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Differential Tensor Algebras and their Module Categories

R. Bautista, L. Salmerón and R. Zuazua

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R. BAUTISTA, L. SALMERÓN AND R. ZUAZUA

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Preface

This monograph is concerned with the notions of ditalgebras (an acronym for "differential tensor algebras") and the study of their categories of modules. It involves reduction techniques which have proved to be very useful in the development of the theory of representation of finite-dimensional algebras. Our aim has been to present in a systematic, elementary and self-contained as possible way some of the main results obtained with these methods. They were originally introduced by the Kiev School in representation theory of algebras, in an attempt to formalize and generalize matrix problems methods.

The presentation given here has many common features with the original one of A. V. Roiter and M. Kleiner [46], in terms of differential graded categories, as well as with the formulation given by Y. Drozd [28] (and further developed by W. Crawley-Boevey [19] and [20]), in terms of bocses. It is clear that some applications of these techniques, notably in the study of coverings in representation theory of algebras, will require the categorical formulation of the theory, as suggested in [30]. However, for the sake of simplicity, we preferred to work here in the more concrete ring theoretical context of ditalgebras. We assume from the reader some familiarity with the basics of representation theory of algebras and homological algebra (including the basics of the theory of additive categories with exact structures), which can be obtained from the first sections of [29], [47] and [3] (respectively, [32] and [27]).

In the representation theory of finite-dimensional algebras, the notions of finite, tame and wild representation type play a central role. An algebra is of finite representation type if it has finitely many pairwise non-isomorphic indecomposable modules. It is of wild representation type, or simply wild, if it contains the problem of finding a normal form for pairs of square matrices over a field under simultaneous conjugation by a non-singular matrix. Finally, it is of tame representation type, or simply tame, if the pairwise non-isomorphic indecomposable modules in each dimension can be described by a finite number of one-parameter families. For precise definitions, see Sections 22 and 27. This monograph includes a fresh point of view of well-known facts on tame and wild ditalgebras, on tame and wild algebras, and on their modules. But there are also some new results and some new proofs.

We will review, for instance, Drozd's Tame and Wild Theorem, stating that a finite-dimensional algebra over an algebraically closed field is either tame or wild, but not both. We review also Crawley's Theorem on the existence of generic modules for tame finite-dimensional algebras over an algebraically closed field, and his Structure Theorem for its Auslander–Reiten quiver.

Our approach presents a formal alternative to the use of bocses with underlying additive categories and pull-back reduction constructions. This is replaced by the use of some special dual basis and what we call "reduction by an admissible module". This approach permits to perform explicit calculations with a reasonable effort. As an illustration of this, Section 24 includes a more conceptual proof of the fact that critical bocses are wild than the original proof of Drozd (see [19]) or than some of its subsequent simplifications (see [49]), where an explicit bimodule which produces wildness is exhibited.

The presentation given here of the reduction by an admissible module is more general than the one given in [6]. We believe that this approach has some promising potential since it provides a systematic treatment of a wide variety of reductions.

Let us comment on one interesting new result proved in Section 31. Let K denote a field extension of our ground field k. As usual, if Λ is some k-algebra, Λ-Mod denotes the category of left Λ-modules. We consider the induced K-algebra $\Lambda^K = \Lambda \otimes_k K$. Recall that the *endolength* of a Λ-module M is by definition the length of the right $\mathrm{End}_\Lambda(M)^{op}$-module M. A *generic* Λ-module is an indecomposable Λ-module with finite endolength and not finitely generated over Λ.

We will prove that if Λ is a finite-dimensional algebra over an algebraically closed field k and the induced algebra Λ^K is not wild, then every generic Λ^K-module is *rationally induced* from a generic Λ-module. More precisely, any generic Λ^K-module is of the form $G \otimes_{k(t)} K(t)$, where G is some generic Λ-module equipped with a natural structure of Λ-$k(t)$-bimodule. This is related to the study in [37], where it is shown that the extension of a field to its algebraic closure preserves generic tameness.

Now assume that k is algebraically closed and let $K = k(t)$, the rational function field of k. It has been proved in [11] that Λ^K is of finite representation type if and only if every indecomposable Λ^K-module is induced from a Λ-module. In Section 31, we show that Λ^K is not wild if and only if every generic Λ^K-module is rationally induced from a generic Λ-module. Our proof is derived from the compatibility of the scalar extension with reduction operations and a "scalar extended version" of Crawley's article [20] on the existence and description of generic modules for tame algebras Λ over an algebraically closed field k.

We have included a series of exercises in order to illustrate and enrich the content of these notes. As usual, some of them contain parts of various research works. We have added reference paragraphs at the end of some sections, where we tried to provide fair recognition of previous work on the subject from which these notes developed.

R. Bautista, L. Salmerón and R. Zuazua

Contents

1

t-algebras and differentials

From now on k denotes a fixed ground field. Whenever we consider a k-algebra or a bimodule, we always assume that the field k acts centrally on them. We start with some basic notions and notation, and some elementary remarks.

Definition 1.1. *We say that the k-algebra T is freely generated by the pair (A, V) if A is a k-subalgebra of T, V is a A-A-subbimodule of T, and the following universal property is satisfied: for any k-algebra B, any morphism of k-algebras $A \xrightarrow{\phi_0} B$ and any morphism of A-A-bimodules $V \xrightarrow{\phi_1} B$, where the structure of A-A-bimodule of B is obtained by restriction through ϕ_0, there exists a unique morphism of k-algebras $T \xrightarrow{\phi} B$, which extends both ϕ_0 and ϕ_1.*

Definition 1.2. *Consider a k-algebra A and any A-A-bimodule V. For $i \geq 2$, we write $V^{\otimes i}$ for the tensor product $V \otimes_A V \otimes_A \cdots \otimes_A V$ of i copies of V, and set $V^{\otimes 0} = A$ and $V^{\otimes 1} = V$. The vector space $T_A(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ admits a natural structure of k-algebra with product determined by the canonical isomorphism $V^{\otimes i} \otimes_A V^{\otimes j} \longrightarrow V^{\otimes (i+j)}$. The algebra $T_A(V)$ is called the tensor algebra of V over A.*

Lemma 1.3. *Consider a k-algebra A and any A-A-bimodule V. Then:*

(1) The tensor algebra $T_A(V)$ is freely generated by (A, V).
(2) If the algebra T is freely generated by (A, V), the morphism $T_A(V) \longrightarrow T$ determined by the inclusions of A and V in T is an isomorphism.
(3) Assume that T is an algebra freely generated by (A, V), and that $V = V' \bigoplus V''$ is a bimodule decomposition of the A-A-bimodule V. Then, the subalgebra A' of T generated by $A \cup V'$ is freely generated by (A, V') and T is freely generated by $(A', A'V''A')$.

Proof. (1) and (2) are easy to show. We show (3): let $\pi_1 : V \longrightarrow V'$ be the projection and consider the algebra morphism $T \xrightarrow{\pi} T_A(V')$ determined by the inclusion of A in $T_A(V')$ and π_1. Consider also the morphism $T_A(V') \xrightarrow{\sigma} T$ determined by the inclusion of A and V' in T. Then, clearly, $\pi\sigma$ is the identity map, and the restriction of σ to its image provides the isomorphism $T_A(V') \cong A'$. Now, consider the morphism $T_{A'}(A'V''A') \xrightarrow{\phi} T$ determined by the inclusions of A' and $A'V''A'$ in T. Consider also the morphism of A-A-bimodules $V \xrightarrow{\psi_1} T_{A'}(A'V''A')$, which maps each $v' \in V'$ onto $v' \in A'$, and each $v'' \in V''$ onto $v'' \in A'V''A'$. Then, the morphism $T \xrightarrow{\psi} T_{A'}(A'V''A')$ determined by the inclusion of A in A' and ψ_1 is an inverse for ϕ. □

Definition 1.4. *We say that T is a graded k-algebra if T is a k-algebra which admits a vector space decomposition $T = \oplus_{i \geq 0}[T]_i$ such that $[T]_i[T]_j \subseteq [T]_{i+j}$, for all i, j. Thus, $[T]_0$ is a subalgebra of T and each $[T]_i$ is a $[T]_0$-subbimodule of T. The elements $a \in [T]_i$ are called homogeneous of degree i, and we write* $\deg(a) = i$.

We say that T is a t-algebra if T is a graded k-algebra and T is freely generated by the pair $([T]_0, [T]_1)$.

Remark 1.5. *Consider a k-algebra A and any A-A-bimodule V:*

(1) The tensor algebra $T_A(V)$ with its standard grading given by $[T_A(V)]_i = V^{\otimes i}$ is a graded algebra.

(2) If T is a t-algebra, and we make $A = [T]_0$ and $V = [T]_1$, then there is an isomorphism of graded k-algebras $T_A(V) \cong T$ if we consider the standard grading on $T_A(V)$. In particular, for each n, the product of n elements induces an isomorphism

$$[T]_1 \otimes_{[T]_0} [T]_1 \otimes_{[T]_0} \cdots \otimes_{[T]_0} [T]_1 \xrightarrow{\cong} [T]_n.$$

We often identify both bimodules.

Definition 1.6. *Assume T is a graded k-algebra. Then we say that δ is a differential on T if $\delta : T \to T$ is a linear transformation such that $\delta([T]_i) \subseteq [T]_{i+1}$, for all i, and δ satisfies Leibniz rule: $\delta(ab) = \delta(a)b + (-1)^{\deg(a)}a\delta(b)$, for all homogeneous elements $a, b \in T$.*

Remark 1.7. *If T is a graded k-algebra and δ is a differential on T, then:*

(1) By induction, we obtain the following formula, for any homogeneous elements $t_1, t_2, \ldots, t_n \in T$

$$\delta(t_1 t_2 \cdots t_n) = \sum_{i=1}^{n} (-1)^{(\sum_{j=1}^{i-1} \deg(t_j))} t_1 t_2 \cdots t_{i-1} \delta(t_i) t_{i+1} t_{i+2} \cdots t_n.$$

(2) *The linear map* $\delta^2 : T \longrightarrow T$ *satisfies* $\delta^2(ab) = \delta^2(a)b + a\delta^2(b)$, *for any homogeneous elements* $a, b \in T$. *Again, by induction, we obtain the following formula, for any homogeneous elements* $t_1, \ldots, t_n \in T$

$$\delta^2\,(t_1 t_2 \cdots t_n) = \sum_{i=1}^{n} t_1 t_2 \cdots t_{i-1} \delta^2(t_i) t_{i+1} t_{i+2} \cdots t_n .$$

(3) *From (1), (2) and (1.5), we obtain that if* T *is a t-algebra, the differential* δ *and its square* δ^2 *are determined by their values on* $A = [T]_0$ *and on* $V = [T]_1$. *In particular, we can also derive that* $\delta^2(A) = 0$ *and* $\delta^2(V) = 0$ *imply* $\delta^2 = 0$.

Lemma 1.8. *Let* T *be a t-algebra. Denote* $A = [T]_0$ *and* $V = [T]_1$. *Assume we have a pair of linear maps* $\delta_0 : A \longrightarrow [T]_1$ *and* $\delta_1 : V \longrightarrow [T]_2$ *such that* $\delta_0(ab) = \delta_0(a)b + a\delta_0(b)$, $\delta_1(av) = \delta_0(a)v + a\delta_1(v)$ *and* $\delta_1(va) = \delta_1(v)a - v\delta_0(a)$, *for* $a, b \in A$ *and* $v \in V$. *Then, these maps extend uniquely to a differential* $\delta : T \longrightarrow T$.

Proof. Since T is a t-algebra, freely generated by (A, V), we may assume that $T = T_A(V)$, with its standard grading. We shall define a linear map δ_n from each of the direct summands $V^{\otimes n}$ of T to T. We use the same symbols δ_0 and δ_1 to denote their compositions with the inclusions to T. Then, for $n \geq 2$, define δ_n by the formula

$$\delta_n\,(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (-1)^{(i-1)} v_1 v_2 \cdots v_{i-1} \delta_1(v_i) v_{i+1} v_{i+2} \cdots v_n,$$

where each $v_i \in V$. See Remark (1.7). This formula yields a well-defined linear map $\delta_n : V^{\otimes n} \longrightarrow T$, because $\delta_1(av_i) = \delta_0(a)v_i + a\delta_1(v_i)$ and $\delta_1(v_{i-1}a) = \delta_1(v_{i-1})a - v_{i-1}\delta_0(a)$, for any $a \in A$. Then, there is a linear map $\delta : T \longrightarrow T$ which extends all these maps δ_n. It is clear that $\delta([T]_i) \subseteq [T]_{i+1}$. It remains to show that δ satisfies Leibniz rule. By assumption, δ already satisfies Leibniz rule for products of the form ab, av and va, with $a, b \in A$ and $v \in V$. From this and the definition of δ, it follows that δ satisfies Leibniz rule for products of the form aw and wa, with $a \in A$ and $w \in V^{\otimes n}$. To finish our proof, it is enough to show that given $u_n = \otimes_{s=1}^{n} v_s$ and $w_m = \otimes_{r=1}^{m} v'_r$, with $v_s, v'_r \in V$, then $\delta(u_n \otimes w_m) = \delta(u_n) \otimes w_m + (-1)^{\deg(u_n)} u_n \otimes \delta(w_m)$. This is a straightforward calculation using the definition of δ. □

2

Ditalgebras and modules

In this section, we introduce the basic objects studied in these notes. Namely, ditalgebras and their categories of modules. Its study constitutes a natural generalization of the theory of algebras and their categories of modules. At the same time, it has proved to be a useful tool in establishing some deep results in representation theory of algebras.

Definition 2.1. *A differential t-algebra or ditalgebra $\mathcal{A}$ is by definition a pair $\mathcal{A} = (T, \delta)$, where T is a t-algebra and δ is a differential on T satisfying $\delta^2 = 0$.*

A morphism of ditalgebras $\phi : (T, \delta) \rightarrow (T', \delta')$ is a morphism of k-algebras $\phi : T \rightarrow T'$, satisfying $\phi([T]_i) \subseteq [T']_i$, for all i, and $\delta'\phi = \phi\delta$.

Clearly, we can consider the category of ditalgebras over k, where the morphisms are composed as maps.

Definition 2.2. *The category of modules (or representations) of the ditalgebra $\mathcal{A} = (T, \delta)$, denoted by $\mathcal{A}$-Mod, is defined as follows. Denote by $A = A_{\mathcal{A}} = [T]_0$, a k-subalgebra of T, and by $V = V_{\mathcal{A}} = [T]_1$, an A-A-subbimodule of T. The objects of $\mathcal{A}$-Mod are all the A-modules. Given $M, N \in \mathcal{A}$-Mod, a morphism $f : M \rightarrow N$ in $\mathcal{A}$-Mod is a pair $f = (f^0, f^1)$, with $f^0 \in \mathrm{Hom}_k(M, N)$ and $f^1 \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, N))$ satisfying that*

$$af^0(m) = f^0(am) + f^1(\delta(a))(m),$$

for any $a \in A$ and $m \in M$. The Hom-space in this category is denoted by $\mathrm{Hom}_{\mathcal{A}}(M, N)$. Given $f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(N, L)$ in $\mathcal{A}$-Mod, consider the composition of morphisms of A-A-bimodules

$$V \otimes_A V \xrightarrow{g^1 \otimes f^1} \mathrm{Hom}_k(N, L) \otimes_A \mathrm{Hom}_k(M, N) \xrightarrow{\pi} \mathrm{Hom}_k(M, L),$$

where the last morphism is induced by composition. Since T is a t-algebra, we can identify $V \otimes_A V$ with $[T]_2$. Then the composition gf is defined, for any

$v \in V$, by the following

$$\begin{aligned}(gf)^0 &= g^0 f^0;\\ (gf)^1(v) &= g^0 f^1(v) + g^1(v) f^0 + \pi(g^1 \otimes f^1)(\delta(v)).\end{aligned}$$

*The full subcategory of $\mathcal{A}$-*Mod *consisting of all finite-dimensional objects will be denoted by $\mathcal{A}$-*mod.

Proposition 2.3. *Given a ditalgebra $\mathcal{A}$, the definition above indeed gives rise to a k-category $\mathcal{A}$-*Mod.

Proof. First we see that $gf = ((gf)^0, (gf)^1)$ is indeed a morphism. Clearly, $(gf)^0 \in \mathrm{Hom}_k(M, L)$. Let us verify that $(gf)^1 \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, L))$. For this, take $v \in V$, $a \in A$ and $m \in M$, then

$$\begin{aligned}[(gf)^1(av)](m) &= [g^0 f^1(av)](m) + [g^1(av) f^0](m)\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(av))](m)\\ &= ag^0[f^1(v)(m)] - g^1(\delta(a))[f^1(v)(m)] + [ag^1(v) f^0](m)\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(a)v + a\delta(v))](m)\\ &= ag^0[f^1(v)(m)] - g^1(\delta(a))[f^1(v)(m)] + [ag^1(v) f^0](m)\\ &\quad + [g^1(\delta(a)) f^1(v)](m) + a[\pi(g^1 \otimes f^1)(\delta(v))](m)\\ &= ag^0[f^1(v)(m)] + [ag^1(v) f^0](m)\\ &\quad + a[\pi(g^1 \otimes f^1)(\delta(v))](m)\\ &= a[(gf)^1(v)](m).\end{aligned}$$

Now, take $a \in A$, $v \in V$ and $m \in M$, then

$$\begin{aligned}[(gf)^1(va)](m) &= [g^0 f^1(va)](m) + [g^1(va) f^0](m)\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(va))](m)\\ &= [g^0(f^1(v)a)](m) + (g^1(v)a)[f^0(m)]\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(v)a - v\delta(a))](m)\\ &= g^0[f^1(v)(am)] + (g^1(v))[af^0(m)]\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(v)a - v\delta(a))](m)\\ &= g^0[f^1(v)(am)] + (g^1(v))[f^0(am) + f^1(\delta(a))(m)]\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(v)a)](m) - [g^1(v) f^1(\delta(a))](m)\\ &= g^0[f^1(v)(am)] + (g^1(v))[f^0(am)]\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(v)a)](m)\\ &= [g^0 f^1(v)](am) + [g^1(v) f^0](am)\\ &\quad + [\pi(g^1 \otimes f^1)(\delta(v))](am)\\ &= [(gf)^1(v)](am)\\ &= [(gf)^1(v)a](m).\end{aligned}$$

Finally, $gf \in \mathrm{Hom}_{\mathcal{A}}(M, L)$, because, for $a \in A$ and $m \in M$, we have

$$\begin{aligned}
[(gf)^0](am) &= [g^0 f^0](am) \\
&= g^0[af^0(m) - f^1(\delta(a))(m)] \\
&= g^0[af^0(m)] - g^0[f^1(\delta(a))](m) \\
&= ag^0[f^0(m)] - g^1(\delta(a))[f^0(m)] - g^0[f^1(\delta(a))](m) \\
&= [a(g^0 f^0)](m) \\
&\quad - [g^0 f^1(\delta(a)) + g^1(\delta(a)) f^0 + \pi(g^1 \otimes f^1)(\delta^2(a))][m] \\
&= [a(g^0 f^0)](m) - (gf)^1(\delta(a))[m] \\
&= [a(gf)^0](m) - (gf)^1(\delta(a))[m].
\end{aligned}$$

Clearly, for each $M \in \mathcal{A}$-Mod, the morphism $I_M = (I_M, 0)$ plays the role of an identity. Now we show that the composition is associative (it is clearly bilinear). Consider the morphisms $M \xrightarrow{f} N$, $N \xrightarrow{g} L$ and $L \xrightarrow{h} K$ in $\mathcal{A}$-Mod.

It is clear that $[h(gf)]^0 = [(hg)f]^0$. In order to show that $[h(gf)]^1 = [(hg)f]^1$, having in mind our identification $V \otimes_A V = [T]_2$, consider $v \in V$ and let $\delta(v) = \sum_a u_a \otimes w_a$, with $u_a, w_a \in V$. Assume that, for each a

$$\delta(u_a) = \sum_b u^1_{ab} \otimes u^2_{ab} \quad \text{and} \quad \delta(w_a) = \sum_c w^1_{ac} \otimes w^2_{ac}.$$

Then

$$\begin{aligned}
[h(gf)]^1(v) &= h^0(gf)^1(v) + h^1(v)(gf)^0 + \textstyle\sum_a h^1(u_a)(gf)^1(w_a) \\
&= h^0[g^0 f^1(v) + g^1(v) f^0 + \textstyle\sum_a g^1(u_a) f^1(w_a)] + h^1(v) g^0 f^0 \\
&\quad + \textstyle\sum_a h^1(u_a)[g^0 f^1(w_a) + g^1(w_a) f^0 + \sum_c g^1(w^1_{ac}) f^1(w^2_{ac})],
\end{aligned}$$

and

$$\begin{aligned}
[(hg)f]^1(v) &= (hg)^0 f^1(v) + (hg)^1(v) f^0 + \textstyle\sum_a (hg)^1(u_a) f^1(w_a) \\
&= h^0 g^0 f^1(v) + [h^0 g^1(v) + h^1(v) g^0 + \textstyle\sum_a h^1(u_a) g^1(w_a)] f^0 \\
&\quad + \textstyle\sum_a [h^0 g^1(u_a) + h^1(u_a) g^0 + \sum_b h^1(u^1_{ab}) g^1(u^2_{ab})] f^1(w_a).
\end{aligned}$$

Then, we have to show that

$$\sum_{a,c} h^1(u_a) g^1(w^1_{ac}) f^1(w^2_{ac}) = \sum_{a,b} h^1(u^1_{ab}) g^1(u^2_{ab}) f^1(w_a).$$

We have

$$\sum_a \delta(u_a) w_a + (-1)^{\deg(u_a)} u_a \delta(w_a) = \delta\left(\sum_a u_a \otimes w_a\right) = \delta^2(v) = 0,$$

which implies the following equality in $V \otimes_A V \otimes_A V = [T]_3$ (we use again that T is a t-algebra)

$$\sum_{a,c} u_a \otimes w^1_{ac} \otimes w^2_{ac} = \sum_{a,b} u^1_{ab} \otimes u^2_{ab} \otimes w_a.$$

Then, the following composition applied to the last equality gives the desired result

$$V \otimes_A V \otimes_A V \xrightarrow{h^1 \otimes g^1 \otimes f^1} \mathrm{Hom}_k(L, K) \otimes_A \mathrm{Hom}_k(N, L) \otimes_A \mathrm{Hom}_k(M, N)$$
$$\xrightarrow{\pi} \mathrm{Hom}_k(M, K).$$

□

Lemma 2.4. *Any morphism of ditalgebras $\phi : \mathcal{A} \longrightarrow \mathcal{A}'$ induces, by restriction, a functor $F_\phi : \mathcal{A}'$-Mod$\longrightarrow \mathcal{A}$-Mod. To give the explicit formula on morphisms, denote by $A = A_{\mathcal{A}}$ and $V = V_{\mathcal{A}}$, and with A' and V' the corresponding objects for the ditalgebra $\mathcal{A}'$; consider also the morphisms $\phi_0 : A \to A'$ and $\phi_1 : V \to V'$, induced by ϕ. Then, if $M \in \mathcal{A}'$-Mod, $F_\phi(M)$ is the A-module obtained from M by restriction of scalars through ϕ_0. The receipt on morphisms is given, for any $(f^0, f^1) \in \mathrm{Hom}_{\mathcal{A}'}(M, N)$, by $F_\phi(f^0, f^1) = (f^0, f^1\phi_1)$.*

If ϕ is surjective, then F_ϕ is faithful and injective on objects. Moreover, if $\phi' : \mathcal{A}' \longrightarrow \mathcal{A}''$ is another morphism of ditalgebras, then $F_{\phi'\phi} = F_\phi F_{\phi'}$.

Proof. We first show that $F_\phi(f^0, f^1) \in \mathrm{Hom}_{\mathcal{A}}(F_\phi(M), F_\phi(N))$, whenever we have $(f^0, f^1) \in \mathrm{Hom}_{\mathcal{A}'}(M, N)$. For $m \in M$ and $a \in A$, we have

$$\begin{aligned} F_\phi(f)^0(am) &= f^0(am) \\ &= f^0[\phi_0(a)m] \\ &= \phi_0(a)f^0(m) - f^1(\delta'(\phi_0(a)))[m] \\ &= af^0(m) - f^1\phi_1(\delta(a))[m] \\ &= aF_\phi(f)^0[m] - F_\phi(f)^1(\delta(a))[m]. \end{aligned}$$

In order to show that F_ϕ preserves the composition, take $f \in \mathrm{Hom}_{\mathcal{A}'}(M, N)$ and $g \in \mathrm{Hom}_{\mathcal{A}'}(N, L)$. Therefore, $[F_\phi(gf)]^0 = (gf)^0 = g^0 f^0 = [F_\phi(g)F_\phi(f)]^0$. Moreover, for $v \in V$ with $\delta(v) = \sum_i v_i^1 \otimes v_i^2$, we have $\delta'(\phi(v)) = \phi\delta(v) = \sum_i \phi(v_i^1) \otimes \phi(v_i^2)$. Therefore

$$\begin{aligned} \left[F_\phi(g)F_\phi(f)\right]^1(v) &= F_\phi(g)^0 F_\phi(f)^1(v) + F_\phi(g)^1(v)F_\phi(f)^0 \\ &\quad + \pi(F_\phi(g)^1 \otimes F_\phi(f)^1)(\delta(v)) \\ &= g^0 f^1(\phi(v)) + g^1(\phi(v))f^0 + \textstyle\sum_i g^1(\phi(v_i^1))f^1(\phi(v_i^2)) \\ &= g^0 f^1(\phi(v)) + g^1(\phi(v))f^0 + \pi(g^1 \otimes f^1)(\delta'(\phi(v))) \\ &= (gf)^1(\phi(v)) \\ &= \left[F_\phi(gf)\right]^1(v). \end{aligned}$$

We have seen that $F_\phi(gf) = F_\phi(g)F_\phi(f)$. Clearly, F_ϕ preserves identities. □

Remark 2.5. *Whenever $\mathcal{A}$ is a ditalgebra, there is a canonical embedding*

$$L = L_{\mathcal{A}} : A\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod},$$