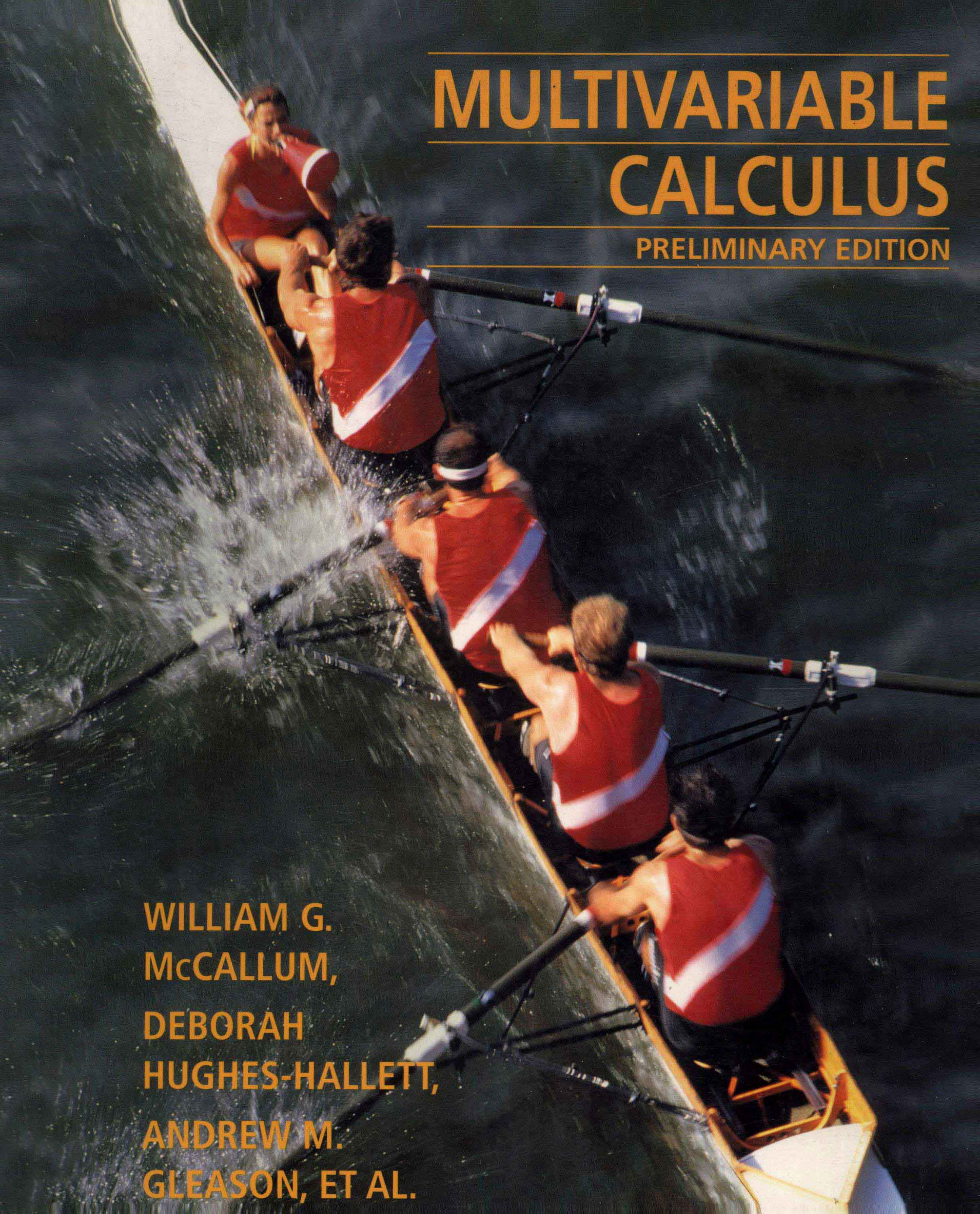
MULTIVARIABLE CALCULUS

PRELIMINARY EDITION



WILLIAM G. McCALLUM, DEBORAH HUGHES-HALLETT, ANDREW M. GLEASON, ET AL.

MULTIVARIABLE CALCULUS

Produced by the Consortium based at Harvard and funded by a National Science Foundation Grant. All proceeds from the sale of this work are used to support the work of the Consortium.

William G. McCallum
University of Arizona

Daniel Flath
University of South Alabama

Andrew M. Gleason
Harvard University

Sheldon P. Gordon
Suffolk County Community College

David Mumford
Harvard University

Brad G. Osgood
Stanford University

Deborah Hughes-Hallett
Harvard University

Douglas Quinney
University of Keele

Wayne Raskind
University of Southern California

Jeff Tecosky-Feldman
Haverford College

Joe B. Thrash
University of Southern Mississippi

Thomas W. Tucker
Colgate University

John Wiley & Sons, Inc.
New York Chichester Brisbane Toronto Singapore

Dedicated to Amy, Nell, Abby, and Sally.

Cover Photo by Greg Pease

This material is based upon work supported by the National Science Foundation under Grant No. DUE-9352905. All royalties from the sale of this book will go toward the furtherance of the project.

ISBN: 0-471-12256-4

Printed in the United States of America

10 9 8 7 6 5 4 3 2

MULTIVARIABLE CALCULUS

PREFACE

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but the rules and procedures – thereby losing sight of both the mathematics and of its practical value. With the generous support of the National Science Foundation, our group set out to create a new calculus curriculum that would restore that insight. This book is the second stage in that endeavor. The first stage is our single variable text.

Basic Principles

The two principles that guided our efforts in developing the single variable book remain valid. The first is our prescription for restoring the mathematical content to calculus:

The Rule of Three: *Every topic should be presented geometrically, numerically and algebraically.*

We continually encourage students to think and write about the geometrical and numerical meaning of what they are doing. It is not our intention to undermine the purely algebraic aspect of calculus, but rather to reinforce it by giving meaning to the symbols. In the homework problems dealing with applications, we continually ask students what their answers mean in practical terms.

The second principle, inspired by Archimedes, is our prescription for restoring practical understanding:

The Way of Archimedes: *Formal definitions and procedures evolve from the investigation of practical problems.*

Archimedes believed that insight into mathematical problems is gained by investigating mechanical or physical problems first.[1] For the same reason, our text is problem driven. Whenever possible, we start with

[1]. . . I thought fit to write out for you and explain in detail . . . the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. From *The Method*, in *The Works of Archimedes* edited and translated by Sir Thomas L. Heath (Dover, NY)

a practical problem and derive the general results from it. By practical problems we usually, but not always, mean real world applications. These two principles have led to a dramatically new curriculum - more so than a cursory glance at the table of contents might indicate.

Technology

In multivariable calculus, even more so than in single variable calculus, computer technology can be put to great advantage to help students learn to think mathematically. For example, looking at surface graphs and contour diagrams is enormously helpful in understanding functions of many variables. Furthermore, the ability to use technology effectively as a tool in itself is of the greatest importance. Students are expected to use their judgment to determine where technology is useful.

However, the book does not require any specific software or technology, and we have accommodated those without access to sufficiently powerful technology by providing supplementary master copies for overhead slides, showing surface graphs, contour diagrams, parametrized curves, and vector fields. Ideally, students should have access to technology with the ability to draw surface graphs, contour diagrams, and vector fields, and to calculate multiple integrals and line integrals numerically. Failing that, however, the combination of hand held graphing calculators and the overhead transparencies is quite satisfactory, and has been used successfully by test sites.

What Student Background is Expected?

Students using this book should have successfully completed a course in single variable calculus. It is not necessary for them to have used the single variable book from the same consortium in order for them to learn from this book.

The book is thought-provoking for well-prepared students while still accessible to students with weaker backgrounds. Providing numerical and graphical approaches as well as the algebraic gives students another way of mastering the material. This approach encourages students to persist, thereby lowering failure rates.

Content

Our approach to designing this curriculum was the same as the one we took in our single variable book: we started with a clean slate, and compiled a list of topics that we thought were fundamental to the subject, after discussions with mathematicians, engineers, physicists, chemists, biologists, and economists. In order to meet individual needs or course requirements, topics can easily be added or deleted, or the order changed.

Chapter 11: Functions of Many Variables

We introduce functions of many variables from several points of view, using surface graphs, contour diagrams, and tables. This chapter is as crucial for this course as Chapter 1 is for the single variable course; it gives students the skills to read graphs and contour diagrams and think graphically, to read tables and think numerically, and to apply these skills, along with their algebraic skills, to modeling the real world. We pay particular attention to the idea of a section of a function, obtained by varying one variable independently of the others. It is important that the student thoroughly digest this notion from both a graphical and a numerical point of view, before being exposed to the ideas of partial derivatives and gradients. We study linear functions in detail from all points of view, in preparation for the notion of local linearity.

Chapter 12: A Fundamental Tool: Vectors

We define vectors as geometric objects having direction and magnitude, with displacement vectors as the model, and then show how to resolve vectors into components. We define the dot and cross product of two vectors purely in terms of their direction and magnitude, and then give the formulas in terms of components. We continue this approach to vectors throughout the book; the geometric definition first, and the formula in terms of components immediately afterward.

Chapter 13: Differentiating Functions of Many Variables

We introduce the basic notions of partial derivative, directional derivative, gradient, and differential. In keeping with the spirit of the single variable book, we put all the different notions of derivative in the framework of local linearity. We also use local linearity as the basis for the multivariable chain rule. We discuss higher order partial derivatives, their interpretation in partial differential equations, and their application to quadratic Taylor approximations.

Chapter 14: Optimization

We apply the ideas of the previous chapter to optimization problems, both constrained and unconstrained. We derive the second derivative test for local extrema by first considering the case of quadratic polynomials, and then appealing to the quadratic Taylor approximation. We discuss the existence of global extrema for continuous functions on closed and bounded regions. In the section on constrained optimization, we discuss Lagrange multipliers, equality and inequality constraints, problems with more than one constraint, and the Lagrangian.

Chapter 15: Integrating Functions of Many Variables

We motivate the multivariable definite integral graphically by considering the problem of estimating total population from a contour diagram for population density, using finer and finer grids. We continue with numerical examples using tables, and then give two methods of calculating multiple integrals: analytically, by means of iterated integrals, and numerically, by the Monte Carlo method. We discuss both double and triple integrals in Cartesian, polar, spherical, and cylindrical coordinates. We also discuss applications to multivariate probability.

Chapter 16: Parametric Curves and Surfaces

We start with the problem of representing motion in space. This leads in two different directions, each discussed in a separate section: the problem of representing curves parametrically, and the study of velocity and acceleration of moving particles. In keeping with our approach to vectors, we define velocity and acceleration geometrically, then give the formulas in terms of components. We conclude with a section parameterizing surfaces. The chapter leads to an appendix with one of the original, and still the most inspiring, applications of calculus; the derivation of Kepler's laws of motion from Newton's laws.

Chapter 17: Vector Fields

In this brief chapter we introduce vector-valued functions of many variables, or vector fields. This chapter lays the foundation for the geometric approach in the next three chapters to line integrals, flux integrals, divergence, and curl. We start with physical examples, such as velocity vector fields and force fields, and include many sketches of vector fields to help build geometric intuition. We also discuss flow lines of vector fields and their relation with systems of differential equations.

Chapter 18: Line Integrals

We present the concept of integrating a vector field along a path with a coordinate-free definition. We spend some time building intuition using sketches of vector fields with paths superimposed, before introducing the method of calculating line integrals using parametrizations. We then discuss conservative fields, gradient fields, and the Fundamental Theorem of Calculus for Line Integrals. We conclude with a section on non-conservative vector fields and Green's Theorem.

Chapter 19: Flux Integrals

We introduce the flux integral of a vector field through a parameterized surface in the same way as we introduced line integrals. First we give a coordinate-free definition, then we discuss examples where the flux integral (or at least its sign) can be calculated geometrically. Then we show how to calculate flux integrals, first over surface graphs, and then over arbitrary parameterized surfaces.

Chapter 20: Calculus of Vector Fields

We introduce divergence and curl in a coordinate-free way; the divergence in terms of flux density, and curl in terms of circulation density. We then give the formulas in Cartesian coordinates. In the single variable book we derived the Fundamental Theorem of Calculus by pointing out that the integral of the rate of change is the total change. In much the same way, we derive the divergence theorem by showing that the integral of flux density over a volume is the total flux out of the volume, and Stokes' theorem by showing that the integral of circulation density over a surface is the total circulation around its boundary.

Changes in This Edition

We have incorporated suggestions from users of the Draft Version into this new Preliminary Edition. These changes include the following:

- *Chapter 11.* We have rewritten the material in 11.6 on the relation between surface graphs and level surfaces for greater clarity.
- *Chapter 12.* We have reorganized the first two sections: Components of vectors are now in 12.1, and vectors other than displacement vectors are now in 12.2. We have moved the material on the area vector to Chapter 18, where it is first used.
- *Chapter 13.* We have reorganized and rewritten parts of Chapter 13 to make the unifying theme of local linearity clearer. Differentials and local linearity are now together in 13.3, which now also includes a discussion of differentiability. We give a different treatment of the gradient, using local linearity rather than the previous geometric argument. Partial differential equations now come directly after higher order partial derivatives, and quadratic Taylor approximations have been moved to the end of the chapter.
- *Chapter 14.* We have added material on inequality constraints, multiple constraints, Lagrangians, and the existence of global extrema for continuous functions on closed and bounded regions.
- *Chapter 15.* We have added a section on applications of multiple integration to multivariate probability.
- *Chapter 16.* We have extensively reorganized this chapter, to make clearer the distinction between parametric equations used to describe motion in space, and parametric equations used to represent

curves. We have merged the sections on velocity and acceleration, and have removed the material on curvature. Parameterized surfaces now come at the end of the chapter. We have moved the material on area vectors to Chapter 18.

- *Chapters 17-20.* The old Chapters 17 and 18 have been split into four chapters, as follows:
  - Chapter 17: Vector Fields
  - Chapter 18: Line Integrals
  - Chapter 19: Flux Integrals
  - Chapter 20: Calculus of Vector Fields

  Our purpose was to make the text more flexible for a one-semester course, and to make it easier to pick out a fast track through the material. For example, instructors have the following two choices: They can stop at the end of Chapter 18, allowing time for a thorough treatment of parameterized curves and surfaces, line integrals, and Green's theorem; or they can continue to Chapter 20, covering only the earlier sections in Chapters 16, 17, 18, and 19, which will yield a brief treatment of line and flux integrals from a geometric point of view and give students enough background to understand the divergence theorem and Stokes' theorem.
- *Chapter 17.* This is the first two sections of the old Chapter 17.
- *Chapter 18.* We have reorganized for clarity the material on conservative fields, gradient fields, and circulation, and we have added material on nonconservative vector fields, the two-dimensional curl criterion, and Green's theorem.
- *Chapter 19.* This is the material on flux integrals first two sections of the old Chapter 18. We have split the old section on flux integrals over parameterized surfaces into two sections, on integrals over surface graphs and over general parameterized surfaces.
- *Chapter 20.* We have moved the formulas for divergence and curl in Cartesian coordinates closer to their definitions, and moved the more detailed arguments justifying these formulas to a new section at the end of the chapter.

Supplementary Materials

- **Instructor's Manual** with teaching tips, calculator programs, some overhead transparency masters and sample exams and quizzes.
- **Instructor's Solution Manual** with complete solutions to all problems.
- **Student's Solution Manual** with complete solutions to every other odd-numbered problem.
- **Answer Manual** with brief answers to all odd-numbered problems.
- **MultiGraph** for Windows based surface plotting software.

Our Experiences

In the process of developing the ideas incorporated in this book, we have been conscious of the need to test the materials thoroughly in a wide variety of institutions serving many different types of students. Consortium members have used previous versions of the book at a broad range of institutions. During the 1994–1995 academic year we were assisted by colleagues at over 100 schools who class-tested the Draft Version and reported their experiences and those of their students. This diverse group of schools used the book in semester and quarter systems, in computer labs, small groups, and traditional settings, and with a number of different technologies. We appreciate the valuable suggestions they made, which we have tried to incorporate into this Preliminary Edition of the text.

Acknowledgements

Thanks to Ed Alexander, Carole Anderson, Kevin Anderson, Ralph Baierlein, Roxann Batiste, Jerrie Beiberstein, Shelina Bhojani, Paul Blanchard, Melkana Brakalova, Otto Bretscher, John Brillhart, Ruvim Breydo, Chris Bowman, David Bressoud, Will Brockman, Theresa Broderick, Edward Chandler, Phil Cheifetz, C. K. Cheung, Dave Chen, Dave Chua, Robert Condon, Eric Connally, Radu Constantinescu, Josh Cowley, Greg Crow, Jie Cui, Caspar Curjel, John Drabicki, Bill Dunn, Pavel Etingof, Bill Faris, Paul Feehan, George Fennemore, Hermann Flaschka, Katy Flint, Leonid Friedlander, Deborah Gaines, Avijit Gangopadhyay, Liwei Gao, Scott Gilbert, Nikki Grant, David Grazer, Marty Greenlee, John Hagood, Robert Hanson, Angus Hendrick, Tricia Hersh, Randy Ho, Greg Holmberg, Sharon Hurst, David Hurtubise, Luke Hunsberger, Brady Hunsaker, Robert Indik, Utith Inprasit, Adrian Iovita, Jack Jackson, Jerry Johnson, Millie Johnson, Calvin Jongsma, Georgia Kamvosoulis, Joe Kanapka, Alex Kasman, Matthias Kawski, Misha Kazhdan, Charlie Kerr, Mike Klucznik, Dmitri Kountourgiannis, Matt Kruse, Robert Kuhn, Kam Kwong, Ted Laetsch, Sylvain Laroche, Janny Leung, Dave Levermore, Lei Li, Weiye Li, Li Liu, Carlos Lizzaraga, Patti Frazer Lock, John Lucas, Alex Mallozzi, James Mark, Ricardo Martinez, Mark McConnell, Dan McGee, Andrew Metrick, Karen Millstone, Michal Mlejnek, Kathy Mosher, Marshall Mundt, Don Myers, Jeff Nelson, Alan Newell, Huy Nguyen, John Olson, Myriam Oviedo, James Osterburg, Ed Park, Howard Penn, Tony Phillips, Jessica Polito, Steve Prothero, Amy Rabb-Liu, Fred Richman, Renee Robles, David Royster, W. R. Salzman, Bill Schultz, Barbara Shipman, Michael Stringer, Noah Syroid, Mike Tabor, Sulian Tay, Tepache, Denise Todd, Jose Torres, Elias Toubassi, Jerry Uhl, Doug Ulmer, Steve Uurtamo, Bill Velez, Faye Villalobos, Alice Wang, Joseph Watkins, Eric Wepsic, Steve Wheaton, Maciej Wojtkowski, Xianbao Xu, and Bruce Yoshiwara.

William G. McCallum, Deborah Hughes-Hallett, Daniel E. Flath, Andrew M. Gleason, Sheldon P. Gordon, David Mumford, Brad G. Osgood, Douglas Quinney, Wayne Raskind, Jeff Tecosky-Feldman, Joe B. Thrash, Thomas W. Tucker

To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. This book emphasizes at every stage the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of multivariable calculus in plain English. Your success in using this book will depend on your reading, questioning, and thinking hard about the ideas presented. Although you may not have done this with other books, you should plan on reading the text in detail, not just the worked examples.
- There are very few examples in the text that are exactly like the homework problems. This means that you can't just look at a homework problem and search for a similar-looking "worked out" example. Success with the homework will come by grappling with the ideas of calculus.
- Many of the problems that we have included in the book are open-ended. This means that there may be more than one approach and more than one solution, depending on your analysis. Many times, solving a problem relies on common sense ideas that are not stated in the problem but which you will know from everyday life.
- This book assumes that you have access to a graphing calculator or computer; preferably one that can draw surface graphs, contour diagrams, and vector fields, and can compute multivariable integrals and line integrals numerically. There are many situations where you may not be able to find an exact solution to a problem, but you can use a calculator or computer to get a reasonable approximation. An answer obtained this way is usually just as useful as an exact one. However, the problem does not always state that a calculator is required, so use your judgement.
- This book attempts to give equal weight to three methods for describing functions: graphical (a picture), numerical (a table of values) and algebraic (a formula). Sometimes you may find it easier to translate a problem given in one form into another. For example, if you have to find the maximum of a function, you might use a contour diagram to estimate its approximate position, use its formula to find equations that give the exact position, then use a numerical method to solve the equations. The best idea is to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups very helpful. There are a great many problems which are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of one of the most important accomplishments of the millennium – calculus – as well as a real sense of how mathematics is used in the age of technology.

CONTENTS

CHAPTER ELEVEN

FUNCTIONS OF MANY VARIABLES

Many quantities depend on more than one variable: the amount of food grown depends on the amount of rain and the amount of fertilizer used; the rate of a chemical reaction depends on the temperature and the pressure of the environment in which it proceeds; the strength of the gravitational attraction between two bodies depends on their masses and their distance apart; and the rate of fallout from a volcanic explosion depends on the distance from the volcano and the time since the explosion. In this chapter we will see the many different ways of looking at functions of many variables.

11.1 FUNCTIONS OF TWO VARIABLES

Function Notation

Suppose you are planning to take out a five-year loan to buy a car and you need to calculate what your monthly payment will be; this depends on both the amount of money you borrow and the interest rate. These quantities can vary separately: the loan amount can change while the interest rate remains the same, or the interest rate can change while the loan amount remains the same. To calculate your monthly payment you need to know both. If the monthly payment is $\$m$, the loan amount is $\$L$, and the interest rate is $r\%$, then we express the fact that m is a function of L and r by writing:

$$m = f(L, r).$$

This is just like the function notation of one-variable calculus. The variable m is called the dependent variable , and the variables L and r are called the independent variables. The letter f stands for the *function* or rule that gives the value of m corresponding to given values of L and r.

A function of two variables can be represented pictorially, numerically by a table of values, or algebraically by a formula. In this section we will give examples of each of these three ways of viewing a function.

Graphical Example: A Weather Map

Figure 11.1 shows a weather map from a newspaper. What information does it convey? It is displaying the predicted high temperature, T, in degrees Fahrenheit (°F), at any point in the US on that day. The curving lines on the map, called *isotherms*, separate the country into zones, according to whether T is in the 60s, 70s, 80s, 90s, or 100s. (*iso* means same and *therm* means heat.) Notice that the isotherm separating the 80s and 90s zones connects all the points where the temperature is exactly 90°F.

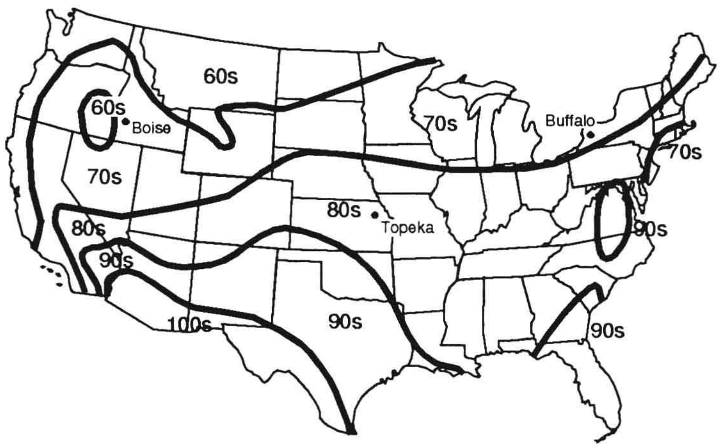


Figure 11.1: Weather map showing predicted high temperatures, T, for June 30, 1992

Example 1 Estimate the value of T in Boise, Idaho; Topeka, Kansas; and Buffalo, New York.

Solution Boise and Buffalo are in the 70s region, and Topeka is in the 80s region. Thus, the temperature in Boise and Buffalo is between 70 and 80; the temperature in Topeka is between 80 and 90.