Introduction to

Abstract Algebra

W. Keith Nicholson

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University of Calgary

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Preface

This book is a self-contained introduction to the basic structures of abstract algebra: groups, rings, and fields. It is designed to be used in a one- or two-semester course (see the chapter summaries that follow) and may also serve for self-study. In addition, it contains several optional sections on special topics and applications.

Because many students will not have had much experience with abstract thinking, I introduce a number of important concrete examples (complex numbers, two-by-two matrices, integers modulo n, and permutations) at the beginning and refer to them throughout the book. I chose these examples for their importance and intrinsic interest and also because the student can do actual computations almost immediately even though the examples are, in the student's view, quite abstract. Thus they provide a bridge to the abstract theory and serve as prototype examples of the abstract structures themselves. For example, the student will encounter composition and inverses of permutations before having to fit these notions into the general framework of group theory.

I also emphasize the axiomatic development of these structures. Modern algebra provides one of the best illustrations of the power of abstraction to strip concrete examples of nonessential aspects, so as to reveal similarities between ostensibly different objects and to suggest that a theorem about one structure may have an analogue for a different structure. Achieving this sort of facility with abstraction is one of the goals of the book, which goes hand in hand with another goal: to teach the student how to do proofs. The proofs of most theorems are at least as important for the techniques as for the theorems themselves. Hence, whenever possible, I introduce techniques and use them in examples before giving them in the general case as a proof. This approach explains the large number of examples (nearly 500) in the book.

Of course, a generous supply of exercises is essential if this subject is to have a lasting impact on students, and the book contains nearly 1500 exercises (many with separate parts). Computational exercises appear first for the most part, and the exercises are more or less in ascending order of difficulty. Hints are given for the less straightforward problems. On the whole, I do not use exercises to develop results that are needed later in the text, so not all exercises need to be solved in order to continue with the book. Answers are provided to odd-numbered (parts of) computational exercises and to selected theoretical exercises.

An increasing number of students of abstract algebra come from outside mathematics, and, for many of them, the lure of pure abstraction is not as strong as for mathematicians. Therefore, I include applications of the theory that make the subject more meaningful and lively for these students (and for the mathematicians!). These include cryptography, linear codes, cyclic and BCH codes, and combinatorics, as well as "theoretical" applications within mathematics, such as the impossibility of the classical geometric constructions. The inclusion of short historical notes and biographies should help the reader put the material into perspective. In the same spirit, some classical "gems" appear in optional sections (one example is the elegant proof of the fundamental theorem of algebra in Section 6.6, using the structure theorem for symmetric polynomials). In addition, I convey the modern flavor of the subject by mentioning some of the unsolved problems in abstract algebra and by occasionally stating more advanced theorems that extend beyond the results of the book.

Apart from that, the material is quite standard. The aim is to reveal the basic facts about groups, rings, and fields and to give the student the working tools for further study. The level of exposition rises slowly throughout the text, and no prior knowledge of abstract algebra is required. Even linear algebra is not needed. Except for a few well-marked instances, the aspects of linear algebra that are needed are developed in the text. Calculus is completely unnecessary. Some preliminary topics that are needed are covered in Chapter 0 (including complex numbers and two-by-two matrices).

Although the chapters are necessarily arranged in a linear order, this is by no means true of the contents, and the student (as well as the instructor) should keep the chapter dependency diagram on page xiv in mind. A glance at that diagram shows that Chapters 1–4 are the core of the book but that there is enough flexibility in the remaining chapters to accommodate an instructor who wants to include more than just the basics. The jump from Chapter 6 to Chapter 10 deserves mention. The student has a choice at the end of Chapter 6: either change the subject and do some more group theory or continue with fields in Chapter 10 (solvable groups are adequately reviewed in Section 10.3, so Chapter 9 is not necessary).

The chapter summaries that follow and the chapter dependency diagram can assist in the preparation of a course syllabus. Our course of 36 lectures touches Sections 0.3 and 0.4 lightly and then covers Chapters 1–4, except for Sections 1.5, 2.11, 3.5, and 4.4–4.6.

FEATURES

This book offers the following significant features:

- Self-contained treatment
- Preliminary material available in Chapter 0 for self-study or review
- Integers modulo n and permutations done first as a bridge to abstraction
- Nearly 500 worked examples to guide the student
- Wide variety of exercises with selected answers
- Gradual increase in level throughout the text
- Applications to number theory, combinatorics, geometry, coding, and equations
- Flexibility in syllabus construction and choice of optional topics (see chapter dependency diagram)
- Historical notes and biographies
- Several special topics (for example, symmetric polynomials, nilpotent groups, and finite-dimensional algebras)
- Solutions manual containing answers or solutions to all exercises

CHAPTER SUMMARIES

Chapter 0 ***Preliminaries*** This chapter should be viewed as a primer on mathematics, as it consists of material essential to any mathematics major. The treatment is self-contained. I personally let students read Sections 0.1 and 0.2 on their own, I touch the highlights in Sections 0.3 and 0.4, and review Section 0.5 briefly (our students have had elementary linear algebra so I omit Section 0.6).

Chapter 1 ***Integers and Permutations.*** This chapter covers the fundamental properties of the integers and the two prototype examples of rings and groups: the integers modulo n and the permutation group S_n. These are done naïvely and allow the students to do ring and group calculations in a concrete setting.

Chapter 2 ***Groups.*** This chapter gives the basic facts of group theory, including cyclic groups, Lagrange's theorem, normal subgroups, factor groups, homomorphisms, and the isomorphism theorem. An optional application to linear codes is included. Section 2.7 on groups of motions is also optional.

Chapter 3 ***Rings.*** The basic properties of rings are developed: integral domains, characteristic, rings of quotients, ideals, factorization, homomorphisms, and the isomorphism theorem. The analogy between these notions and the corresponding group-theoretic concepts is noted.

Chapter 4 ***Polynomials.*** After the usual elementary facts are developed, irreducible polynomials are discussed and the unique factorization of polynomials over a field is proved. Then factor rings of polynomials over a field are described and

some finite fields are constructed. In an optional section, symmetric polynomials are discussed and the fundamental structure theorem is proved.

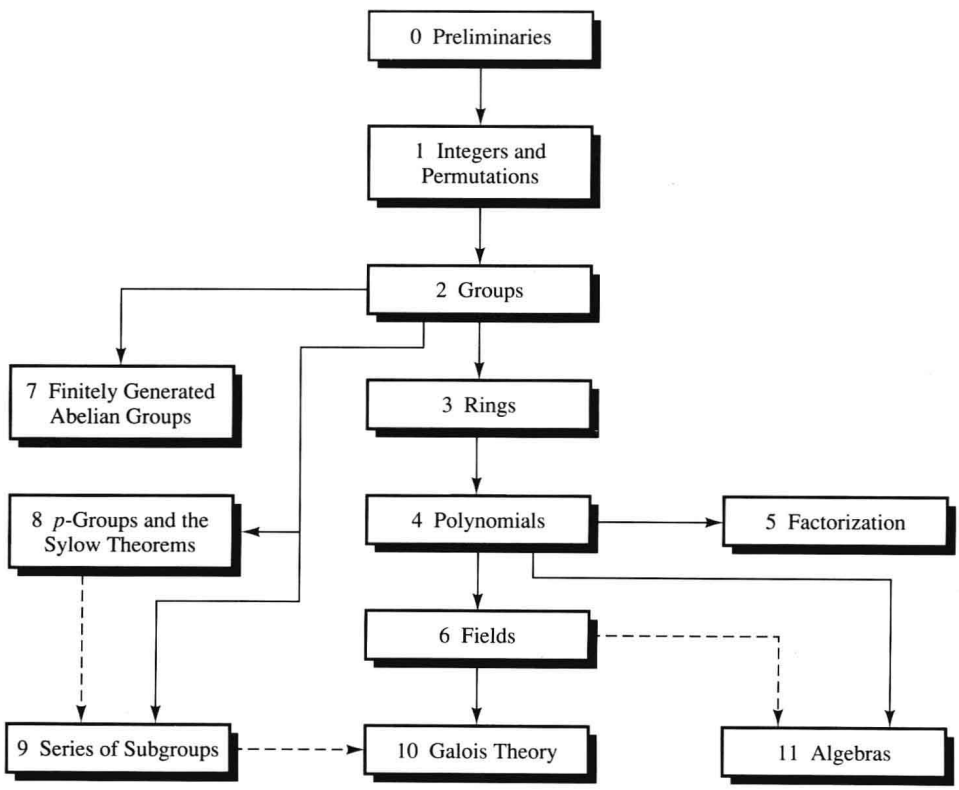
Chapter 5 ***Factorization in Integral Domains.*** Unique factorization domains are characterized in terms of irreducibles and primes, and the fact that the property is inherited by polynomial rings is proved. Principal ideal domains and Euclidean domains are discussed. The chapter is self-contained, and the material presented is not required elsewhere.

Chapter 6 ***Fields.*** After a minimal amount of vector space theory is developed, splitting fields are constructed and used to completely describe finite fields. This topic is a direct continuation of Section 4.3. In optional sections, the classical results on geometric constructions are derived, the fundamental theorem of algebra is proved, and the theory of cyclic (and BCH) codes is developed.

Chapter 7 ***Finitely Generated Abelian Groups.*** The fundamental theorem for finite abelian groups is proved and then extended to the finitely generated case. This material is self-contained and is not required elsewhere.

Chapter 8 ***p-Groups and the Sylow Theorems.*** This chapter is a direct continuation of Section 2.10. The class equation is given and is used to prove Cauchy's theorem and to derive the basic properties of p-groups. Then group actions

Chapter Dependency Diagram



A broken arrow indicates minor dependency.

are introduced, motivated by the class equation and an extended Cayley theorem, and used to prove the Sylow theorems. An optional application to combinatorics is also included.

Chapter 9. ***Series of Subgroups.*** The chapter begins with composition series and the Jordan–Hölder theorem. Then solvable series are introduced, including the derived series, and the basic properties of solvable groups are proved. Finally, central series are discussed and nilpotent groups are characterized as direct products of p-groups. Sections 9.1 and 9.2 depend on Chapter 8 only in the statement of some results, and so could be studied before Chapter 8.

Chapter 10 ***Galois Theory.*** Galois groups of field extensions are defined, separable elements are introduced, and the main theorem of Galois theory is proved. Then the fact that polynomials of degree 5 or more are not solvable in radicals is proved. All this requires only Chapter 6 (the reference to solvable groups in Section 10.3 is adequately reviewed there). Finally, cyclotomic polynomials are discussed and used, with the class equation, to prove Wedderburn's theorem that every finite division ring is a field.

Chapter 11 ***Algebras.*** Finite-dimensional algebras are defined and the regular representation is given. Then the Wedderburn structure theorems are derived. Chapter 6 is needed only for the notion of dimension in a vector space.

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W. Keith Nicholson

Notation used in the Text

Symbol	Description	First Used
$\Rightarrow$	implication	2
$\neg$	negation of a statement	3
$\Leftrightarrow$	logical equivalence	4
$\in$	set membership	6
$\subseteq$	set containment	7
$\subset$	proper set containment	7
$\mathbb{N}$	set of natural numbers	7
$\mathbb{Z}$	set of integers	7
$\mathbb{Q}$	set of rational numbers	7
$\mathbb{R}$	set of real numbers	7
$\mathbb{C}$	set of complex numbers	7
$\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$	positive elements in these sets	7
$\|A\|$	number of elements in set A	8
$\varnothing$	empty set	8
$\cup$	union of sets	8, 9
$\cap$	intersection of sets	8, 9
A^c	complement of a set A	10
$A - B$	difference set	11
(a, b)	ordered pair	11
$A \times B$	Cartesian product of sets A and B	11, 12
$(a_1, a_2, \ldots, a_n)$	ordered n-tuple	12
$\alpha: A \to B$ $A \xrightarrow{\alpha} B$	mapping α from A to B	14, 15
$\alpha(x)$	image of x under mapping α	15
$\operatorname{im} \alpha$	image of the mapping α	17
$\alpha\beta$	composite of mappings α and β	18
1_A	identity mapping on set A	19
α^{-1}	inverse of mapping α	20
$\equiv$	equivalence relation	24
$[a]$	equivalence class of a	24
$A_{\equiv}$	quotient set of an equivalence $\equiv$	27
$\operatorname{re} z$	real part of z	31
$\operatorname{im} z$	imaginary part of z	31
$\bar{z}$	conjugate of a complex number z	32
$\|z\|$	absolute value of z	32
$e^{i\theta}$	notation for $\cos\theta + i \sin\theta$	35

Symbol	Description	First Used
$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$	$m \times n$ matrix	40
I, I_n	identity matrix	43
$\det A$	determinant of a square matrix A	45
$n!$	n factorial	51
$\binom{n}{r}$	binomial coefficient	52
$d \mid n$	d is a divisor of n	61
$\gcd(m, n)$ $\gcd(n_1, \ldots, n_r)$	greatest common divisor	62, 68
$\operatorname{lcm}(m, n)$ $\operatorname{lcm}(n_1, \ldots, n_r)$	least common multiple	68
$a \equiv b \pmod{n}$	congruence modulo n	74
$\bar{a}$	residue class of an integer a	74
$\mathbb{Z}_n$	integers modulo n	74
S_n	symmetric group of degree n	87, 90
$\begin{pmatrix} 1 & 2 \cdots & n \\ \sigma 1 & \sigma 2 \cdots & \sigma n \end{pmatrix}$	permutation σ in S_n	87
ε	identity permutation in S_n	88
$(k_1\, k_2 \cdots k_n)$	cycle permutation in S_n	92
$\operatorname{sgn} \sigma$	sign of permutation σ	101
$\mathbb{C}^0$	circle group	115
U_n	group of nth roots of unity	115
M^*	groups of units of M	116
S_X	group of permutations of X	117
$GL_n(R)$	general linear group over R	118
C_n	cyclic group of order n	121
K_4	Klein 4-group	123
$SL_n(R)$	special linear group over R	127
$Z(G)$	center of group G	128
$\langle g \rangle$	cyclic subgroup generated by g	134
$\lvert g \rvert$	order of group element g	137
$\langle X \rangle$	subgroup generated by X	143
$\operatorname{aut} G$	automorphism group of G	151
$\operatorname{inn} G$	inner automorphism group of G	152
Ha, aH	right, left cosets	159
$\lvert G : H \rvert$	index of subgroup H in G	163

Symbol	Description	First Used
D_n	dihedral group	164
$H \lhd G$	H is a normal subgroup of G	177
Q	quaternion group	181
G/K	factor group of G by K	189
G'	derived (commutator) subgroup of G	192
$\ker \alpha$	kernel of α	203
B^n	set of binary n-tuples	214
$F(X, R)$	ring of functions $X \to R$	234
$M_n(R)$	ring of $n \times n$ matrices over R	236
$R[x]$	ring of polynomials in x over R	237
$\deg p(x)$	degree of $p(x)$	237
$\operatorname{char} R$	characteristic of a ring R	238
$\mathbb{H}$	quaternions	254
R^1	ring extension of a general ring R	282
$a \sim b$	associates in an integral domain	362
$\operatorname{span}\{v_1, \ldots, v_n\}$	space spanned by $v_1, \ldots, v_n$	391
$\dim V$	dimension of vector space V	394
$[E:F]$	dimension of E over a subfield F	399
$F(u_1, \ldots, u_n)$	field generated over F by $u_1, \ldots, u_n$	400
$\mathrm{GF}(p^n)$	Galois field of order p^n	422
class a	conjugacy class of a	475
$N(X)$	normalizer of X	476
core H	core of a subgroup H	484
$G \cdot x$	orbit of x generated by G	488
$S(x)$	stabilizer of x	488
length G	composition length of G	511
$\mathrm{gal}(E:F)$	Galois group of E over F	534

Contents

† This is an optional section.

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