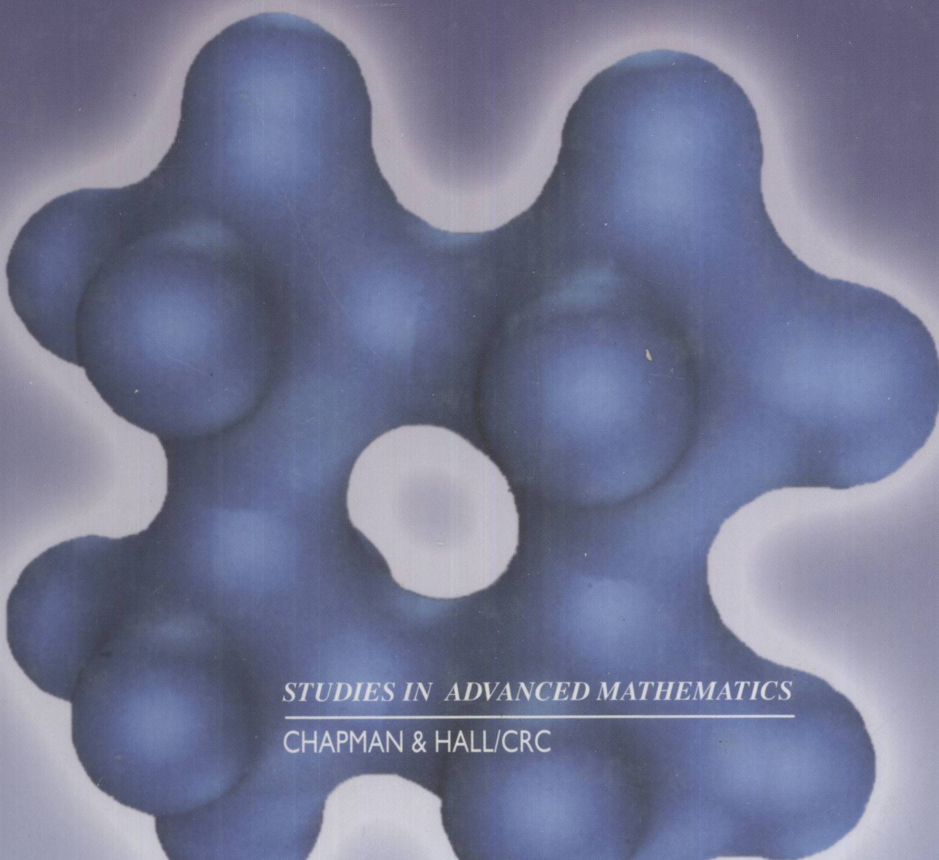
*Keith Burns*
*Marian Gidea*

# Differential Geometry and Topology

## With a View to Dynamical Systems



*STUDIES IN ADVANCED MATHEMATICS*

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# Differential Geometry and Topology

## With a View to Dynamical Systems

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# Differential Geometry and Topology

## With a View to Dynamical Systems

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*To Claudia – M.G.*

## Preface

This book grew out of notes from a differential geometry course taught by the second author at Northwestern University. It aims to provide an introduction, at the level of a beginning graduate student, to differential topology and Riemannian geometry. The theory of differentiable dynamics has close relations to these subjects. We introduce basic concepts from dynamical systems and try to emphasize interactions of dynamics, geometry and topology.

We have attempted to introduce important concepts by intuitive discussions or suggestive examples and to follow them by significant applications, especially those related to dynamics. Where this is beyond the scope of the book, we have tried to provide references to the literature.

We have not attempted to give a comprehensive introduction to dynamical systems as this would have required a much longer book. The reader who wishes to learn more about dynamical systems should turn to one of the textbooks in that area. Three excellent recent books, with different emphases, are the texts by Brin and Stuck (2002), by Katok and Hasselblatt (1995), and by Robinson (1998).

The illustrations in this book were produced with Adobe Illustrator, DPGraph, Dynamics Solver, Maple, and Sierpinski Curve Generator. We thank Victor Donnay, Josep Masdemont, and John M. Sullivan for permission to reproduce some of the illustrations.

# Contents

# Chapter 1

# Manifolds

## 1.1 Introduction

A manifold is usually described by a collection of 'patches' sewed together in some 'smooth' way. Each patch is represented by some parametric equation, and the smoothness of the sewing means that there are no cusps, corners or self-crossings.

As an example, we consider a hyperboloid of one sheet $x^2+y^2-z^2=1$ (see Figure 1.1.1 (a)). The hyperboloid is a surface of revolution, obtained by rotating the hyperbola $x^2 - z^2 = 1$, lying in the $(x, z)$-plane, about the $z$-axis. The hyperbola can be parametrized by $t \to (\cosh t, 0, \sinh t)$, so the hyperboloid of revolution is given by the differentiable parametrization

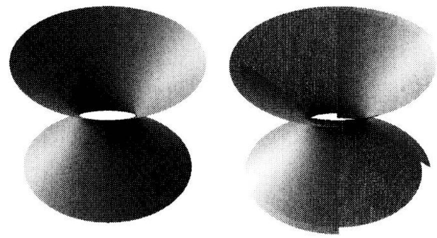
$$\phi(t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, \sinh t), \ -\infty < t < \infty, \ -\infty < \theta < \infty.$$

We would like to have each point $(x, y, z)$ of the hyperboloid uniquely determined by its coordinates $(t, \theta)$ and, conversely, each pair of coordinates $(t, \theta)$ uniquely assigned to a point. This does not work for the above parametrization, since the points of the hyperbola $x^2 - z^2 = 1$, $y = 0$, correspond to all $(t, \theta)$ with $\theta$ an integer multiple of $2\pi$. We can get parametrizations that are one-to-one by restricting the mapping $\phi$ to certain open subsets of $\mathbb{R}^2$:

$$\phi_1(t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, \sinh t), \ -\infty < t < \infty, \ 0 < \theta < 3\pi/2,$$

$$\phi_2(t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, \sinh t), \ -\infty < t < \infty, \ \pi < \theta < 5\pi/2.$$

Note that the image of each $\phi_i$ is the intersection of the hyperboloid with some open set in $\mathbb{R}^3$. In cylindrical coordinates $(r, \theta, z)$ on $\mathbb{R}^3$, the



(a) (b)

**FIGURE 1.1.1**
**Hyperboloid of one sheet.**

image of $\phi_1$ represents the portion of the hyperboloid inside the open region $0 < \theta < 3\pi/2$, and the image of $\phi_2$ represents the portion of the hyperboloid inside the open region $\pi < \theta < 5\pi/2$.

Since the mappings $\phi_1$ and $\phi_2$ are differentiable, the images of $\phi_1$ and $\phi_2$ are smooth patches of surface.

The following properties are at the core of the general definition of a manifold:

- Each $\phi_i$ is an injective map, and $\phi_i^{-1}$ is continuous, that is, $\phi_i^{-1}$ is the restriction to the hyperboloid of a continuous map defined on an open set in $\mathbb{R}^3$. This condition ensures that the surface does not self-intersect.

- For each $\phi_i$, the vectors $\partial\phi_i/\partial t$, $\partial\phi_i/\partial\theta$ are linearly independent. This condition ensures that there is a well defined tangent plane to the surface, spanned by these two vectors, at each point.

A subset $S$ of $\mathbb{R}^3$ together with a collection of smooth parametrizations whose images cover $S$ and which satisfy the above properties is called a regular surface.

The images of $\phi_1$ and $\phi_2$ are sewed together along two regions corresponding to $0 < \theta < \pi/2$ and to $\pi < \theta < 3\pi/2$, in the following sense:

- In the regions where the images of $\phi_1$ and $\phi_2$ overlap, the mapping $\phi_1$ can be obtained from the mapping $\phi_2$ by a smooth change of coordinates, and $\phi_2$ can be obtained from $\phi_1$ by a smooth change of coordinates. This means that there exist mappings $\theta_{12}$ and $\theta_{21}$, defined on appropriate open domains in $\mathbb{R}^2$, such that $\phi_2 = \phi_1 \circ \theta_{12}$ and $\phi_1 = \phi_2 \circ \theta_{21}$. Moreover, $\theta_{12}$ and $\theta_{21}$ are each the inverse mapping of the other.

Indeed, $\phi_2(t,\theta) = \phi_1(t,\theta)$ for all $(t,\theta)$ with $t \in \mathbb{R}$ and $\pi < \theta < \pi/2$, and $\phi_2(t,\theta) = \phi_1(t,\theta - 2\pi)$ for all $(t,\theta)$ with $t \in \mathbb{R}$ and $2\pi < \theta < 5\pi/2$. The corresponding smooth change of coordinates

$$\theta_{12} : \mathbb{R} \times [(\pi, 3\pi/2) \cup (2\pi, 5\pi/2)] \to \mathbb{R} \times [(\pi, 3\pi/2) \cup (0, \pi/2)]$$

is given by

$$\theta_{12}(t,\theta) = \begin{cases} (t,\theta), & \text{for } t \in \mathbb{R} \text{ and } \pi < \theta < 3\pi/2, \\ (t,\theta - 2\pi), & \text{for } t \in \mathbb{R} \text{ and } 2\pi < \theta < 5\pi/2. \end{cases}$$

Similarly, the change of coordinates

$$\theta_{21} : \mathbb{R} \times [(\pi, 3\pi/2) \cup (0, \pi/2)] \to \mathbb{R} \times [(\pi, 3\pi/2) \cup (2\pi, 5\pi/2)]$$

is given by

$$\theta_{21}(t,\theta) = \begin{cases} (t,\theta), & \text{for } t \in \mathbb{R} \text{ and } \pi < \theta < 3\pi/2, \\ (t,\theta + 2\pi), & \text{for } t \in \mathbb{R} \text{ and } 0 < \theta < \pi/2. \end{cases}$$

These coordinate changes provide essential information about the surface. As we will see in the next section, the above property is a cornerstone of the definition of a manifold.

Finally, we notice that same surface can be described through different collections of parametrizations. For example, consider a third local parametrization of the hyperboloid, which agrees with the previous ones:

$$\phi_3(t,\theta) = (\cosh t \cos\theta, \cosh t \sin\theta, \sinh t), \; -\infty < t < \infty, \; \pi/2 < \theta < 2\pi.$$

This patch lies on top of the other two, and it does not supply any new information about the surface. Indeed, $\phi_3(t,\theta) = \phi_1(t,\theta)$ for all $(t,\theta)$ with $t \in \mathbb{R}$ and $\pi/2 < \theta < 3\pi/2$, and $\phi_3(t,\theta) = \phi_2(t,\theta)$ for all $(t,\theta)$ with $t \in \mathbb{R}$ and $\pi < \theta < 2\pi$. Thinking of the hyperboloid as a collection of smooth parametrizations, we have two equivalent representations: one consisting of $\{\phi_1, \phi_2\}$, and a second one consisting of $\{\phi_1, \phi_2, \phi_3\}$. There are, in fact, infinitely many collections of equivalent parametrizations describing the same surface. We can always choose one collection of parametrizations as a representative.

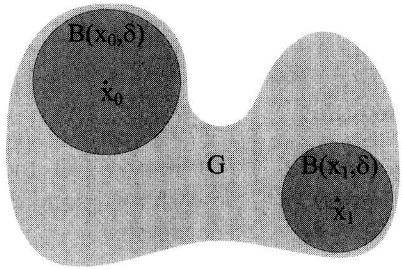



**FIGURE 1.2.1**
**Open set in the plane.**

## 1.2 Review of topological concepts

In the next section we will define manifolds. Unlike surfaces in Section 1.1, manifolds are not necessarily embedded in Euclidean spaces. Therefore, the ideas of nearness and continuity on a manifold need to be expressed in some intrinsic way.

Recall that a mapping $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is continuous at a point $x_0 \in U$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for each $x \in U$,

$$\|f(x) - f(x_0)\| < \epsilon \quad \text{provided} \quad \|x - x_0\| < \delta.$$

A mapping is said to be a continuous if it is continuous at every point of its domain. A sufficient (but not necessary) condition for a mapping $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ defined on an open set $U$ in $\mathbb{R}^m$ to be continuous is that $f$ is differentiable at every point $x_0 \in U$.

Continuity can be expressed in terms of open sets. A set $G \subseteq \mathbb{R}^m$ is open provided that for every $x_0 \in G$ one can find an open ball

$$B(x_0, \delta) = \{x \in \mathbb{R}^m \mid \|x - x_0\| < \delta\},$$

that is contained in $G$, for some $\delta > 0$. See Figure 1.2.1. If $X$ is a subset of $\mathbb{R}^m$, a set $G \subseteq X$ is said to be (relatively) open in $X$ if there exists an open set $H \subseteq \mathbb{R}^m$ such that $G = H \cap X$. A set is said to be closed if its complement is an open set. One can easily verify that a mapping is continuous on its domain if and only if, for every open set $V \subseteq \mathbb{R}^n$, the set $f^{-1}(V)$ is an open set in $U$. Equivalently, a mapping is continuous on its domain if and only if for every closed set $F \subseteq \mathbb{R}^n$, the set $f^{-1}(F)$ is a closed set in $U$.

Open sets are therefore essential in studying continuity. It turns out that all familiar properties of continuous mappings can be proved directly from only a few properties of open sets, with no reference to the $\epsilon - \delta$ definition. Those properties are at the core of the concept of a topological space:

***DEFINITION 1.2.1***

*A topological space is a set $X$ together with a collection $\mathcal{G}$ of subsets of $X$ satisfying the following properties:*

*(i) The empty set $\emptyset$ and the 'total space' $X$ are in $\mathcal{G}$;*

*(ii) The union of any collection of sets in $\mathcal{G}$ is a set in $\mathcal{G}$;*

*(iii) The intersection of any finite collection of sets in $\mathcal{G}$ is a set in $\mathcal{G}$.*

*The sets in $\mathcal{G}$ are called the open sets of the topological space. The collection $\mathcal{G}$ of all open sets is referred to as the topology on $X$.*

We will often omit specific mention of $\mathcal{G}$ and refer to a topological space only by the total space $X$. Given a set $A \subseteq X$, the union of all open sets contained in $A$ is called the interior of $A$ and is denoted by $\text{int}(A)$. The interior of a set is always an open set, possibly empty.

***Example 1.2.2***

(i) The Euclidean space $\mathbb{R}^m$ with the open sets defined as above is a topological space.

(ii) If $X$ is any set, the collection of all subsets of $X$ is a topology on $X$; it is called the discrete topology.

(iii) If $X$ is a topological space and $S$ is a subset of $X$, then the set $S$ together with the collection of all sets of the type $\{S \cap G \mid G \in \mathcal{G}\}$ is a topological space. This topology is referred to as the relative topology induced by $X$ on $S$.

(iv) If $X$ and $Y$ are topological spaces, then the collection of all unions of sets of the form $G \times H$, with $G$ an open set in $X$ and $H$ an open set in $Y$, is a topology on the product space $X \times Y$. This is called the product topology. This definition extends naturally to the case of finitely many topological spaces.

(v) Assume that $X$ is a topological space and $\sim$ is an equivalence relation on $X$. We define the quotient set $X/\sim$ as the set of all equivalence classes on $X$, and the canonical projection $\pi : X \to X/\sim$ that sends every element $x \in X$ into its equivalence class $[x] \in X/\sim$. The set of all $U \subseteq X/\sim$ for which $\pi^{-1}(U)$ is an open set in $X$ defines a topology on $X/\sim$, called the quotient topology.

(vi) A distance function (also called a metric) on a set $X$ is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $p, q, r \in X$:

(1) positive definiteness and non-degeneracy: $d(p,q) \geq 0$ and $d(p,q) = 0$ if and only if $p = q$;

(2) symmetry: $d(p,q) = d(q,p)$;

(3) triangle inequality: $d(p,q) \leq d(p,r) + d(r,q)$.

A set $X$ together with a distance function on it is called a metric space. A metric space has a natural topology: a set $G$ in $X$ is defined to be open provided that for every $x_0 \in G$ there exists $\delta > 0$ such that the open ball

$$B(x_0, \delta) = \{x \in X \mid d(x, x_0) < \delta\}$$

is contained in $G$. A metric space is said to be complete if every Cauchy sequence is convergent.

The natural topology on a Euclidean space is the topology induced by the Euclidean distance. ▯

In many instances, it is easier to describe the topology of a space by specifying a certain sub-collection of open sets. A basis for a topology $\mathcal{G}$ of $X$ is a collection $\mathcal{B} \subseteq \mathcal{G}$ of open sets with the property that every set in $\mathcal{G}$ can be obtained as a union of sets from $\mathcal{B}$. As an example, in a metric space, the collection of all open balls is a basis for the metric space topology. A sub-basis for a topology $\mathcal{G}$ of $X$ is a collection $\mathcal{S} \subset \mathcal{G}$ with the property that the collection of all finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{G}$.

***Example 1.2.3***

Let $X_i$ be an infinite collection on topological spaces, whose topologies are denoted by $\mathcal{G}_i$, respectively. On the cartesian product $\Pi_i X_i$ we define a topological basis as the collection of all sets of the type $\Pi_i U_i$, where $U_i$ is an open set in $X_i$ for each $i$, and only finitely many of the sets $U_i$ are different from $X_i$. The product topology of $\Pi_i X_i$ is defined as consisting of unions of sets of the above type. In the finite case, this topology is the same as the one described in Example 1.2.2 (iv). ▯

The complement of an open set is said to be a closed set. Given a set $A \subseteq X$, the intersection of all closed sets containing $A$ is called the closure of $A$, and is denoted by $\mathrm{cl}(A)$. The closure of a set is always a closed set. For a set $A$, the boundary set is defined by $\mathrm{bd}(A) = \mathrm{cl}(A) \setminus \mathrm{int}(A)$. The boundary of a set is always a closed set. If the closure of a set is the total space, that set is said to be a dense set. A closed set with an empty interior is called nowhere dense. For example, if $X = \mathbb{R}$ with the natural topology, and $A = \mathbb{Q}$, the set of all rational numbers, then $\mathrm{cl}(A) = X$, so $A$ is dense. The set $B = \mathbb{Z}$ of all integers is a nowhere dense set in $X$.

If $f : X \to \mathbb{R}$ is continuous, the support of $f$ is

$$\mathrm{supp}(f) = \mathrm{cl}\left(\{x \mid f(x) \neq 0\}\right).$$

A set $N \subseteq X$ is said to be a neighborhood of a point $x \in X$ provided that there exists an open set $G$ with $x \in G \subseteq N$. A basis of neighborhoods of a point $x \in X$ is a collection $\mathcal{V}_x$ of neighborhoods of $x$ with the property that every neighborhood of $x$ contains some set from $\mathcal{V}_x$. As an example, the balls of the type $B(x, 1/n)$ in a metric space form a basis of neighborhoods of $x$. One can completely describe a topology by specifying a basis of neighborhoods for each point of the space.

The idea of nearness in a topological space can be expressed through convergent sequences. A sequence $(x_n)_{n\geq 0}$ in $X$ is said to be convergent to a point $z \in X$ provided that for every neighborhood $V$ of $z$, there exists an integer $n_V$ such that all terms of the sequence $x_n$ with $n \geq n_V$ are contained in $V$. Unlike in $\mathbb{R}$, the limit of a convergent sequence in a topological space may not be unique. In order for the limit to be unique, it is sufficient that for every pair of points $x \neq y$ there exits a pair of disjoint neighborhoods $V_x$ of $x$ and $V_y$ of $y$. A topology satisfying this condition is said to be Hausdorff.

There is a natural definition of continuity in the context of topological spaces.

**DEFINITION 1.2.4**
*Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is continuous at a point $x_0$ in $X$ provided that $f^{-1}(V)$ is a neighborhood of $x_0$ for every neighborhood $V$ of $x_0$. A map $f : X \to Y$ is said to be continuous if for each open set $V$ in $Y$, the set $f^{-1}(V)$ is an open set in $X$.*

From calculus, we know that every continuous function on a closed bounded interval is bounded and attains its minimum and maximum