Planned Textbook For University

Advanced Mathematics

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Mingming Chen
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Chemical Industry Press

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Chemical Industry Press
· Beijing ·

The aim of this book is to meet the requirement of bilingual teaching of advanced mathematics. The selection of the contents is in accordance with the fundamental requirements of teaching issued by the Ministry of Education of China. And base on the property of our university, we select some examples about petrochemical industry. These examples may help readers to understand the application of advanced mathematics in petrochemical industry.

This book is divided into two volumes. The first volume contains calculus of functions of a single variable and differential equation. The second volume contains vector algebra and analytic geometry in space, multivariable calculus and infinite series.

This book may be used as a textbook for undergraduate students in the science and engineering schools whose majors are not mathematics, and may also be suitable to the readers at the same level.

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# Preface

English is the most important language in international academia. In order to strengthen academic exchange with western countries, many universities in China pay more and more attention to the bilingual teaching in classrooms in recent years. Considering the importance of advanced mathematics and scarcity of bilingual mathematics textbook, we have written this book.

The main subject of this book is calculus. Besides, it also includes differential equation, analytic geometry in space, vector algebra and infinite series. This book is divided into two volumes. The first volume contains calculus of functions of a single variable and differential equation. The second volume contains vector algebra and analytic geometry in space, multivariable calculus and infinite series.

We have attempted to give this book the following characteristics.

① The content of this book is based on the Chinese textbook "advanced mathematics (sixth edition)" which is written by department of mathematics of Tongji University. The readers may read this book and use the Chinese textbook "advanced mathematics" as a reference. It may help readers to understand the mathematical contents and to improve the level of their English.

② In order to train the mathematical idea and ability of the students, we use some modern idea, language and methods of mathematics. We also bring in some mathematical symbol and logical symbol.

③ We pay more attention to the application of mathematics in practical problems. We have added some other examples and exercises in physics, chemistry, economics and even daily life.

④ Considering the different teaching requirements in different school, we mark some difficult sections and exercises by the symbol " * " . Teachers and students may choose suitable contents as required.

In this volume, Chapter 1 and Chapter 2 are written by Zhenyu Guo, Chapter 3 is written by Jinqiu Li, Chapter 4 and Chapter 5 are written by Jingxian Yu, Chapter 6 and Chapter 7 are written by Mingming Chen. All the chapters are checked and revised by Mingming Chen.

We hope this book can bring readers some help in the studying and teaching of bilingual mathematics. Due to the limit of our ability, it is impossible to avoid some unclear explanations. We would appreciate any constructive criticisms and corrections from readers.

**Authors**
**2010-8**

# Contents

# Chapter 1

# Functions and limits

Basically, the object of research in elementary mathematics is a constant quantity, but in advanced mathematics is a variable. In this chapter, we will introduce three fundamental concepts, which are functions, limits and continuity, and some of their properties.

## 1.1 Mappings and functions

### 1.1.1 Sets

#### 1.1.1.1 Concepts of sets

Set is an important concept and has been learned in high school. A set is defined as the collection of all objects having some specified property. Sets are usually denoted by capital letters $A, B, \cdots$. Each of the objects belonging to a set is called an **element** of the set, usually denoted by small letters $a, b, \cdots$. The relation $a \in A$ means that $a$ is an element of the set $A$, read as "$a$ belongs to the set $A$"; the relation $a \notin A$ means that $a$ is not an element of the set $A$, read as "$a$ does not belong to the set $A$". A set consisting of finite number of elements is called **a finite set.** A set containing no element is called **the empty set**, denoted by $\varnothing$. A set which is neither a finite set nor the empty set is called **an infinite set.** For instance, the set consisting of all natural numbers 0, 1, 2, $\cdots$ is an infinite set and is called **the set of natural numbers**, denoted by $\mathbb{N}$. Similarly, **the integer set** consists of all integers and is denoted by $\mathbb{Z}$; **the set of rational numbers** consists of all rational numbers and is denoted by $\mathbb{Q}$; **the set of real numbers** consists of all real numbers and is denoted by $\mathbb{R}$. The main objects of research about sets are infinite sets.

It is worthwhile to notice that the concept of set is very extensive. The elements of a set may be numbers but may also be many other things. For instance, if we assign a serial number to each student in a university, then the collection of all students in the university can be regarded as a set with the students as elements. To study the shortest route between two points we may construct a set which contains all possible curves connecting these two points, and then each curve is an element of the set.

There are two ways of representing sets. One is the method of citation, that is, to list all the elements of the set. For instance, the set consisting of all solutions of the equation $x^2-1=0$ may be expressed by $S=\{-1,1\}$. Another method is to point out the determining property held in common by all the elements of a set, that is,

$$X=\{x\mid x \text{ has some property } p(x)\}$$

For instance, the set of solutions of the equation $x^2-1=0$, can also be represented by $S=\{x\mid x^2-1=0\}$.

Suppose that $A$ and $B$ are two sets. If each element of the set $A$ is also an element of the set $B$, then $A$ is called a subset of the set $B$, denoted by $A\subseteq B$(or $B\supseteq A$), read as "$A$ is contained by $B$ (or $B$ contains $A$)"; if $A\subseteq B$ and $B\subseteq A$, then $A$ and $B$ are called equal, denoted by $A=B$; if $A\subseteq B$ but $A\neq B$, then, is called a proper subset of $B$, denoted by $A\subsetneqq B$.

#### 1.1.1.2 Operations on sets

There are three basic operations on sets, as follows.

Suppose that $A$ and $B$ are two sets.

The **union** of $A$ and $B$ is the set whose elements are those which belong to at least one of the two sets $A$ and $B$. The union is denoted by $A\cup B$, namely

$$A\cup B=\{x\mid x\in A \text{ or } x\in B\}$$

The **intersection** of $A$ and $B$ is the set whose elements are those which belong to both $A$ and $B$. The intersection is denoted by $A\cap B$, namely

$$A\cap B=\{x\mid x\in A \text{ and } x\in B\}$$

It is important to note that an element of $A\cup B$ may belong to both $A$ and $B$.

The **difference** of $A$ and $B$ is the set whose elements are those which belong to $A$ but not to $B$. The difference is denoted by $A\setminus B$, namely

$$A\setminus B=\{x\mid x\in A, x\notin B\}$$

If $B\subseteq A$, then the difference $A\setminus B$ is called **the complement of** $B$ **with respect** to $A$, denoted by $\complement_A B$.

If the sets which we are discussing are all the subsets of a set $X$(called the **universal set**), then $X\setminus B$ is called **the complement of** $B$, denoted by $\complement_B$ or $B^C$.

For instance, the complement of $A=\{x\mid 0<x\leqslant 1\}$ is $A^C=\{x\mid x\leqslant 0 \text{ or } x>1\}$ in the set of real numbers $\mathbb{R}$.

The union and the intersection of sets may be extended to any finite number of sets.

If $A\cap B=\varnothing$, then $A$ and $B$ are said to be disjoint.

It is easily seen from the geometric meaning that operations on sets satisfy the following rules. The analytic proof of rule 1 is omitted.

**Rule 1** Let $A,B,C$ be any three sets, then

① **commutative law** $A\cup B=B\cup A, A\cap B=B\cap A$;

② **associative law** $(A\cup B)\cup C=A\cup(B\cup C), (A\cap B)\cap C=A\cap(B\cap C)$;

③ **distributive law** $(A\cup B)\cap C=(A\cap C)\cup(B\cap C), (A\cap B)\cup C=(A\cup C)\cap(B\cup C)$, $(A\setminus B)\cap C=(A\cap C)\setminus(B\cap C)$;

④ **idempotent law** $A\cup A=A, A\cap A=A$;

⑤ **absorption law** $A\cup\varnothing=A, A\cap\varnothing=\varnothing$, if $A\subseteq B$, then $A\cup B=B, A\cap B=A$.

**Rule 2 (Dualization law)** If $A,B$ are two subsets of the universal set $X$, then

$$(A\cup B)^C=A^C\cap B^C,\ (A\cap B)^C=A^C\cup B^C$$

**Proof.** We prove only $(A\cup B)^C=A^C\cap B^C$. Since

$$x\in(A\cup B)^C\Rightarrow x\notin A\cup B\Rightarrow x\notin A \text{ and } x\notin B\Rightarrow x\in A^C \text{ and } x\in B^C$$
$$\Rightarrow x\in A^C\cap B^C,\ (A\cup B)^C\subset A^C\cap B^C$$

Since

$$x\in A^C\cap B^C\Rightarrow x\in A^C \text{ and } x\in B^C\Rightarrow x\notin A \text{ and } x\notin B\Rightarrow x\notin A\cup B$$
$$\Rightarrow x\in(A\cup B)^C,\ A^C\cap B^C\subset(A\cup B)^C$$

Therefore, $(A\cup B)^C=A^C\cap B^C$.

In many problems we also encounter the product of sets. Suppose that sets $A,B$ are both nonempty, then the set consisting of all the ordered pairs $(x,y)$, where $x$ and $y$ are any elements of $A$ and $B$ respectively, is called the **product** (or **Descartes**) of the sets $A$ and $B$, denoted by $A\times B$, namely

$$A\times B=\{(x,y)\mid x\in A, y\in B\}$$

For example, let $A=\{x\mid a\leqslant x\leqslant b\}, B=\{y\mid c\leqslant y\leqslant d\}$, then

$$A\times B=\{(x,y)\mid a\leqslant x\leqslant b, c\leqslant y\leqslant d\}$$

is the set consisting of all points of the rectangle in the $xOy$ plane and its four vertices are respectively $(a,c),(b,c),(b,d),(a,d)$.

The set $\mathbb{R}\times\mathbb{R}=\{(x,y)\mid x,y\in\mathbb{R}\}$ is the whole $xOy$ plane, denoted by $\mathbb{R}^2$.

### 1.1.1.3 Intervals and neighborhoods

Intervals are a class of important number sets. Let $a$ and $b$ are real numbers, $a<b$. The number set

$$\{x\mid a<x<b\}$$

is said to be an **open interval**, denoted by $(a,b)$, that is,

$$(a,b)=\{x\mid a<x<b\}$$

$a$ and $b$ are called the **end points** of the open interval $(a,b)$, here, $a\notin(a,b), b\notin(a,b)$. The number set

$$\{x\mid a\leqslant x\leqslant b\}$$

is said to be a **closed interval**, denoted by $[a,b]$, that is,

$$[a,b]=\{x\mid a\leqslant x\leqslant b\}$$

$a$ and $b$ are also called the **end points** of the closed interval $[a,b]$, here, $a\in[a,b], b\in[a,b]$.

Concept of neighborhoods is useful and widely used. Any open interval with center $a$ is said to be a neighborhood of point $a$, denoted by $U(a)$.

Let $\delta$ be any positive number, then the open interval $(a-\delta,a+\delta)$ is a neighborhood of point $a$, which is called $\delta$-neighborhood of point $a$, denoted by $U(a,\delta)$. That is,

$$U(a,\delta)=\{x\mid a-\delta<x<a+\delta\}$$

Point $a$ is the center of the neighborhood, $\delta$ is called the radius of the neighborhood (Figure 1-1).

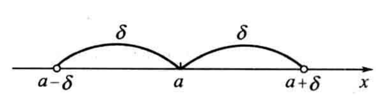

Figure 1-1

Some time, we need to delete the center of a neighborhood, which is said a deleted neighborhood, denoted by

$$\mathring{U}(a,\delta)=\{x\mid 0<|x-a|<\delta\}$$

where $0<|x-a|$ represent $x\neq a$.

It is convenient, the open interval $(a-\delta,a)$ is called the left $\delta$-neighborhood of point $a$, the open interval $(a,a+\delta)$ is called the right $\delta$-neighborhood of point $a$.

## 1.1.2 Mappings

### 1.1.2.1 Concepts of mappings

**Definition 1 (mapping)** Let $X$ and $Y$ be two nonempty sets. If for every $x\in X$, there is a unique $y\in Y$ corresponding to $x$ according to some determined rule $f$, then $f$ is called a mapping of $X$ into $Y$, denoted by

$$f:X\to Y \text{ or } f:x\mapsto y=f(x),x\in X$$

Here, $y$ is called the **image** of $x$ under the mapping $f$, and $x$ is called the **inverse image** of $y$ under the mapping $f$. The set $X$ is called the **domain of definition** of the mapping $f$, denoted by $D_f$. The set consisting of the image $y$ of all elements $x\in X$ is called the **range** of $f$, denoted by $R_f$ or $f(X)$. That is, $R_f=f(X)=\{y\mid y=f(x),x\in X\}$.

A mapping is also called an **operator.** If $X=Y$, namely, if the mapping $f$ maps a set $A$ into itself, then $f$ is called a **transformation** on the set $X$. If a mapping $f$ maps every element of the set $X$ into itself, then $f$ is called the identity mapping or unit mapping, denoted by $I_X$ or $I$, that is $I_X=x$, $\forall x\in X$.

**Note.** There are three essential factors in the definition of a mapping, namely the domain of definition $D_f=X$, the range of function $R_f\subset Y$ and the corresponding rule $f$. The domain of definition describes the region of existence of the mapping, and the corresponding rule $f$ gives the method for determining the corresponding elements of the set $Y$ from the elements of the set $X$, a specific expression of the mapping. For every element $x\in X$, the image of $x$ is unique; for every $y\in R_f$, the inverse image of $y$ is not unique.

**Example 1** let $f:\mathbb{R}\to\mathbb{R}$, if definition $f(x)=x^2$, $\forall x\in\mathbb{R}$ then $f$ is a mapping. The domain of definition of $f$ is $D_f=\mathbb{R}$ and the range of $f$ is $R_f=\{y\mid y\geqslant 0\}\subset\mathbb{R}$.

For $y\in R_f(y\neq 0)$, the inverse image of $y$ is not unique. For instance, the inverse image of $y=4$ are $x=2$ and $x=-2$.

**Example 2** Let $X=\{(x,y)\mid x^2+y^2=1\}$, $Y=\{(x,0)\mid |x|\leqslant 1\}$. If $f:X\to Y$ for every element $(x,y)\in X$, definition$(x,0)\in Y$, then $f$ is a mapping. The domain of definition of $f$ is $D_f=X$ and the range of $f$ is $R_f=Y$. In geometry, the given mapping $f$ maps the point on the unit-circle with the center $O$ into the interval $[-1,1]$ on $x$-axis.

**Example 3** Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[1,1]$, for every element $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, definite $f(x)=\sin x$ then $f$ is a mapping. The domain of definition of $f$ is $D_f=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the range of $f$ is $R_f=[-1,1]$.

Let $f$ is a mapping of $X$ into $Y$, denoted by $f:X\to Y$. If $R_f=Y$, namely, every element $y$ in $Y$ has a corresponding element $x$ in $X$ such that $y=f(x)$, then $f$ is said to be a mapping from $X$ onto $Y$. $f$ is said to be a one-to-one mapping if whenever $f(x_1)=f(x_2)$ for $x_1,x_2$ in $X$, one has $x_1=x_2$. Equivalently, $f$ is a one-to-one mapping if whenever $x_1\neq x_2$, one has

$f(x_1) \neq f(x_2)$.

According to the definition, the mapping of Example 1 is only a mapping but does not one-to-one and onto mapping. The mapping of Example 2 is a one-to-one mapping but does not onto mapping. The mapping of Example 3 is also one-to-one and onto mapping, therefore it is a bijective mapping.

**1. 1. 2. 2 Inverse mappings and composite mappings**

**Definition 2 (one-to-one mapping)** Suppose that $f: X \to Y$ is a mapping, so $\forall x \in X$, $\exists y \in Y$ corresponds to $x \in X$ and $y = f(x)$. If $\forall y \in Y$, $\exists x \in X$ corresponds to $y \in Y$ as well and satisfies $f(x) = y$, then $f$ is called a one-to-one mapping from $X$ to $Y$, and the sets $X$ and $Y$ are said to be in one to one correspondence. (The symbol" $\forall$ "means"arbitrary" and the symbol " $\exists$ " means "there exists a ". )

**Definition 3 (inverse mapping)** Suppose that $f: X \to Y$ is a one-to-one mapping, $\forall y \in R_f$, $\exists x \in X$ such that $f(x) = y$. Then we can define a mapping $g: R_f \to X$ which maps each $y \in R_f$ to $x \in X$, that is $g(y) = x$ and $f(x) = y$. The mapping $g$ is called the **inverse mapping** of the mapping $f$, denoted by $g = f^{-1}$. The domain is $D_{f^{-1}} = R_f$ and the range is $R_{f^{-1}} = X$. And $f$ is said to be **invertible mapping.**

**Definition 4 (composite mappings)** Let $g: X \to Y_1$ and $f: Y_2 \to Z$ are mappings, where $Y_1 \subset Y_2$ . Then for each $x \in X$, by the mapping $g$, there exists a unique $y = g(x) \in Y_1$ which corresponds to $x$. Again by the mapping $f$, there exists a unique $z = f(y) = f(g(x)) \in Z$, which corresponds to $y$. Thus, for each $x \in X$, the corresponding $z \in Z$ can be determined uniquely by the mappings $g$ and $f$. Hence a new mapping from $X \to Z$ is determined. This new mapping is called a composite mapping of $g$ and $f$, denoted by $f \circ g: X \to Z$. That is,

$$(f \circ g)(x) = f(g(x)), \quad x \in X$$

It is not difficult to see that for two given mappings $g$ and $f$, they can be composed if and only if the range of one mapping is a subset of the domain of definition of another mapping. For instance, if $R(g) \subseteq D(f)$, then the mappings can be composed into $f \circ g$.

Note that the composite mapping of $g$ and $f$ ($f \circ g$) is different from the composite mapping of $f$ and $g$ ($g \circ f$).

**Example 4** Let $g: \mathbb{R} \to [-1, 1]$ is a mapping, for each $x \in \mathbb{R}$, we definite $g(x) = \sin x$. $f: [-1, 1] \to [0, 1]$ is also a mapping, for each $u \in [-1, 1]$, we definite $f(u) = \sqrt{1 - u^2}$. Then the composite mapping of $g$ and $f$ ($f \circ g$) $\mathbb{R} \to [0, 1]$, for each $x \in \mathbb{R}$, we have

$$(f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{1 - \sin^2 x} = |\cos x|$$

### 1. 1. 3 Functions

**1. 1. 3. 1 Concepts of functions**

**Definition 5 (function)** If $f: D \to \mathbb{R}$ be a mapping and $D \subset \mathbb{R}$. Then mapping $f$ is called a function of definition on set $D$, denoted by

$$y = f(x), \quad x \in D$$

where $x$ is called the independent variable, and $y$ is called the dependent variable. The set $D$ is called **domain** of $f$, that is $D_f = D$.

In the definition of function, for every $x \in D$, there is a unique value $y \in \mathbb{R}$ corresponding to $x$ according to some determined rule $f$, the value is called functional value of $f$, denoted by $f(x)$, that is $y=f(x)$. The set consists of all functional value of $f$ is called the range of $f$, denoted by $R_f$ or $f(D)$, that is, $R_f=f(D)=\{y|y=f(x), x\in D\}$.

According to this definition, the terminology "function" should refer to the corresponding rule $f$, but we often use "$y=f(x), x\in D$" to express the function $f$ defined in $D$.

Hence, a function is a particular kind of mapping whose domain and range are both real number sets. As function is the central concept of calculus, we will go into more detail in explaining it.

Some notes on functions. Consider a function $y=g(x), x\in D$.

① **Domain** Just like a mapping, the domain is one of the essential factors in the concept of function. It is a set $D$ such that when the independent variable $x\in D$, the dependent variable $y$ can be determined from the corresponding rule $f$.

If the domain of function is an interval $[a,b]$, then it is often called the interval of definition.

When we investigate a practical problem and the function has some practical meaning, then the domain sometimes will be determined by this practical meaning. For instance, we consider the law of a freely falling body: $s=\frac{1}{2}gt^2$. This is a function of time $t$. If we consider only the expression, then the interval of definition is $(-\infty,+\infty)$. But for the motion of a free falling body the time runs from $t=0$ to $t=T$ when the body reaches the ground. Therefore, the interval of definition for this motion should be $[0,T]$.

If functions do not appear in some practical problems, but is given by a mathematical expression, the domain is usually understood to consist of the numbers for which the mathematical expression is defined. For instance, the domain of function $y=\sqrt{1-x^2}$ is close interval $[-1,1]$; the domain of function $y=\frac{1}{\sqrt{1-x^2}}$ is open interval $(-1,1)$.

**Example 5** Find the domain of the function $y=\sqrt{4-x^2}+\frac{1}{\sqrt{x-1}}$.

**Solution.** For the function $y=\sqrt{4-x^2}+\frac{1}{\sqrt{x-1}}$ to be meaningful, the square roots of $4-x^2$ and $x-1$ must make sense and $\sqrt{x-1}\neq 0$. Thus, the domain consists of all numbers $x$ such that

$$\begin{cases} 4-x^2\geqslant 0 \\ x-1>0 \end{cases}$$

Equivalently, $1<x\leqslant 2$. That is, the domain is the interval $(1,2]$.

② **Correspondence rule** This is another essential factor in the concept of function. It is the specific manifestation of the relation between $y$ and $x$. To express a function is mainly to express its correspondence rule. There are many methods to express the correspondence rule, the following three are often used.

**Case 1 Method of tabulation** Tabulate the values of the independent variable $x$ and the dependent variable $y$, like a table of trigonometrically functions or a table of logarithms. The correspondent rule is shown explicitly by the table.

**Case 2 Method shown by graphs** The relationship between $y$ and $x$ is shown by a graph. For instance, the temperature curve recorded by some instruments expresses the relationship between the temperature and time.

**Case 3 Analytic representation** The relation between $y$ and $x$ is expressed by an analytic representation. Those functions which we have learned in high school, such as trigonometric functions, inverse trigonometric functions, exponential functions, logarithm functions and power functions are all described by analytic representations. The above five types of functions and constants are called by the joint name basic elementary functions. Functions described by analytic representations are the main forms used in calculus.

When we study a function given by an analytic representation, sketching its graph will often help us to obtain some intuitive information about the function. In the coordinate system, the point set

$$\{(x,y)\mid y=f(x), x\in D(f)\}$$

often a curve, is called the **graph of the function** $f$ (Figure 1-2).

It should be noted that the analytic representation of a function sometimes consists of several components on different subsets of the domain of the function. A function expressed by this kind of representations is called a **piecewise defined function.**

**Example 6** The function $y=|x|=\begin{cases} x, & x\geqslant 0, \\ -x, & x<0. \end{cases}$

The domain of function is $D=(-\infty,+\infty)$ , the range of function is $R_f=[0,+\infty)$. The function is called **absolute function.** Its graph is shown in Figure 1-3.

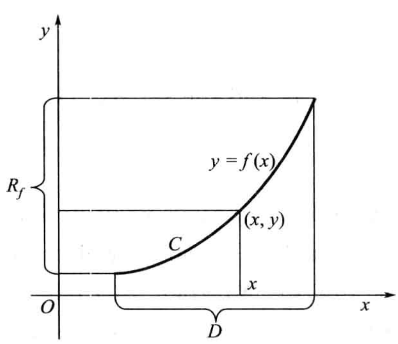


Figure 1-2

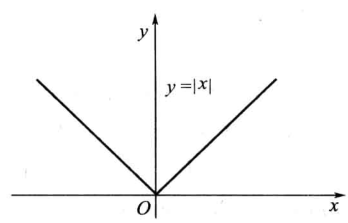


Figure 1-3

**Example 7** The function $y=\operatorname{sgn}x=\begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$ is called **sign function.** The domain of the function is $D=(-\infty,+\infty)$, the range of function is $R_f=\{-1,0,1\}$. Its graph is shown in Figure 1-4. Obviously, for any $x\in D$, we have $x=|x|\cdot\operatorname{sgn}x$ or $|x|=x\cdot\operatorname{sgn}x$.

**Example 8** The function whose value at any number $x$ is the largest integer smaller than or equal to $x$ is called **the greatest integer function**, denoted by $[x], x\in(-\infty,+\infty)$. For instance, $\left[\frac{5}{7}\right]=0, [\sqrt{2}]=1, [\pi]=3, [-1]=-1, [-3.5]=-4$. The

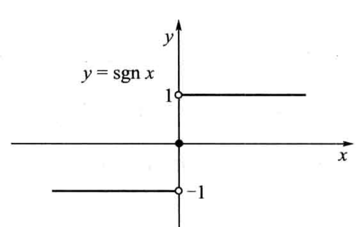


Figure 1-4

range of function is $R_f=\mathbb{Z}$ and its graph is shown in Figure 1-5.

**Example 9** The function $y=f(x)=\begin{cases}2\sqrt{x}, & 0\leqslant x\leqslant 1\\ 1+x, & x>1\end{cases}$ is a piecewise defined function. Its domain is $D=[0,+\infty)$.

If $x\in[0,1]$, $f(x)=2\sqrt{x}$; if $x\in(1,+\infty)$, $f(x)=1+x$.

For instance, $\frac{1}{2}\in[0,1]$, $f(\frac{1}{2})=\sqrt{2}$; $1\in[0,1]$, $f(1)=2\sqrt{1}=2$; $3\in(1,+\infty)$, $f(3)=1+3=4$.

The graph is shown in Figure 1-6.

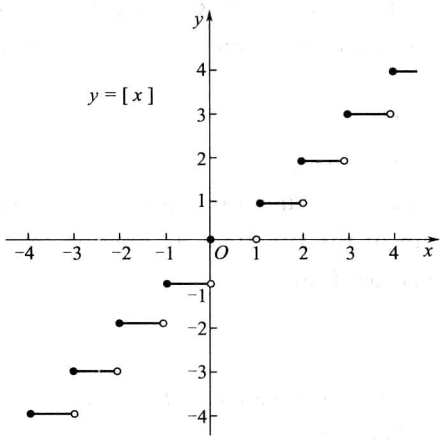


Figure 1-5

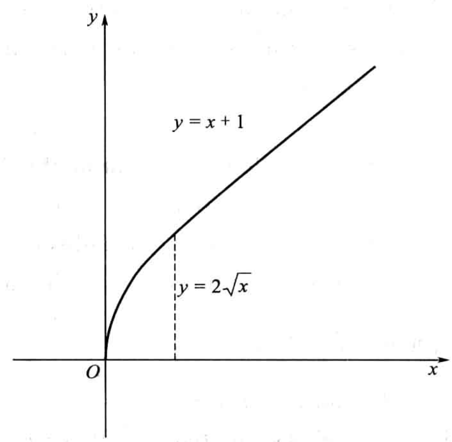


Figure 1-6

### 1.1.3.2 Properties of functions

**Bounded functions** Let the domain of function $f(x)$ is $D$ and the set $X\subset D$. If there exists a positive number $M$ such that

$$|f(x)|\leqslant M$$

for any $x\in X$, then $f(x)$ is said to be **bounded** on $X$. If for any such positive number $M$, there exists $x_1\in X$ such that $|f(x_1)|>M$, then $f(x)$ is said to be **unbounded** on $X$.

If there exists $M_1>0$ such that $f(x)\leqslant M_1$, for any $x\in X$, then $f(x)$ is said to be **bounded above** on $X$. The positive number $M_1$ is called the upper bound of the function $f(x)$.

If there exists $M_2>0$ such that $f(x)\geqslant M_2$, for any $x\in X$, then $f(x)$ is said to be **bounded below** on $X$. The positive number $M_2$ is called the lower bound of the function $f(x)$.

For example, $f(x)=\sin x$ is bounded on $(-\infty,+\infty)$. In fact, $|\sin x|\leqslant 1$ for any real number $x$. Of course, any number larger than 1 can be chosen as $M$.

$f(x)=\frac{1}{x}$ is not above bounded but below bounded in interval $(0,1)$, for instance, 1 is a below bounded of the given function. Because there does not exist such positive number $M$, such that $|\frac{1}{x}|\leqslant M$ for all $x\in(0,1)$, the function $f(x)=\frac{1}{x}$ is unbounded in interval $(0,1)$.

**Monotone functions** Let the domain of function $f(x)$ is $D$ and interval $I\subset D$.

If $f(x_1)<f(x_2)$ for any $x_1,x_2\in I$ with $x_1<x_2$, then $f(x)$ is said to be **monotone increasing** on the interval $I$(Figure 1-7).

If $f(x_1)>f(x_2)$ for any $x_1,x_2\in I$ with $x_1<x_2$, then $f(x)$ is said to be **monotone de-**