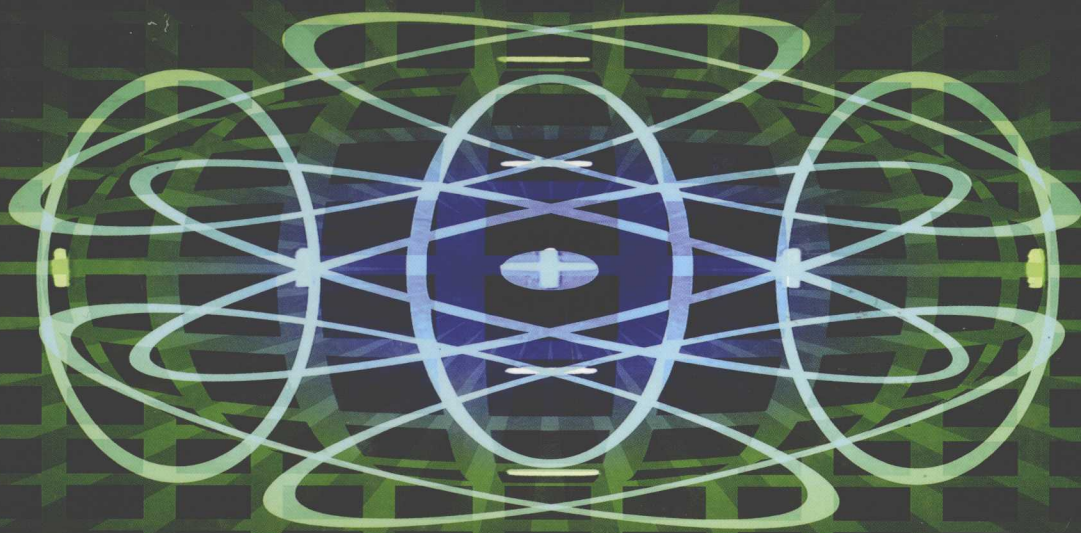


The Finite Element Method for **Electromagnetic Modeling**

Edited by Gérard Meunier



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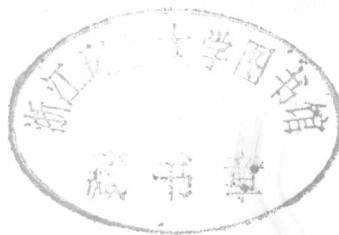
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First published in France in three volumes by Hermes Science/Lavoisier entitled "Electromagnétisme et éléments finis Vol. 1, 2 et 3"

First published in Great Britain and the United States in 2008 by ISTE Ltd and John Wiley & Sons, Inc.

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Library of Congress Cataloging-in-Publication Data

Electromagnétisme et éléments finis. English

The finite element method for electromagnetic modeling / edited by Gérard Meunier.

p. cm.

Includes bibliographical references and index.

ISBN: 978-1-84821-030-1

1. Electromagnetic devices--Mathematical models. 2. Electromagnetism--Mathematical models. 3. Engineering mathematics. 4. Finite element method. I. Meunier, Gérard.

TK7872.M25E4284 2008

621.301'51825--dc22

2007046086

British Library Cataloguing-in-Publication Data

A CIP record for this book is available from the British Library

ISBN: 978-1-84821-030-1

Printed and bound in Great Britain by CPI Antony Rowe, Chippenham, Wiltshire.



Cert no. SGS-COC-2953
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Table of Contents

Chapter 1. Introduction to Nodal Finite Elements	1
Jean-Louis COULOMB	
1.1. Introduction	1
1.1.1. The finite element method	1
1.2. The 1D finite element method	2
1.2.1. A simple electrostatics problem	2
1.2.2. Differential approach	3
1.2.3. Variational approach	4
1.2.4. First-order finite elements	6
1.2.5. Second-order finite elements	9
1.3. The finite element method in two dimensions	10
1.3.1. The problem of the condenser with square section.	10
1.3.2. Differential approach	12
1.3.3. Variational approach	14
1.3.4. Meshing in first-order triangular finite elements	15
1.3.5. Finite element interpolation	17
1.3.6. Construction of the system of equations by the Ritz method.	19
1.3.7. Calculation of the matrix coefficients	21
1.3.8. Analysis of the results	25
1.3.9. Dual formations, framing and convergence	42
1.3.10. Resolution of the nonlinear problems.	44
1.3.11. Alternative to the variational method: the weighted residues method	45
1.4. The reference elements.	47
1.4.1. Linear reference elements	48
1.4.2. Surface reference elements.	49
1.4.3. Volume reference elements	52
1.4.4. Properties of the shape functions	53
1.4.5. Transformation from reference coordinates to domain coordinates .	54
1.4.6. Approximation of the physical variable	56

1.4.7. Numerical integrations on the reference elements	60
1.4.8. Local Jacobian derivative method	63
1.5. Conclusion	66
1.6. References	66
Chapter 2. Static Formulations: Electrostatic, Electrokinetic, Magnetostatics	69
Patrick DULAR and Francis PIRIOU	
2.1. Problems to solve	70
2.1.1. Maxwell's equations	70
2.1.2. Behavior laws of materials	71
2.1.3. Boundary conditions	71
2.1.4. Complete static models	74
2.1.5. The formulations in potentials	75
2.2. Function spaces in the fields and weak formulations	82
2.2.1. Integral expressions: introduction	82
2.2.2. Definitions of function spaces	82
2.2.3. Tonti diagram: synthesis scheme of a problem	84
2.2.4. Weak formulations	86
2.3. Discretization of function spaces and weak formulations	91
2.3.1. Finite elements	91
2.3.2. Sequence of discrete spaces	93
2.3.3. Gauge conditions and source terms in discrete spaces	106
2.3.4. Weak discrete formulations	109
2.3.5. Expression of global variables	114
2.4. References	115
Chapter 3. Magnetodynamic Formulations	117
Zhuoxiang REN and Frédéric BOUILLAULT	
3.1. Introduction	117
3.2. Electric formulations	119
3.2.1. Formulation in electric field	119
3.2.2. Formulation in combined potentials $a - \Psi$	120
3.2.3. Comparison of the formulations in field and in combined potentials	121
3.3. Magnetic formulations	123
3.3.1. Formulation in magnetic field	123
3.3.2. Formulation in combined potentials $t - \phi$	124
3.3.3. Numerical example	125
3.4. Hybrid formulation	127
3.5. Electric and magnetic formulation complementarities	128
3.5.1. Complementary features	128
3.5.2. Concerning the energy bounds	129
3.5.3. Numerical example	129

3.6. Conclusion	133
3.7. References	134
Chapter 4. Mixed Finite Element Methods in Electromagnetism	139
Bernard BANDELIER and Françoise RIOUX-DAMIDAU	
4.1. Introduction	139
4.2. Mixed formulations in magnetostatics.	140
4.2.1. Magnetic induction oriented formulation	141
4.2.2. Formulation oriented magnetic field	144
4.2.3. Formulation in induction and field	146
4.2.4. Alternate case	147
4.3. Energy approach: minimization problems, searching for a saddle-point.	147
4.3.1. Minimization of a functional calculus related to energy	147
4.3.2. Variational principle of magnetic energy	149
4.3.3. Searching for a saddle-point	151
4.3.4. Functional calculus related to the constitutive relationship.	154
4.4. Hybrid formulations	154
4.4.1. Magnetic induction oriented hybrid formulation	154
4.4.2. Hybrid formulation oriented magnetic field.	156
4.4.3. Mixed hybrid method	157
4.5. Compatibility of approximation spaces – inf-sup condition	157
4.5.1. Mixed magnetic induction oriented formulation	158
4.5.2. Mixed formulation oriented magnetic field	160
4.5.3. General case	160
4.6. Mixed finite elements, Whitney elements.	161
4.6.1. Magnetic induction oriented formulation	162
4.6.2. Magnetic field oriented formulation	163
4.7. Mixed formulations in magnetodynamics.	164
4.7.1. Magnetic field oriented formulation	164
4.7.2. Formulation oriented electric field	167
4.8. Solving techniques	167
4.8.1. Penalization methods	168
4.8.2. Algorithm using the Schur complement	171
4.9. References	174
Chapter 5. Behavior Laws of Materials	177
Frédéric BOUILLAUT, Afef KEDOUS-LEBOUC, Gérard MEUNIER, Florence OSSART and Francis PIRIOU	
5.1. Introduction	177
5.2. Behavior law of ferromagnetic materials	178
5.2.1. Definitions	178
5.2.2. Hysteresis and anisotropy	179
5.2.3. Classification of models dealing with the behavior law	180
5.3. Implementation of nonlinear behavior models	183

5.3.1. Newton method	183
5.3.2. Fixed point method	187
5.3.3. Particular case of a behavior with hysteresis	191
5.4. Modeling of magnetic sheets	192
5.4.1. Some words about magnetic sheets	192
5.4.2. Example of stress in the electric machines	192
5.4.3. Anisotropy of sheets with oriented grains	194
5.4.4. Hysteresis and dynamic behavior under uniaxial stress	200
5.4.5. Determination of iron losses in electric machines: nonlinear isotropic finite element modeling and calculation of the losses <i>a posteriori</i>	209
5.4.6. Conclusion	215
5.5. Modeling of permanent magnets	216
5.5.1. Introduction.	216
5.5.2. Magnets obtained by powder metallurgy	216
5.5.3. Study of linear anisotropic behavior	218
5.5.4. Study of nonlinear behavior	220
5.5.5. Implementation of the model in finite element software	223
5.5.6. Validation: the experiment by Joel Chavanne	224
5.5.7. Conductive magnet subjected to an AC field	225
5.6. Modeling of superconductors	226
5.6.1. Introduction.	226
5.6.2. Behavior of superconductors	227
5.6.3. Modeling of electric behavior of superconductors	230
5.6.4. Particular case of the Bean model.	232
5.6.5. Examples of modeling	237
5.7. Conclusion	240
5.8. References	241
Chapter 6. Modeling on Thin and Line Regions	245
Christophe GUÉRIN	
6.1. Introduction	245
6.2. Different special elements and their interest	245
6.3. Method for taking into account thin regions without potential jump	249
6.4. Method for taking into account thin regions with potential jump	250
6.4.1. Analytical integration method	251
6.4.2. Numerical integration method.	252
6.5. Method for taking thin regions into account	255
6.6. Thin and line regions in magnetostatics.	256
6.6.1. Thin and line regions in magnetic scalar potential formulations.	256
6.6.2. Thin and line regions in magnetic vector potential formulations	257
6.7. Thin and line regions in magnetoharmonics	257
6.7.1. Solid conducting regions presenting a strong skin effect	258
6.7.2. Thin conducting regions	265

6.8. Thin regions in electrostatic problems: “electric harmonic problems” and electric conduction problems	272
6.9. Thin thermal regions	272
6.10. References	273
Chapter 7. Coupling with Circuit Equations	277
G�rard MEUNIER, Yvan LEFEVRE, Patrick LOMBARD and Yann LE FLOCH	
7.1. Introduction	277
7.2. Review of the various methods of setting up electric circuit equations	278
7.2.1. Circuit equations with nodal potentials	278
7.2.2. Circuit equations with mesh currents.	279
7.2.3. Circuit equations with time integrated nodal potentials	280
7.2.4. Formulation of circuit equations in the form of state equations	281
7.2.5. Conclusion on the methods of setting up electric equations	283
7.3. Different types of coupling	284
7.3.1. Indirect coupling.	285
7.3.2. Integro-differential formulation	285
7.3.3. Simultaneous resolution	285
7.3.4. Conclusion	285
7.4. Establishment of the “current-voltage” relations	286
7.4.1. Insulated massive conductor with two ends: basic assumptions and preliminary relations	286
7.4.2. Current-voltage relations using the magnetic vector potential	287
7.4.3. Current-voltage relations using magnetic induction	288
7.4.4. Wound conductors.	290
7.4.5. Losses in the wound conductors	291
7.5. Establishment of the coupled field and circuit equations.	292
7.5.1. Coupling with a vector potential formulation in 2D	292
7.5.2. Coupling with a vector potential formulation in 3D	303
7.5.3. Coupling with a scalar potential formulation in 3D	310
7.6. General conclusion	317
7.7. References	318
Chapter 8. Modeling of Motion: Accounting for Movement in the Modeling of Magnetic Phenomena	321
Vincent LECONTE	
8.1. Introduction	321
8.2. Formulation of an electromagnetic problem with motion	322
8.2.1. Definition of motion	322
8.2.2. Maxwell equations and motion	325
8.2.3. Formulations in potentials	329
8.2.4. Eulerian approach	335
8.2.5. Lagrangian approach	338

8.2.6. Example application	342
8.3. Methods for taking the movement into account	346
8.3.1. Introduction	346
8.3.2. Methods for rotating machines	346
8.3.3. Coupling methods without meshing and with the finite element method	348
8.3.4. Coupling of boundary integrals with the finite element method	350
8.3.5. Automatic remeshing methods for large distortions	355
8.4. Conclusion	362
8.5. References	363
Chapter 9. Symmetric Components and Numerical Modeling	369
Jacques LOBRY, Eric NENS and Christian BROCHE	
9.1. Introduction	369
9.2. Representation of group theory	371
9.2.1. Finite groups	371
9.2.2. Symmetric functions and irreducible representations	374
9.2.3. Orthogonal decomposition of a function	378
9.2.4. Symmetries and vector fields	379
9.3. Poisson's problem and geometric symmetries	384
9.3.1. Differential and integral formulations	384
9.3.2. Numerical processing	387
9.4. Applications	388
9.4.1. 2D magnetostatics	388
9.4.2. 3D magnetodynamics	394
9.5. Conclusions and future work	403
9.6. References	404
Chapter 10. Magneto-thermal Coupling	405
Mouloud FÉLIACHI and Javad FOULADGAR	
10.1. Introduction	405
10.2. Magneto-thermal phenomena and fundamental equations	406
10.2.1. Electromagnetism	406
10.2.2. Thermal	408
10.2.3. Flow	408
10.3. Behavior laws and couplings	409
10.3.1. Electrmagnetic phenomena	409
10.3.2. Thermal phenomena	409
10.3.3. Flow phenomena	409
10.4. Resolution methods	409
10.4.1. Numerical methods	409
10.4.2. Semi-analytical methods	410
10.4.3. Analytical-numerical methods	411
10.4.4. Magneto-thermal coupling models	411

10.5. Heating of a moving work piece	413
10.6. Induction plasma	417
10.6.1. Introduction	417
10.6.2. Inductive plasma installation	418
10.6.3. Mathematical models	418
10.6.4. Results	426
10.6.5. Conclusion	427
10.7. References	428
Chapter 11. Magneto-mechanical Modeling	431
Yvan LEFEVRE and Gilbert REYNE	
11.1. Introduction	431
11.2. Modeling of coupled magneto-mechanical phenomena	432
11.2.1. Modeling of mechanical structure	433
11.2.2. Coupled magneto-mechanical modeling	437
11.2.3. Conclusion	442
11.3. Numerical modeling of electromechanical conversion in conventional actuator	442
11.3.1. General simulation procedure	443
11.3.2. Global magnetic force calculation method	444
11.3.3. Conclusion	447
11.4. Numerical modeling of electromagnetic vibrations	447
11.4.1. Magnetostriction vs. magnetic forces	447
11.4.2. Procedure for simulating vibrations of magnetic origin	449
11.4.3. Magnetic forces density	449
11.4.4. Case of rotating machine teeth	452
11.4.5. Magnetic response modeling	453
11.4.6. Model superposition method	455
11.4.7. Conclusion	458
11.5. Modeling strongly coupled phenomena	459
11.5.1. Weak coupling and strong coupling from a physical viewpoint	459
11.5.2. Weak coupling or strong coupling problem from a numerical modeling analysis	460
11.5.3. Weak coupling and intelligent use of software tools	461
11.5.4. Displacement and deformation of a magnetic system	463
11.5.5. Structural modeling based on magnetostrictive materials	465
11.5.6. Electromagnetic induction launchers	469
11.6. Conclusion	470
11.7. References	471
Chapter 12. Magnetohydrodynamics: Modeling of a Kinematic Dynamo	477
Franck PLUNIAN and Philippe MASSÉ	
12.1. Introduction	477
12.1.1. Generalities	477

12.1.2. Maxwell's equations and Ohm's law	481
12.1.3. The induction equation	482
12.1.4. The dimensionless equation	483
12.2. Modeling the induction equation using finite elements	485
12.2.1. Potential (A, ϕ) quadric-vector formulation	485
12.2.2. $2D^{1/2}$ quadri-vector potential formulation	488
12.3. Some simulation examples.	491
12.3.1. Screw dynamo (Ponomarenko dynamo)	491
12.3.2. Two-scale dynamo without walls (Roberts dynamo).	495
12.3.3. Two-scale dynamo with walls	498
12.3.4. A dynamo at the industrial scale.	502
12.4. Modeling of the dynamic problem	503
12.5. References	504
Chapter 13. Mesh Generation	509
Yves DU TERRAIL COUVAT, François-Xavier ZGAINSKI and Yves MARÉCHAL	
13.1. Introduction	509
13.2. General definition.	510
13.3. A short history	512
13.4. Mesh algorithms.	512
13.4.1. The basic algorithms.	512
13.4.2. General mesh algorithms	518
13.5. Mesh regularization	526
13.5.1. Regularization by displacement of nodes	526
13.5.2. Regularization by bubbles	528
13.5.3. Adaptation of nodes population	530
13.5.4. Insertion in meshing algorithms	530
13.5.5. Value of bubble regularization.	531
13.6. Mesh processor and modeling environment.	533
13.6.1. Some typical criteria.	533
13.6.2. Electromagnetism and meshing constraints	534
13.7. Conclusion	541
13.8. References	541
Chapter 14. Optimization	547
Jean-Louis COULOMB	
14.1. Introduction	547
14.1.1. Optimization: who, why, how?	547
14.1.2. Optimization by numerical simulation: is this reasonable?	548
14.1.3. Optimization by numerical simulation: difficulties.	549
14.1.4. Numerical design of experiments (DOE) method: an elegant solution	549

14.1.5. Sensitivity analysis: an “added value” accessible by simulation	550
14.1.6. Organization of this chapter	551
14.2. Optimization methods	551
14.2.1. Optimization problems: some definitions	551
14.2.2. Optimization problems without constraints	553
14.2.3. Constrained optimization problems	559
14.2.4. Multi-objective optimization	560
14.3. Design of experiments (DOE) method.	562
14.3.1. The direct control of the simulation tool by an optimization algorithm: principle and disadvantages	562
14.3.2. The response surface: an approximation enabling indirect optimization	563
14.3.3. DOE method: a short history.	565
14.3.4. DOE method: a simple example.	565
14.4. Response surfaces	572
14.4.1. Basic principles	572
14.4.2. Polynomial surfaces of degree 1 without interaction: simple but sometimes useful	573
14.4.3. Polynomial surfaces of degree 1 with interactions: quite useful for screening	573
14.4.4. Polynomial surfaces of degree 2: a first approach for nonlinearities.	574
14.4.5. Response surfaces of degrees 1 and 2: interests and limits	576
14.4.6. Response surfaces by combination of radial functions.	576
14.4.7. Response surfaces using diffuse elements	577
14.4.8. Adaptive response surfaces.	579
14.5. Sensitivity analysis	579
14.5.1. Finite difference method	579
14.5.2. Method for local derivation of the Jacobian matrix	580
14.5.3. Steadiness of state variables: steadiness of state equations	581
14.5.4. Sensitivity of the objective function: the adjoint state method	583
14.5.5. Higher order derivative	583
14.6. A complete example of optimization.	584
14.6.1. The problem of optimization	584
14.6.2. Determination of the influential parameters by the DOE method	585
14.6.3. Approximation of the objective function by a response surface	587
14.6.4. Search for the optimum on the response surface	587
14.6.5. Verification of the solution by simulation	587
14.7. Conclusion	588
14.8. References	588
List of Authors	595
Index	599

Chapter 1

Introduction to Nodal Finite Elements

1.1. Introduction

1.1.1. *The finite element method*

The finite element method, resulting from the matrix techniques of calculation of the discrete or semi-discrete mechanical structures (assembly of beams), is a tool for resolving problems with partial differential equations involved in physics problems. We will thus tackle this method accordingly because it is useful in modeling mechanical, thermal, neutron and electromagnetic problems [ZIE 79], [SIL 83], [DHA 84], [SAB 86], [HOO 89].

The aim of this chapter is to present the principles of this method which have become essential in the panoply of the engineer. For this presentation, we will only deal with electrostatics. Indeed, this field has a familiar formulation in scalar potential, particularly suitable for the presentation of nodal finite elements, which will be the only ones discussed here.

We will develop two examples of increasing complexity which are manageable "by hand", 1D in a first part and 2D in a second. As it is very close to physical considerations, the variational approach will most of the time be favored. However, the more general method of weighted residues will also be presented. In our examples, we will see how to solve the problems at issue, but also how, using the obtained fields, to extract more relevant information.

In the third and last part, we will present the concept of a reference element and the principles that make it possible to pass from the local coordinates to the domain coordinates. We will see that beyond the possibility of handling curvilinear elements, which is quite convenient for the discretization of manufactured objects, this technique leads to a general tool for working with geometric deformations.

1.2. The 1D finite element method

1.2.1. A simple electrostatics problem

In order to present the finite element method, we propose, initially, to implement it on a simple 1D electrostatics example, borrowed from [HOO 89]. We will first formulate this problem in its differential form, then in its variation form. This form of integral will enable us to introduce the concept of first-order finite elements and then second-order finite elements.

We thus consider the problem of Figure 1.1 where two long distant parallel plates of 10 m are: one with the electric potential of 0 V and the other with the potential of 100 V . Between the two plates, the density of electric charges and the dielectric permittivity are assumed to be constant. This problem could represent a hydrocarbon storage tank in which we wish to know the distribution of the electric potential. The lower plate corresponds to the free surface of the liquid, the upper plate to the ceiling of the tank and the intermediate part to the electrically charged vapors.

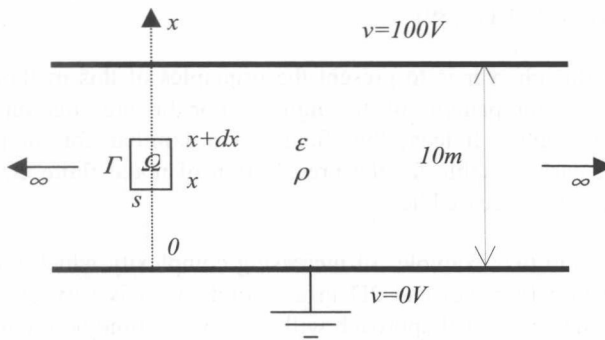


Figure 1.1. The cloud of electric charges between the two plates

1.2.2. Differential approach

The physical and geometric quantities varying only according to one direction, this problem is 1D in the interval $x \in [0, 10]$ and the electric field E and electric flux density $D = \epsilon E$ vectors have only one non-zero component E_x and D_x .

Let us consider a parallelepipedic elementary volume of constant section s in the direction perpendicular to x and of length dx . The flux of the electric density vector, leaving its border Γ , and the internal electric charge to its volume Ω are respectively:

$$\oint_{\Gamma} D_x d\Gamma = [D_x(x+dx) - D_x(x)]s \quad [1.1]$$

$$\iiint_{\Omega} \rho d\Omega = \rho \cdot s \cdot dx \quad [1.2]$$

The Gaussian electric law implies the equality of these two integrals, which gives, for the electric flux density, the following differential equation:

$$\frac{dD_x}{dx} = \rho \quad [1.3]$$

This equation is specifically one of Maxwell's equations:

$$\text{div}D = \rho \quad [1.4]$$

applied to a 1D problem in which the variations in the orthogonal directions to the x axis are zero.

On the terminals of the domain, the boundary conditions are expressed in terms of electric potential $v(0) = 0 \text{ V}$ and $v(10) = 100 \text{ V}$. It is thus judicious to specify the problem entirely in terms of v which is connected to the electric field by the relation

$E_x = -\text{grad } v$, which, in our 1D case, gives $E_x = -\frac{dv}{dx}$. The equation and the

boundary conditions governing the distribution of the electric potential are thus

$$\frac{d}{dx} \left[-\epsilon \frac{dv}{dx} \right] = \rho \quad \text{for } x \in [0, 10] \quad [1.5]$$

$$v = 0 \quad \text{for } x = 0$$

$$v = 100 \quad \text{for } x = 10$$

In our case, the electric permittivity is constant, which simplifies the equation and becomes

$$\frac{d^2v}{dx^2} = -\frac{\rho}{\varepsilon}, \quad v(0) = 0, \quad v(10) = 100 \quad [1.6]$$

This problem has the following analytical solution

$$v(x) = -\frac{\rho}{2\varepsilon}x^2 + \left[1 + \frac{\rho}{2\varepsilon}\right]10x \quad [1.7]$$

the knowledge of which will be useful for us when evaluating the quality of the solution given by the finite element method, which we will present below.

1.2.3. Variational approach

In fact, the finite element method does not directly use the previous differential form, but is based on an equivalent integral form. For this reason we will develop the *variational* approach which here is connected to the internal energy of the device. This approach is based on a functional (i.e. a function of the unknown function $v(x)$) which is extremal when $v(x)$ is the solution. The functional, called coenergy for reasons which will be explained later, corresponding to electrostatics problem [1.5] is

$$W_c(v) = \frac{1}{2} \int_0^{10} \varepsilon \left[\frac{dv}{dx} \right]^2 dx - \int_0^{10} \rho v dx \quad [1.8]$$

We will show that, if it exists, a continuous and derivable function $v_m(x)$ which fulfills the boundary conditions $v_m(0) = 0$ and $v_m(10) = 100$ and which makes functional [1.8] extremal is also the solution of problem [1.5].

For that, let us consider a function $v(x)$ built on the basis of $v_m(x)$ as follows

$$v(x) = v_m(x) + \alpha\varphi(x) \quad [1.9]$$

where α is an unspecified real number and $\varphi(x)$ is an arbitrary continuous and derivable function which becomes zero at the boundary of the domain ($\varphi(0) = 0$ and $\varphi(10) = 0$). By construction, function $v(x)$ automatically verifies the boundary conditions $v(0) = 0$ and $v(10) = 100$.

The introduction into [1.8] of this function $v(x)$ defines a simple function of α

$$W_c(\alpha) = \frac{1}{2} \int_0^{10} \varepsilon \left[\frac{d}{dx} [v_m + \alpha\varphi] \right]^2 dx - \int_0^{10} \rho [v_m + \alpha\varphi] dx \quad [1.10]$$

Note that, by assumption, for $\alpha = 0$ this function is extremal. Let us now express the increase of W_c with respect to its extremum,

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2} \int_0^{10} \varepsilon \left[\frac{d\varphi}{dx} \right]^2 dx + \alpha \int_0^{10} \varepsilon \frac{dv_m}{dx} \frac{d\varphi}{dx} dx - \alpha \int_0^{10} \rho \varphi dx \quad [1.11]$$

The integration by parts of the second integral gives

$$\begin{aligned} \int_0^{10} \varepsilon \frac{dv_m}{dx} \frac{d\varphi}{dx} dx &= \left[\varepsilon \frac{dv_m}{dx} \varphi \right]_0^{10} - \int_0^{10} \frac{d}{dx} \left[\varepsilon \frac{dv_m}{dx} \right] \varphi dx \\ \int_0^{10} \varepsilon \frac{dv_m}{dx} \frac{d\varphi}{dx} dx &= - \int_0^{10} \frac{d}{dx} \left[\varepsilon \frac{dv_m}{dx} \right] \varphi dx \end{aligned} \quad [1.12]$$

because the arbitrary function $\varphi(x)$ is zero on the boundaries of the domain.

We thus obtain for the increase of the functional

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2} \int_0^{10} \varepsilon \left[\frac{d\varphi}{dx} \right]^2 dx - \alpha \int_0^{10} \left\{ \frac{d}{dx} \left[\varepsilon \frac{dv_m}{dx} \right] + \rho \right\} \varphi dx \quad [1.13]$$

This polynomial of the second-degree is extremum for $\alpha = 0$, therefore the coefficient of α must be zero. This coefficient is an integral, to be zero whatever the arbitrary function $\varphi(x)$, and it is necessary that the weighting coefficient of this function becomes zero for any X

$$\frac{d}{dx} \left[\varepsilon \frac{dv_m}{dx} \right] + \rho = 0 \quad \forall x \in [0, 10] \quad [1.14]$$

which corresponds precisely to equation [1.5], which we want to solve. Therefore, if function $v_m(x)$ exists, it is indeed the solution of the specified problem. Moreover, the coefficient of α^2 being positive, the extremum is a minimum.

The result that we have just obtained is a particular case of a proof that is much more general of the calculus of variations. Equation [1.14] is in fact the *Euler*