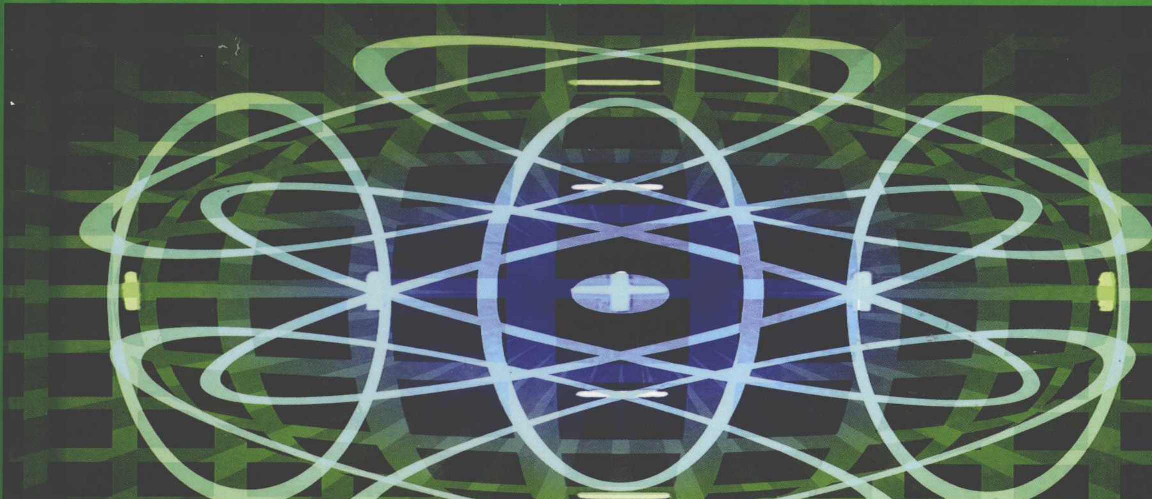
The Finite Element Method for Electromagnetic Modeling

Edited by Gérard Meunier



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Chapter 1

Introduction to Nodal Finite Elements

1.1. Introduction

1.1.1. *The finite element method*

The finite element method, resulting from the matrix techniques of calculation of the discrete or semi-discrete mechanical structures (assembly of beams), is a tool for resolving problems with partial differential equations involved in physics problems. We will thus tackle this method accordingly because it is useful in modeling mechanical, thermal, neutron and electromagnetic problems [ZIE 79], [SIL 83], [DHA 84], [SAB 86], [HOO 89].

The aim of this chapter is to present the principles of this method which have become essential in the panoply of the engineer. For this presentation, we will only deal with electrostatics. Indeed, this field has a familiar formulation in scalar potential, particularly suitable for the presentation of nodal finite elements, which will be the only ones discussed here.

We will develop two examples of increasing complexity which are manageable "by hand", 1D in a first part and 2D in a second. As it is very close to physical considerations, the variational approach will most of the time be favored. However, the more general method of weighted residues will also be presented. In our examples, we will see how to solve the problems at issue, but also how, using the obtained fields, to extract more relevant information.

Chapter written by Jean-Louis COULOMB.

In the third and last part, we will present the concept of a reference element and the principles that make it possible to pass from the local coordinates to the domain coordinates. We will see that beyond the possibility of handling curvilinear elements, which is quite convenient for the discretization of manufactured objects, this technique leads to a general tool for working with geometric deformations.

1.2. The 1D finite element method

1.2.1. *A simple electrostatics problem*

In order to present the finite element method, we propose, initially, to implement it on a simple 1D electrostatics example, borrowed from [HOO 89]. We will first formulate this problem in its differential form, then in its variation form. This form of integral will enable us to introduce the concept of first-order finite elements and then second-order finite elements.

We thus consider the problem of Figure 1.1 where two long distant parallel plates of *10 m* are: one with the electric potential of *0 V* and the other with the potential of *100 V*. Between the two plates, the density of electric charges and the dielectric permittivity are assumed to be constant. This problem could represent a hydrocarbon storage tank in which we wish to know the distribution of the electric potential. The lower plate corresponds to the free surface of the liquid, the upper plate to the ceiling of the tank and the intermediate part to the electrically charged vapors.

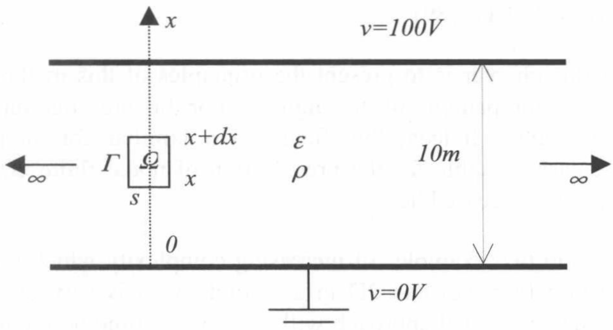

Figure 1.1. *The cloud of electric charges between the two plates*

1.2.2. *Differential approach*

The physical and geometric quantities varying only according to one direction, this problem is 1D in the interval $x \in [0, 10]$ and the electric field E and electric flux density $D = \varepsilon E$ vectors have only one non-zero component E_x and D_x.

Let us consider a parallelepipedic elementary volume of constant section s in the direction perpendicular to x and of length dx. The flux of the electric density vector, leaving its border Γ, and the internal electric charge to its volume Ω are respectively:

$$\oiint_{\Gamma} D.d\Gamma = \left[D_x(x+dx) - D_x(x)\right].s \qquad [1.1]$$

$$\iiint_{\Omega} \rho d\Omega = \rho.s.dx \qquad [1.2]$$

The Gaussian electric law implies the equality of these two integrals, which gives, for the electric flux density, the following differential equation:

$$\frac{dD_x}{dx} = \rho \qquad [1.3]$$

This equation is specifically one of Maxwell's equations:

$$divD = \rho \qquad [1.4]$$

applied to a 1D problem in which the variations in the orthogonal directions to the x axis are zero.

On the terminals of the domain, the boundary conditions are expressed in terms of electric potential $v(0) = 0\ V$ and $v(10) = 100\ V$. It is thus judicious to specify the problem entirely in terms of v which is connected to the electric field by the relation $E_x = -grad\ v$, which, in our 1D case, gives $E_x = -\frac{dv}{dx}$. The equation and the boundary conditions governing the distribution of the electric potential are thus

$$\frac{d}{dx}\left[-\varepsilon\frac{dv}{dx}\right] = \rho \qquad \text{for } x \in [0, 10] \qquad [1.5]$$

$$v = 0 \qquad \text{for } x = 0$$

$$v = 100 \qquad \text{for } x = 10$$

In our case, the electric permittivity is constant, which simplifies the equation and becomes

$$\frac{d^2v}{dx^2} = -\frac{\rho}{\varepsilon}, \qquad v(0) = 0, \qquad v(10) = 100 \qquad [1.6]$$

This problem has the following analytical solution

$$v(x) = -\frac{\rho}{2\varepsilon}x^2 + \left[1 + \frac{\rho}{2\varepsilon}\right]10x \qquad [1.7]$$

the knowledge of which will be useful for us when evaluating the quality of the solution given by the finite element method, which we will present below.

1.2.3. *Variational approach*

In fact, the finite element method does not directly use the previous differential form, but is based on an equivalent integral form. For this reason we will develop the *variational* approach which here is connected to the internal energy of the device. This approach is based on a functional (i.e. a function of the unknown function $v(x)$) which is extremal when $v(x)$ is the solution. The functional, called coenergy for reasons which will be explained later, corresponding to electrostatics problem [1.5] is

$$W_c(v) = \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{dv}{dx}\right]^2 dx - \int_0^{10} \rho v \, dx \qquad [1.8]$$

We will show that, if it exists, a continuous and derivable function $v_m(x)$ which fulfills the boundary conditions $v_m(0) = 0$ and $v_m(10) = 100$ and which makes functional [1.8] extremal is also the solution of problem [1.5].

For that, let us consider a function $v(x)$ built on the basis of $v_m(x)$ as follows

$$v(x) = v_m(x) + \alpha\varphi(x) \qquad [1.9]$$

where α is an unspecified real number and $\varphi(x)$ is an arbitrary continuous and derivable function which becomes zero at the boundary of the domain ($\varphi(0) = 0$ and $\varphi(10) = 0$). By construction, function $v(x)$ automatically verifies the boundary conditions $v(0) = 0$ and $v(10) = 100$.

The introduction into [1.8] of this function *v(x)* defines a simple function of α

$$W_c(\alpha) = \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d}{dx}\left[v_m + \alpha\varphi\right]\right]^2 dx - \int_0^{10} \rho\left[v_m + \alpha\varphi\right] dx \quad [1.10]$$

Note that, by assumption, for $\alpha = 0$ this function is extremal. Let us now express the increase of W_c with respect to its extremum,

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d\varphi}{dx}\right]^2 dx + \alpha \int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx} dx - \alpha \int_0^{10} \rho\varphi dx \quad [1.11]$$

The integration by parts of the second integral gives

$$\int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx} dx = \left[\varepsilon \frac{dv_m}{dx}\varphi\right]_0^{10} - \int_0^{10} \frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right]\varphi dx$$

$$\int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx} dx = -\int_0^{10} \frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right]\varphi dx \quad [1.12]$$

because the arbitrary function $\varphi(x)$ is zero on the boundaries of the domain.

We thus obtain for the increase of the functional

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d\varphi}{dx}\right]^2 dx - \alpha \int_0^{10} \left\{\frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right] + \rho\right\}\varphi dx \quad [1.13]$$

This polynomial of the second-degree is extremum for $\alpha = 0$, therefore the coefficient of α must be zero. This coefficient is an integral, to be zero whatever the arbitrary function $\varphi(x)$, and it is necessary that the weighting coefficient of this function becomes zero for any X

$$\frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right] + \rho = 0 \qquad \forall x \in [0, 10] \quad [1.14]$$

which corresponds precisely to equation [1.5], which we want to solve. Therefore, if function $v_m(x)$ exists, it is indeed the solution of the specified problem. Moreover, the coefficient of α^2 being positive, the extremum is a minimum.

The result that we have just obtained is a particular case of a proof that is much more general of the calculus of variations. Equation [1.14] is in fact the *Euler*