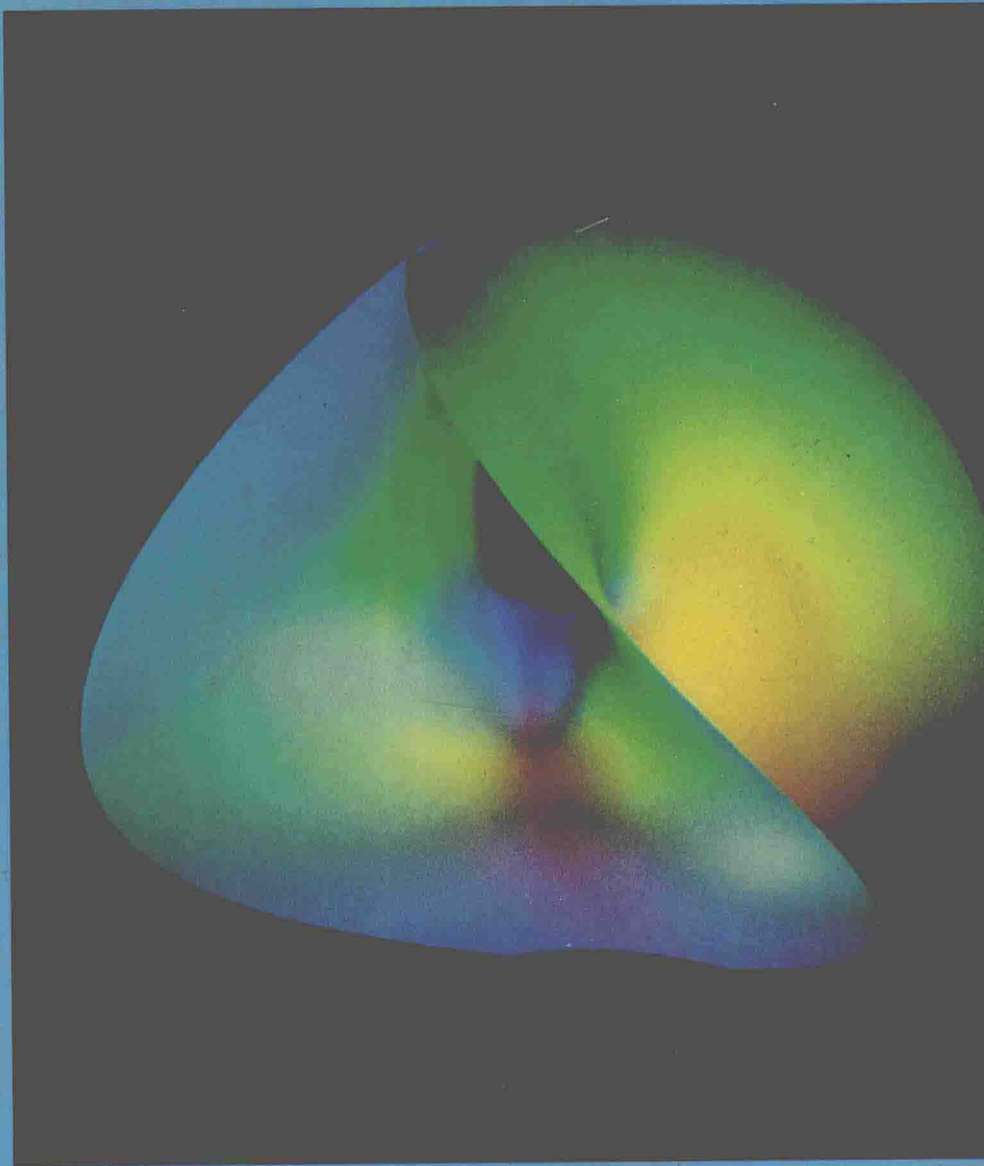
JERROLD E. MARSDEN

ANTHONY J. TROMBA

VECTOR CALCULUS

THIRD EDITION



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THIRD EDITION

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CORNELL UNIVERSITY AND UNIVERSITY OF CALIFORNIA, BERKELEY

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W. H. FREEMAN AND COMPANY / NEW YORK

Politics is for the moment.
An equation is for eternity.

A. EINSTEIN

Some calculus tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics seldom take the trouble to show you how easy the easy calculations are.

SILVANUS P. THOMPSON *Calculus Made Easy*, Macmillan (1910)

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Marsden, Jerrold E.
Vector calculus.

Includes index.
1. Calculus. 2. Vector analysis. I. Tromba, Anthony. II. Title.
QA303.M338 1988 515′.63 87-24595
ISBN 0-7167-1856-1

Printed in the United States of America

4 5 6 7 8 9 0 HA 9 9 8 7 6 5 4 3 2 1 0

VECTOR CALCULUS

PREFACE

This text is intended for a one-semester course in the calculus of functions of several variables and vector analysis taught at the sophomore or junior level. Sometimes the course is preceded by a beginning course in linear algebra, but this is not an essential prerequisite. We require only the bare rudiments of matrix algebra, and the necessary concepts are developed in the text. However, we do assume a knowledge of the fundamentals of one-variable calculus—differentiation and integration of the standard functions.

The text includes most of the basic theory as well as many concrete examples and problems. Teaching experience at this level shows that it is desirable to omit many of the technical proofs; they are difficult for beginning students and are included in the text mainly for reference or supplementary reading. In particular, some of the technical proofs for theorems in Chapters 2 and 5 are given in the optional Sections 2.7 and 5.5. Section 2.2, on limits and continuity, is designed to be treated lightly and is deliberately brief. More sophisticated theoretical topics, such as compactness and delicate proofs in integration theory, have been omitted, because they usually belong to and are better treated in a more advanced course.

Computational skills and intuitive understanding are important at this level, and we have tried to meet this need by making the book as concrete and student-oriented as possible. For example, although we formulate the definition of the derivative correctly, it is done by using matrices of partial derivatives rather than linear transformations. This device alone can save one or two weeks of teaching time and can spare those students whose linear algebra is not in top form from constant headaches. Also, we include a large number of physical illustrations. Specifically, we have included examples from such areas of physics as fluid mechanics, gravitation, and electromagnetic theory, and from economics as well, although prior knowledge of these subjects is not assumed.

A special feature of the text is the early introduction of vector fields, divergence, and curl in Chapter 3, before integration. Vector analysis usually suffers in a course of this type, and the present arrangement is designed to offset this tendency. To go even further, one might consider teaching Chapter 4 (Taylor's theorem, maxima and minima, Lagrange multipliers) after Chapter 8 (vector analysis).

This third edition retains the balance between theory, applications, optional material, and historical notes that was present in the second edition. The bulk of the changes for the third edition are as follows.

The exercises have been thoroughly reworked in conjunction with the writing of a *Study Guide* by Fred Soon and Karen Pao. This guide contains complete solutions to select exercises in the text (the numbers or letters of these exercises are shaded for quick identification) as well as study hints and sample exams. The *Study Guide* may be ordered by your bookstore from the publisher.

The exercises have been improved by a better progression according to level of difficulty and a wider coverage of topics. Optional technical theorems on differentiation and integration theorems have been moved from the appendixes to Chapters 2 and 5 and set in smaller type. The long chapter on integration theory has been split into two, and a new section on applications of multiple integrals has been added. Additional material on cylindrical and spherical coordinates has been included, and the section on the geometric meaning of the divergence and curl has been simplified. Other changes and corrections that improve the exposition have been made throughout text. Many of these have come from readers of the second edition, and we are indebted to them collectively for improving the book for the benefit of the student.

PREREQUISITES AND NOTATION

We assume that students have studied the calculus of functions of a real variable, including analytic geometry in the plane. Some students may have had some exposure to matrices as well, although what we shall need is given in Sections 1.3 and 1.5.

We also assume that students are familiar with functions of elementary calculus, such as $\sin x$, $\cos x$, e^x, and $\log x$ (we write $\log x$ for the natural logarithm, which is sometimes denoted $\ln x$ or $\log_e x$). Students are expected to know, or to review as the course proceeds, the basic rules of differentiation and integration for functions of one variable, such as the chain rule, the quotient rule, integration by parts, and so forth.

We shall now summarize the notations to be used later, often without explicit mention. Students can read through these quickly now, then refer to them later if the need arises.

The collection of all real numbers is denoted **R**. Thus **R** includes the *integers*, . . . , $-3, -2, -1, 0, 1, 2, 3, \ldots$; the *rational numbers*, p/q, where p and q are integers ($q \neq 0$); and the *irrational numbers*, such as $\sqrt{2}$, π, and e. Members of **R** may be visualized as points on the real-number line, as shown in Figure 0.1.

When we write $a \in \mathbf{R}$ we mean that a is a member of the set **R**; in other words, that a is a real number. Given two real numbers a and b with $a < b$ (that is, with a less than b), we can form the *closed interval* $[a, b]$, consisting of all x such that

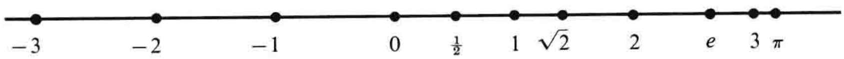

Figure 0.1 The geometric representation of points on the real number line.

$a \le x \le b$, and the *open interval* (a, b), consisting of all x such that $a < x < b$. Similarly, we may form half-open intervals $(a, b]$ and $[a, b)$ (Figure 0.2).

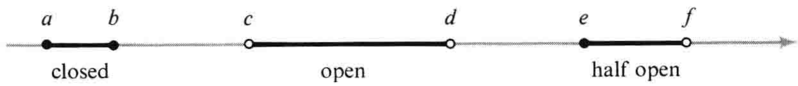

Figure 0.2 The geometric representation of the intervals $[a, b]$, (c, d), and $[e, f)$.

The *absolute value* of a number $a \in \mathbf{R}$ is written $|a|$ and is defined as

$$|a| = \begin{cases} a & \text{if} \quad a \ge 0 \\ -a & \text{if} \quad a < 0. \end{cases}$$

For example, $|3| = 3$, $|-3| = 3$, $|0| = 0$, and $|-6| = 6$. The inequality $|a + b| \le |a| + |b|$ always holds. The *distance from a to b* is given by $|a - b|$. Thus, the distance from 6 to 10 is 4 and from -6 to 3 is 9.

If we write $A \subset \mathbf{R}$, we mean A is a *subset* of $\mathbf{R}$. For example, A could equal the set of integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Another example of a subset of $\mathbf{R}$ is the set $\mathbf{Q}$ of rational numbers. Generally, for two collections of objects (that is, sets) A and B, $A \subset B$ means A is a subset of B; that is, every member of A is also a member of B.

The symbol $A \cup B$ means the *union* of A and B, the collection whose members are members of either A or B. Thus

$$\{\ldots, -3, -2, -1, 0\} \cup \{-1, 0, 1, 2, \ldots\} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}.$$

Similarly, $A \cap B$ means the *intersection* of A and B; that is, this set consists of those members of A and B that are in both A and B. Thus the intersection of the two sets above is $\{-1, 0\}$.

We shall write $A \backslash B$ for those members of A that are not in B. Thus

$$\{\ldots, -3, -2, -1, 0\} \backslash \{-1, 0, 1, 2, \ldots\} = \{\ldots, -3, -2\}.$$

We can also specify sets as in the following examples:

$$\{a \in \mathbf{R} \mid a \text{ is an integer}\} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$$

$$\{a \in \mathbf{R} \mid a \text{ is an even integer}\} = \{\ldots, -2, 0, 2, 4, \ldots\}$$

$$\{x \in \mathbf{R} \mid a \le x \le b\} = [a, b].$$

A *function* $f: A \to B$ is a rule that assigns to each $a \in A$ one specific member $f(a)$ of B. The fact that the function f sends a to $f(a)$ is denoted symbolically by $a \mapsto f(a)$. For example, $f(x) = x^3/(1 - x)$ assigns the number $x^3/(1 - x)$ to each $x \neq 1$ in $\mathbf{R}$. We can specify a function f by giving the rule for $f(x)$. Thus, the above function f can be defined by the rule $x \mapsto x^3/(1 - x)$.

If $A \subset \mathbf{R}$, $f: A \subset \mathbf{R} \to \mathbf{R}$ means that f assigns a value in $\mathbf{R}$, $f(x)$, to each $x \in A$. The set A is called the *domain* of f, and we say f has *range* $\mathbf{R}$, since that is where the values of f are taken. The *graph* of f consists of all the points $(x, f(x))$ in the plane (Figure 0.3). Generally, a *mapping* (=function = transformation = map) $f: A \to B$, where A and B are sets, is a rule that assigns to each $x \in A$ a specific point $f(x) \in B$.

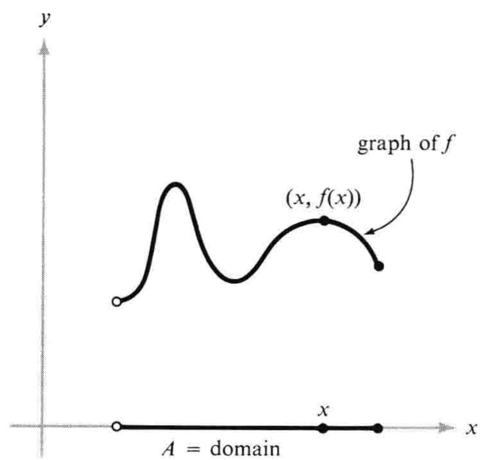

Figure 0.3 The graph of a function with the half-open interval A as domain.

The notation $\sum_{i=1}^{n} a_i$ means $a_1 + \cdots + a_n$, where $a_1, \ldots, a_n$ are given numbers. The sum of the first n integers is

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

The *derivative* of a function $f(x)$ is denoted $f'(x)$ or

$$\frac{df}{dx},$$

and the *definite integral* is written

$$\int_a^b f(x)\, dx.$$

If we set $y = f(x)$, the derivative is also denoted by

$$\frac{dy}{dx}.$$

Readers are assumed to be familiar with the chain rule, integration by parts, and other rules that govern the calculus of functions of one variable. In particular, they should know how to differentiate and integrate exponential, logarithmic, and trigonometric functions. Short tables of derivatives and integrals, which are adequate for the needs of this text, are printed at the front and back of the book.

The following notations are used synonymously: $e^x = \exp x$, $\ln x = \log x$, and $\sin^{-1} x = \arcsin x$.

The end of a proof is denoted by the symbol ■, while the end of an example or remark is denoted by the symbol ▲. Optional material, more theoretical or harder exercises are preceded by a star: *.

ACKNOWLEDGMENTS

Many colleagues and students in the mathematical community have made valuable contributions and suggestions since this book was begun. An early draft of the book was written in collaboration with Ralph Abraham. We thank him for allowing us to draw upon his work. It is impossible to list all those who assisted with this book, but we wish especially to thank Michael Hoffman and Joanne Seitz for their help on earlier editions. We also received valuable comments from Mary Anderson, John Ball, Frank Gerrish, Jenny Harrison, David Knudson, Richard Koch, Andrew Lenard, Gordon McLean, David Merriell, Jeanette Nelson, Dan Norman, Keith Phillips, Anne Perleman, Kenneth Ross, Ray Sachs, Diane Sauvageot, Joel Smoller, Melvyn Tews, Ralph and Bob Tromba, Steve Wan, Alan Weinstein, and John Wilker.

We are grateful to the following instructors who provided detailed reviews of the manuscript for this edition: David Bao, University of Houston; Stanley M. Lukawecki, Clemson University; John F. Pierce, West Virginia University; and Herb Walum, The Ohio State University.

A final word of thanks goes to those who helped in the preparation of the manuscript and the production of the book. We especially thank Connie Calica, Nora Lee, Marnie McElhiney, Rosemarie Stampful, Ruth Suzuki, Ikuko Workman, and Esther Zack for their excellent typing of various versions and revisions of the manuscript; Herb Holden of Gonzaga University and Jerry Kazdan of the University of Pennsylvania for suggesting and preparing the computer-generated figures; Jerry Lyons for his role as our mathematics editor; Richard K. Mickey for his expert copyediting; and Philip McCaffrey for his editorial supervision.

We will be maintaining an up-to-date list of corrections and suggestions concerning this third edition. We will be happy to mail this list to any user of the text. Please send your request to either Jerrold Marsden at the Department of Mathematics, Cornell University, Ithaca, NY 14853-7901 or Anthony Tromba at the Department of Mathematics, University of California, Santa Cruz, CA 95064.

Jerrold E. Marsden

Anthony J. Tromba

CONTENTS

1 THE GEOMETRY OF EUCLIDEAN SPACE

> Quaternions came from Hamilton . . . and have been an unmixed evil to those who have touched them in any way. Vector is a useless survival . . . and has never been of the slightest use to any creature.
>
> Lord Kelvin

In this chapter we consider the basic operations on vectors in three-dimensional space: vector addition, scalar multiplication, and the dot and cross products. In Section 1.5 we generalize some of these notions to n-space and review properties of matrices that will be needed in Chapters 2 and 3.

1.1 VECTORS IN THREE-DIMENSIONAL SPACE

Points P in the plane are represented by ordered pairs of real numbers (a, b); the numbers a and b are called the *Cartesian coordinates of* P. We draw two perpendicular lines, label them x and y axes, and drop perpendiculars from P to these axes as in Figure 1.1.1. After designating the intersection of the x and y axis as the origin and choosing units on these axes, we produce two directed distances a and b as shown in the figure; a is called the x *component* of P, and b is called the y *component*.

Points in space may be similarly represented as ordered triples of real numbers. To construct such a representation, we choose three mutually perpendicular lines that meet at a point in space. These lines are called the x axis, y axis, and z axis, and the point at which they meet is called the *origin* (this is our reference point). We

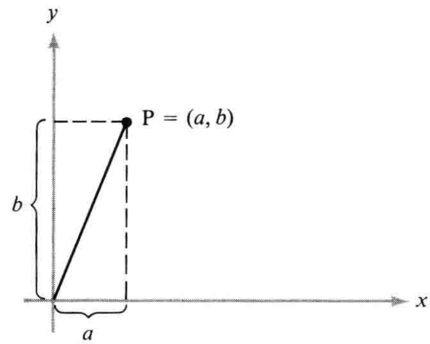

Figure 1.1.1 Cartesian coordinates in the plane.

choose a scale on these axes. The set of axes is often referred to as a *coordinate system*, and it is drawn as shown in Figure 1.1.2.

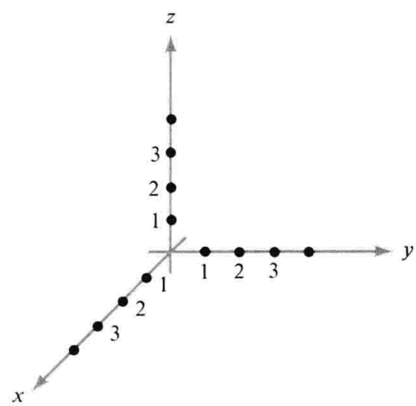

Figure 1.1.2 Cartesian coordinates in space.

We may assign to each point P in space a unique (ordered) triple of real numbers (a, b, c); and conversely, to each triple we may assign a unique point in space, just as we did for points in the plane. Let the triple (0, 0, 0) correspond to the origin of the coordinate system, and let the arrows on the axes indicate the positive directions. Then, for example, the triple (2, 4, 4) represents a point 2 units from the origin in the positive direction along the x axis, 4 units in the positive direction along the y axis, and 4 units in the positive direction along the z axis (Figure 1.1.3).

Because we can associate points in space with ordered triples in this way, we often use the expression "the point (a, b, c)" instead of the longer phrase "the point

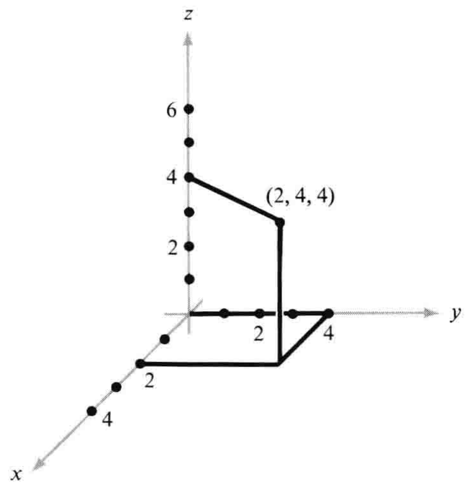

Figure 1.1.3 Geometric representation of the point (2, 4, 4) in Cartesian coordinates.

P that corresponds to the triple (a, b, c)." If the triple (a, b, c) corresponds to P, we say that a is the x coordinate (or first coordinate), b is the y coordinate (or second coordinate), and c is the z coordinate (or third coordinate) of P. With this method of representing points in mind, we see that the x axis consists of the points of the form $(a, 0, 0)$, where a is any real number; the y axis consists of the points $(0, a, 0)$; and the z axis consists of the points $(0, 0, a)$. It is also common to denote points in space with the letters x, y, and z in place of a, b, and c. Thus the triple (x, y, z) represents a point whose first coordinate is x, second coordinate is y, and third coordinate is z.

We employ the following notation for the line, the plane, and three-dimensional space.

(i) The real line is denoted $\mathbf{R}^1$ (thus, $\mathbf{R}$ and $\mathbf{R}^1$ are identical).

(ii) The set of all ordered pairs (x, y) of real numbers is denoted $\mathbf{R}^2$.

(iii) The set of all ordered triples (x, y, z) of real numbers is denoted $\mathbf{R}^3$.

When speaking of $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$ collectively, we write $\mathbf{R}^n$, $n = 1, 2$, or 3; or $\mathbf{R}^m$, $m = 1, 2, 3$.

The operation of addition can be extended from $\mathbf{R}$ to $\mathbf{R}^2$ and $\mathbf{R}^3$. For $\mathbf{R}^3$, this proceeds as follows. Given the two triples (x, y, z) and (x', y', z'), we define their *sum* by

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z').$$

EXAMPLE 1

$$(1, 1, 1) + (2, -3, 4) = (3, -2, 5)$$

$$(x, y, z) + (0, 0, 0) = (x, y, z)$$

$$(1, 7, 3) + (2, 0, 6) = (3, 7, 9). \quad \blacktriangle$$

The element $(0, 0, 0)$ is called the *zero element* (or just *zero*) of $\mathbf{R}^3$. The element $(-x, -y, -z)$ is called the *additive inverse* (or *negative*) of (x, y, z), and we write $(x, y, z) - (x', y', z')$ for $(x, y, z) + (-x', -y', -z')$.

There are important product operations in $\mathbf{R}^3$. One of these, called the *inner product*, assigns a real number to each pair of elements of $\mathbf{R}^3$. We shall discuss the inner product in detail in Section 1.2. Another product operation for $\mathbf{R}^3$ is called *scalar multiplication* (the word "scalar" is a synonym for "real number"). This product combines scalars (real numbers) and elements of $\mathbf{R}^3$ (ordered triples) to yield elements of $\mathbf{R}^3$ as follows: given a scalar α and a triple (x, y, z), we define the *scalar multiple* or *scalar product* by

$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z).$$

EXAMPLE 2

$$2(4, e, 1) = (2 \cdot 4, 2 \cdot e, 2 \cdot 1) = (8, 2e, 2)$$

$$6(1, 1, 1) = (6, 6, 6)$$

$$1(x, y, z) = (x, y, z)$$

$$0(x, y, z) = (0, 0, 0)$$

$$\begin{aligned}(\alpha + \beta)(x, y, z) &= ((\alpha + \beta)x, (\alpha + \beta)y, (\alpha + \beta)z) \\ &= (\alpha x + \beta x, \alpha y + \beta y, \alpha z + \beta z) \\ &= \alpha(x, y, z) + \beta(x, y, z). \quad \blacktriangle\end{aligned}$$

It is a consequence of the definitions that addition and scalar multiplication for $\mathbf{R}^3$ satisfy the following identities:

(i) $(\alpha\beta)(x, y, z) = \alpha[\beta(x, y, z)]$ (associativity)

(ii) $(\alpha + \beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributivity)

(iii) $\alpha[(x, y, z) + (x', y', z')] = \alpha(x, y, z) + \alpha(x', y', z')$ (distributivity)

(iv) $\alpha(0, 0, 0) = (0, 0, 0)$ (properties of zero elements)

(v) $0(x, y, z) = (0, 0, 0)$ (properties of zero elements)

(vi) $1(x, y, z) = (x, y, z)$ (property of the identity element)

For $\mathbf{R}^2$, addition is defined just as in $\mathbf{R}^3$, by

$$(x, y) + (x', y') = (x + x', y + y'),$$