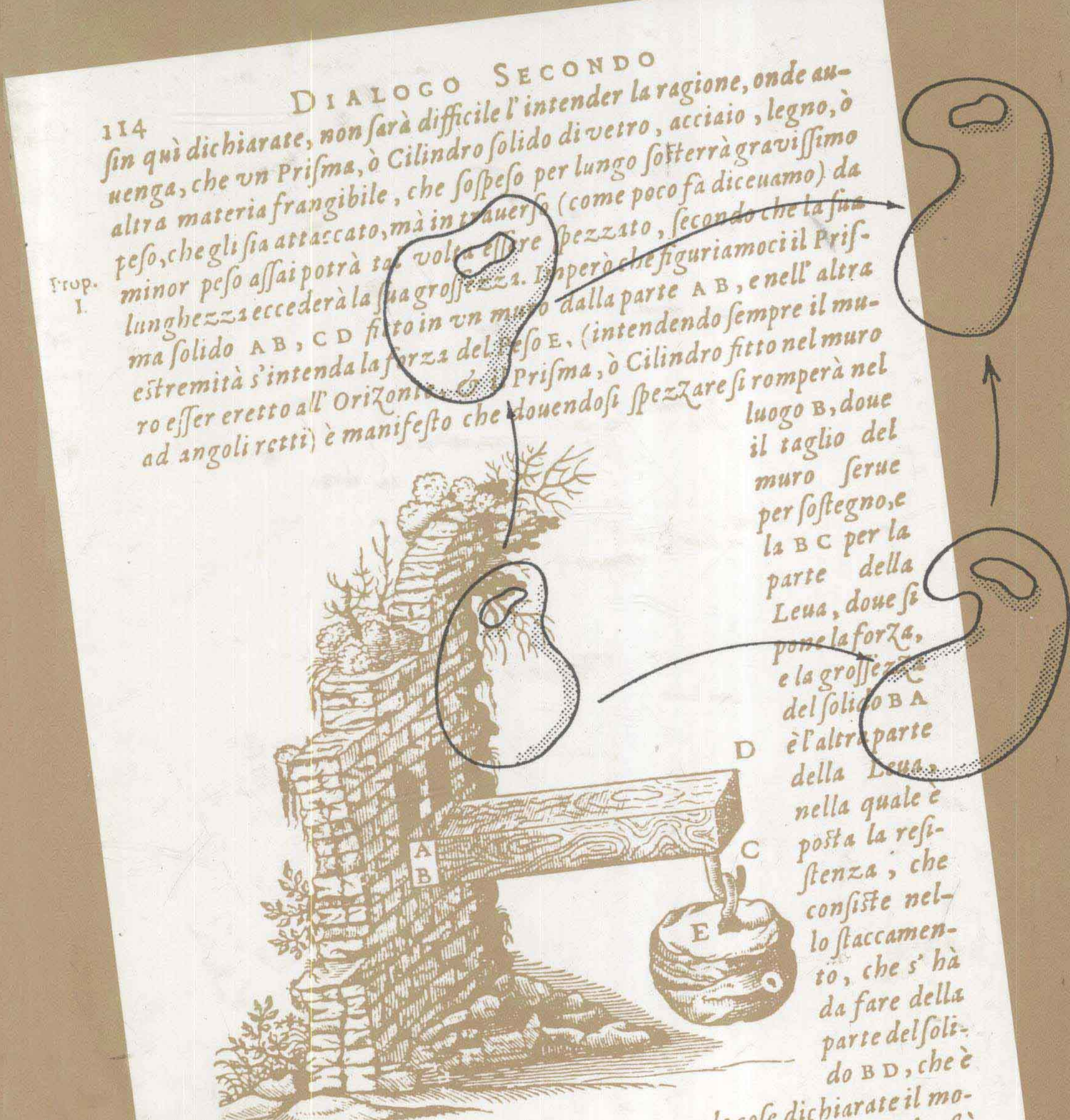
JERROLD E. MARSDEN

THOMAS J. R. HUGHES

Mathematical Foundations of ELASTICITY



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ſin quì dichiarate, non ſarà difficile l'intender la ragione, onde auuenga, che vn Priſma, ò Cilindro ſolido di vetro, acciaio, legno, ò altra materia frangibile, che ſoſpeſo per lungo ſoſterrà grauiſſimo peſo, che gli ſia attaccato, mà in trauerſo (come poco fà diceuamo) da minor peſo aſſai potrà tal volta eſſere ſpezzato, ſecondo che la ſua lunghezza eccederà la ſua groſſezza. Imperò che figuriamoci il Priſma ſolido AB, CD fitto in vn muro dalla parte AB, e nell' altra eſtremità s'intenda la forza del peſo E, (intendendo ſempre il muro eſſer eretto all' Orizonte, ò il Priſma, ò Cilindro fitto nel muro ad angoli retti) è manifeſto che douendoſi ſpezzare ſi romperà nel luogo B, doue il taglio del muro ſerue per ſoſtegno, e la BC per la parte della Leua, doue ſi pone la forza, e la groſſezza del ſolido BA è l'altra parte della Leua, nella quale è poſta la reſiſtenza, che conſiſte nello ſtaccamento, che s'hà da fare della parte del ſolido BD, che è

Prop. I.

MATHEMATICAL FOUNDATIONS OF ELASTICITY

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MATHEMATICAL FOUNDATIONS OF ELASTICITY

Cover illustration: Developed from an old drawing of Galileo's beam as it appears in *Essays in the History of Mechanics* by C. A. Truesdell and published by Springer-Verlag in 1968.

To Nancy, Susan, Chris, Emily, Alison, Ian, and Elizabeth

PREFACE

This book treats parts of the mathematical foundations of three-dimensional elasticity using modern differential geometry and functional analysis. It is intended for mathematicians, engineers, and physicists who wish to see this classical subject in a modern setting and to see some examples of what newer mathematical tools have to contribute.

Disclaimer There are three things that every beginner in elasticity theory should know. The first is that "Kirchhoff" has two h's in it. The second is that Hooke's law will not be found as a basic axiom (it "really" means you are working with the linearized theory). The third is that researchers in elasticity theory are very opinionated, even when they are wrong. During our own work in this field we have refused to fight, and in keeping with this pacifist approach, we now issue these general disclaimers: This book is neither complete nor unbiased. Furthermore, we have not mentioned many deep and highly erudite works, nor have we elucidated alternative approaches to the subject. Any historical comments we make on subjects prior to 1960 are probably wrong, and credits to some theorems may be incorrectly assigned. Excellent historical sketches are available in the works of Truesdell [1968], Sokolnikoff [1956], and Sneddon [1980] cited in the bibliography.

The Two-Track and Box Approach To a mathematician, a tensor $\mathbf{t}$ is a section of a certain bundle over a manifold. To an engineer or physicist, a tensor $\mathbf{t}^{ijk}$ is an object dressed in indices. This is one of many unfortunate paper barriers that have retarded the growth of, and interest in, mathematical elasticity. The beginner should learn to speak both languages and to ignore notational disputes.

For example, beginners who are already trained in some geometry and who realize that ∇f is a vector, while df is a one-form, will recognize at once that the deformation gradient $\boldsymbol{F}$ is not a gradient at all, but is simply the derivative of the deformation. They may also recognize that the rate of deformation tensor is just the Lie derivative of the Riemannian metric on space, and that the Cauchy–Green tensor is the pull-back of the Riemannian metric on space by the deformation.

To aid the reader in this linguistic endeavor we have tried to present as many formulas as possible in both languages. This is done through numerous boxes that summarize the important formulas both ways. These boxes are also used to isolate more advanced or optional material.

Subjects Covered The first two chapters cover the background geometry—which is developed as it is needed—and use this discussion to obtain the basic results on kinematics and dynamics of continuous media. Chapter 3 narrows the discussion to elastic materials. Chapter 4 on linearization gives a systematic way to linearize a nonlinear field theory along with a basic mathematical tool—the inverse function theorem. Chapter 5 deals with variational principles. Chapter 6 presents a relatively self-contained account of the use of functional analysis (such as elliptic theory and semigroups) in elasticity. Chapter 7 introduces bifurcation theory. We originally planned to include a chapter on numerical methods as well, but space and timeliness did not allow us to do so.

Level and Background The book is written at the beginning graduate level, with occasional excursions to the research frontier. Some parts, such as the first five chapters and parts of the remainder, are accessible to good undergraduates. To read this book one should have a solid background in advanced calculus (for example, J. Marsden [1974a] is adequate). One should also be prepared to invest considerable time in learning geometry and functional analysis as the book is read. Most of what is needed is in this book, but it may be useful to consult some of the references that follow.

The Use of Geometry and Functional Analysis We have found differential geometry helpful in sorting out the foundations of the subject. Deeper analytical facts about elasticity require a serious knowledge of functional analysis, including partial differential equations. The reader should realize that many researchers understand one or the other of these subjects, but very few understand both because of the large investment of time and effort involved. Therefore, one should adjust one's aspirations and depth of reading accordingly. For example, if one's goal is to get to modern research in the buckling of shells as fast as possible, it may be a mistake to start on page 1. It is obvious that a large part of any book is irrelevant to such a specific endeavor. Rather, one should jump directly into the current literature (for example, see Section 7.2) and use this book to complete the necessary background. On the other hand, if one has the time to go through the requisite geometry, the insights gained into nonlinear elasticity will be worthwhile. Examples of how geometry is used in elasticity are discussed in Section 6 of the introductory chapter. Likewise, abstract functional analysis is often accused of not shedding any light on "practical" problems of elasticity. Recent progress in constitutive inequalities and numerical methods demonstrates that this view is incorrect.

Point of Departure and Interdependence of Chapters Because of the large amount of geometry involved in the first three chapters, we have written an introductory Chapter to enable readers to bypass parts of Chapters 1–3. After studying Sections 1–5 of the introductory chapter, such readers should be ready to undertake Chapters 4–7. These four chapters do contain some dependence on Chapters 1–3, but this dependence is minimal and may be bypassed if one has a background in elasticity obtained from other sources. We also recommend the introductory chapter for readers who intend to seriously study Chapters 1–3 to keep their work in perspective. Chapters 4–7 are in logical order, but it is not necessary to have full mastery of one before proceeding. To this end, ample cross references are given.

Notation We have adopted a reasonably simple system used by some current workers. This is summarized in a brief glossary. Readers should understand that if they hear a lecture on elasticity, the conventions will probably differ from those here or their own. Here boldface type is used to distinguish abstract tensors from their components. For example, $\boldsymbol{\sigma}$ means the abstract Cauchy stress tensor, while σ^{ab} represents its components. The only other nonstandard notation is the use of block boldface for the fourth-order elasticity tensors, such as $\mathbf{C}$, whose components are denoted C^{ABCD}, and $\mathbf{A}$, whose components are denoted A^{aAbB}. Occasionally the same symbol has two meanings in the book, when the intended meaning is clear from the context. We find this preferable to a multitudinous proliferation of alphabets and fonts that are impossible to reproduce in the classroom.

Things We Fuss Over; Things We Don't Most mathematicians, physicists and engineers now agree that the distinction between a linear transformation and a matrix is worth fussing over. We believe that one should also distinguish tensors from tensor components. However, we do not fuss over whether Euclidean space should be written as $\mathbb{R}^3$ or not. To abstract $\mathbb{R}^3$ properly, we believe that manifolds should be used. They are unquestionably the appropriate setting for tensor analysis.

Resistance to the use of abstract manifolds is frequently encountered, simply because most work in elasticity occurs in $\mathbb{R}^3$. In the literature, $\mathbb{R}^3$ is often replaced by abstract vector spaces. This arena is *not* suitable for general tensor analysis.

Indeed, as Einstein has so profoundly taught us, deep insights can be gained by removing one's blinders to see the theory in the grander time-proven context of covariant formulations. This is why we encourage the use of manifolds.

We do not fuss over measure-theoretic questions that are often used to introduce mass and force densities, for example. If one understands the Radon–Nikodym derivative, it takes only a few minutes to understand this even though technical intricacies may be nontrivial. We chose not to go into measure-theoretic formalism because it requires a lengthy exposition that would divert us from our main goal.

Numbering Conventions Within Chapter 1, the eleventh item of the third section is referred to as 3.11 and the third section is referred to as Section 3. In subsequent chapters this item is referred to as 3.11, Chapter 1, and the third section is referred to as Section 1.3. Similar conventions apply to discussions enclosed in boxes. Figures are given their full labels for editorial reasons. Formulas are numbered within a section when it helps the exposition.

References A relatively large bibliography is included at the back of this book. Specific references cited in the text are listed by author and year like this: Ball [1977b]. It is wise for beginning students to consult a few key books regularly. We recommend the following:

(a) an introductory modern text on continuum mechanics, such as Malvern [1969] or Gurtin [1981b];
(b) one of the classical texts on elasticity such as Love [1927], Sokolnikoff [1956], Landau and Lifshitz [1970], Green and Adkins [1970], or Green and Zerna [1968];
(c) the encyclopedic treatise of Truesdell and Noll [1965] (which has a massive bibliography);
(d) a modern book on manifolds and tensor analysis, such as Abraham Marsden and Ratiu [1982], Bishop and Goldberg [1968], Schutz [1980], Spivak [1975], or Warner [1971], and a classical one such as Eisenhart [1926], Schouten [1954], or Synge and Schild [1956];
(e) a book on functional analysis such as Balakrishnan [1976], Oden [1979], or Yosida [1971].

More advanced readers should consult other contemporary works for comparisons and other points of view. For example, we find the following additional references useful:

(a) Kondo [1955] for an early attempt at the use of geometry in elasticity,
(b) Truesdell and Toupin [1960], Rivlin [1966a], and Eringen [1975] on basic principles;
(c) Gurtin [1972a] on linear elasticity;
(d) Knops and Wilkes [1973] on elastic stability;

(e) Fichera [1972a and b] and Knops and Payne [1971] on existence and uniqueness theorems;
(f) Bloom [1979] on the use of geometry in dislocation theory;
(g) Naghdi [1972] on general shell theory and Ciarlet [1983] on the derivation of plate theory from three dimensional elasticity;
(h) Antman [1972a], [1983] on rod theory and bifurcations in elasticity.

Acknowledgments The main part of this book grew out of a course given by us at Berkeley in 1975–76. A preliminary set of notes by us was published in Volume II of *Nonlinear Analysis and Mechanics*, edited by R. Knops (Pitman, 1978). We are indebted to Professor Knops for encouraging this publication and to the readers who sent us comments.

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We are indirectly indebted to the founding masters of the modern theory, especially Ronald Rivlin and Clifford Truesdell. Their works and those of Stuart Antman, John Ball, Morton Gurtin, Robin Knops, Paul Naghdi, and Walter Noll have had a large influence on our development of the subject.

A BRIEF GLOSSARY OF CONVENTIONS AND NOTATIONS

CONVENTIONS: As far as possible, we shall use the following:

(a) sets and manifolds—lightface script; examples: $\mathcal{B}$, $\mathcal{S}$, $\mathcal{M}$

(b) points, point mappings, scalars—lightface italic; examples: $X \in \mathcal{B}$, $\phi: \mathcal{B} \longrightarrow \mathcal{S}$

(c) vectors and vector fields, 2-tensors—boldface italic; examples: $\boldsymbol{v}$, $\boldsymbol{V}$, $\boldsymbol{F}$, $\boldsymbol{C}$, $\boldsymbol{\sigma}$ $\boldsymbol{P}$, $\boldsymbol{S}$

(d) higher-order tensors—boldface block letters; examples: **A, C**

(e) material quantities—upper case

(f) spatial quantities—lower case

NOTATIONS

$\mathbb{R}^n$	Euclidean n-space
$\mathcal{B}$	reference configuration of a body
X	point in $\mathcal{B}$
$\mathcal{S}$	the space in which the body moves (usually $\mathcal{S} = \mathbb{R}^3$)
x	point in $\mathcal{S}$
$\{X^A\}$	coordinates on $\mathcal{B}$
$\{x^a\}$	coordinates on $\mathcal{S}$
Z^I, z^i	Euclidean coordinates on $\mathcal{B} \subset \mathbb{R}^3$, $\mathcal{S} = \mathbb{R}^3$
G_{AB} or $\boldsymbol{G}$	Riemannian metric on $\mathcal{B}$
g_{ab} or $\boldsymbol{g}$	Riemannian metric on $\mathcal{S}$
Γ^A_{BC}	Christoffel symbols for $\boldsymbol{G}$
γ^a_{bc}	Christoffel symbols for $\boldsymbol{g}$
$T\mathcal{M}$	tangent bundle of a manifold $\mathcal{M}$
ϕ^a or $\phi: \mathcal{B} \longrightarrow \mathcal{S}$	a configuration (or deformation)

dV	material volume element as a measure
$\boldsymbol{dV}$	material volume element as a tensor
U^a or $\boldsymbol{U}$	displacement
u^a or $\boldsymbol{u}: \mathcal{B} \longrightarrow T\mathcal{S}$	linearization of a deformation
$F^a{}_A$ or $\boldsymbol{F} = T\phi$	deformation gradient = tangent of ϕ
v^a or $\boldsymbol{v}$	spatial velocity
V^a or $\boldsymbol{V}$	material velocity
a^a or $\boldsymbol{a}$	spatial acceleration
A^a or $\boldsymbol{A}$	material acceleration
C_{AB} or $\boldsymbol{C}$	(left) Cauchy–Green or deformation tensor
t^a or $\boldsymbol{t}$	Cauchy stress vector
σ^{ab} or $\boldsymbol{\sigma}$	Cauchy stress tensor
P^{aA} or $\boldsymbol{P}$	first Piola–Kirchhoff stress tensor
$\hat{P}^{aA}$ or $\hat{\boldsymbol{P}}$	corresponding constitutive function
S^{AB} or $\boldsymbol{S}$	second Piola–Kirchhoff stress tensor
A^{aAbB} or A	first elasticity tensor
C^{ABCD} or C	second elasticity tensor
c^{abcd} or c	classical elasticity tensor (for the linearized theory)
d_{ab} or $\boldsymbol{d} = \boldsymbol{L}_{\boldsymbol{v}}\boldsymbol{g}$	rate of deformation tensor
$\rho(x)$	mass density in the current configuration
$\rho_{\text{Ref}}(X)$	mass density in the reference configuration
W or E or e	stored energy function
$\hat{W}$ or $\hat{E}$ or $\hat{e}$	corresponding constitutive function
Ψ or ψ	free energy
$\hat{\Psi}$ or $\hat{\psi}$	corresponding constitutive function
Θ, N	temperature, entropy
$\phi^*\mathbf{t}$	pull-back of a tensor $\mathbf{t}$ by the map ϕ
$\phi_*\mathbf{t} = (\phi^{-1})^*\mathbf{t}$	push-forward by the map ϕ
$\dot{\mathbf{t}}$	material derivative of a tensor $\mathbf{t}$
$L_v\mathbf{t} = \dfrac{\partial \mathbf{t}}{\partial t} + \mathcal{L}_v\mathbf{t}$	Lie derivative of a tensor $\mathbf{t}$ along a vector field $\boldsymbol{v}$
SO(3)	proper orthogonal transformations of $\mathbb{R}^3$
$I_1(C), I_2(C), I_3(C)$	invariants of a symmetric 3×3 matrix $\boldsymbol{C}$
$f: \mathfrak{X} \longrightarrow \mathfrak{Y}$	map between Banach spaces
$Df(x_0)$	derivative of f at a point $x_0 \in \mathfrak{X}$
e^{tA}	semigroup generated by an operator A
$\boldsymbol{AB}$	multiplication of linear maps or matrices
Ax or $\boldsymbol{A} \cdot x$	linear operator applied to x
$Df(x) \cdot v$	derivative of f at x in direction v
$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \sigma^{ab} n_b$	contraction of tensors
$\boldsymbol{S}: \boldsymbol{D} = S^{AB} D_{AB}$	double contraction of tensors
$f \circ g$	composition of maps

CONTENTS

A POINT OF DEPARTURE

This preliminary chapter provides a quick survey of a few standard topics in elasticity theory from a classical point of view. The treatment is exclusively in Euclidean space $\mathbb{R}^3$, using standard Euclidean coordinates. One of the first tasks we face in the book is to repeat this material in a more general "intrinsic" context. The preview aims to be as elementary as possible, while still getting to a few issues of current interest. It can be read prior to, or in conjunction with, the main body of the text. The only background needed is calculus of several variables and linear algebra; some first-year physics is helpful for motivation.[1]

Warning. This introductory material is not where the book actually starts. This is intended to give certain readers a quick overview of where we are going. It proceeds at a very different tempo from the text and is sometimes chatty and imprecise. The material presented may be good for some readers, especially mathematicians who wish to learn elasticity for the first time. Experienced readers may wish to omit this and turn directly to Chapter 1.

1. KINEMATICS

In continuum mechanics, *kinematics* refers to the mathematical description of the deformation and motion of a piece of material. For example, if the beam shown in (a) of Figure 1.1 is loaded, it will bend. This is an example of a *deforma-*

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[1]We are indebted to John Ball, whose lectures "Elementary Elasticity from Scratch" at Berkeley inspired this preview. Parts of the exposition are taken directly from his lectures, but have been rewritten to conform to the notations of this book. Of course any inaccuracies are the responsibility of the authors.