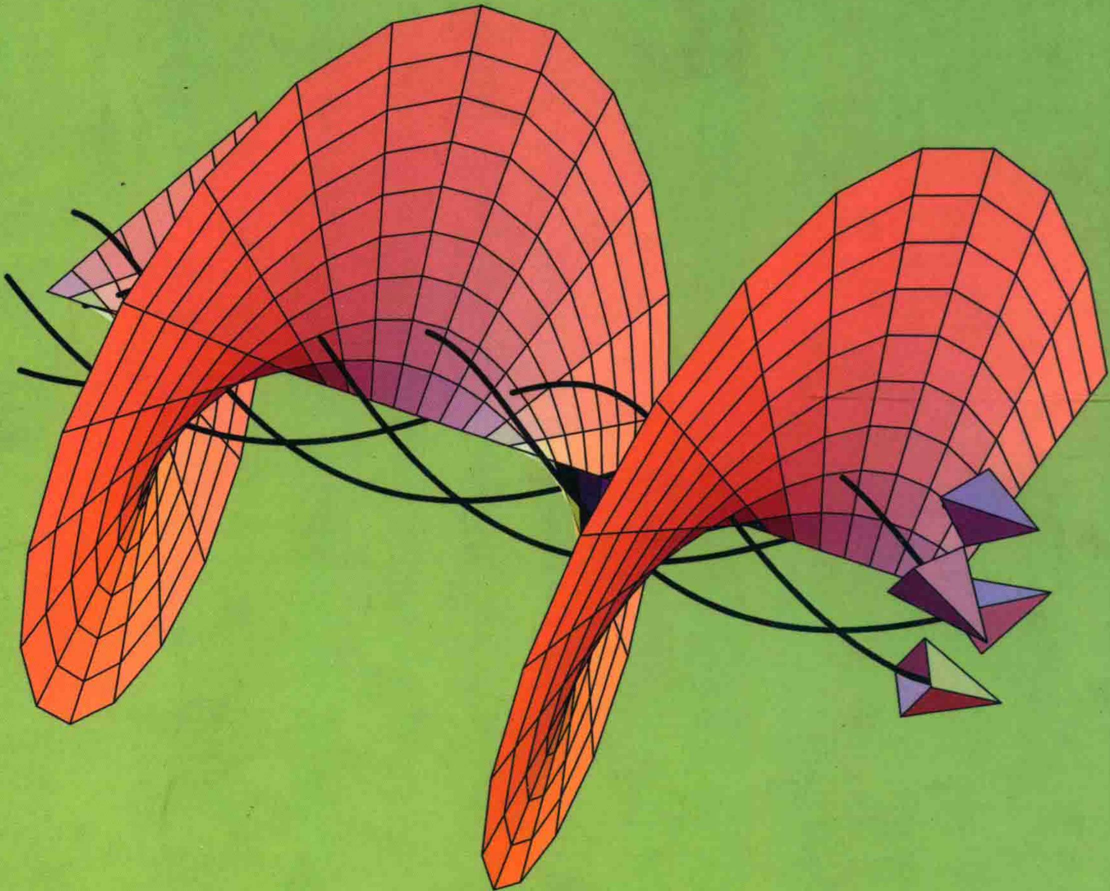

Optical Angular Momentum

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IoP

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Optical Angular Momentum

Preface

It has been recognised for a long time that a photon has spin angular momentum, observable macroscopically in a light beam as polarisation. It is less well known that a beam may also carry orbital angular momentum linked to its phase structure. Although both forms of angular momentum have been identified in electromagnetic theory for very many years, it is only over the past decade that orbital angular momentum has been the subject of intense theoretical and experimental study. The concepts combine neatly into optical angular momentum.

This book is designed to be an accessible survey of the current status of optical angular momentum. It reproduces 44 original papers arranged in eight sections. Each section has a brief introduction to set the reproduced papers in the context of a wider range of related work.

It is a pleasure to thank, in the same order as their papers appear here, G Nienhuis, MV Berry, JM Vaughan, C Tamm, JP Woerdman, L Torner, MS Soskin, NR Heckenberg, A Ashkin, H Rubinsztein-Dunlop, SJ van Enk, DV Petrov, BA Garetz, I Bialynicki-Birula, WJ Firth, M Segev and A Zeilinger for being willing to have their work and that of their co-authors reproduced.

We are pleased, too, to acknowledge the publishers who allowed us to reproduce papers originally published in their journals: the *Royal Society* for paper **1.1**; the *American Physical Society* for papers **1.2**, **2.1**, **2.3**, **2.4**, **2.7**, **3.6**, **3.10**, **4.2**, **4.3**, **5.1**, **5.3**, **6.4**, **6.6**, **7.1**, **7.2**, **7.3**, **8.2** and **8.3**; the *Optical Society of America* for papers **3.2**, **3.3**, **4.1**, **4.4**, **6.1** and **6.3**; *SPIE* – the *International Society for Optical Engineering* for **2.5**; *Elsevier Science B.V* for papers **2.2**, **2.6**, **2.9**, **3.4**, **3.5**, **3.11**, **3.12**, **6.2** and **6.5**; *Kluwer Academic Publishers* for **3.9**; *Nature* for paper **8.1**; *EDP Sciences* for **2.8**; the *American Association of Physics Teachers* and the *American Institute of Physics* for **3.1** and **3.8**; while four papers were published by the *Institute of Physics Publishing,* namely **1.3**, **2.10**, **3.7** and **5.2**.

We wish to express, too, our gratitude to the many friends, too numerous to list here, with whom we have enjoyed exploring the fascinating topic of optical angular momentum.

Optical angular momentum is a new area of physics, but one for which the foundations have been firmly established. It will be very interesting to see how it develops over the next few years.

L Allen
Stephen M Barnett
Miles J Padgett

August 2002

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Section 1

INTRODUCTION

That light should have mechanical properties has been known, or at least suspected, since Kepler proposed that the tails of comets were due to radiation pressure associated with light from the sun. A quantitative theory of such effects became possible only after the development of Maxwell's unified theory of electricity, magnetism and optics. However, although his treatise on electromagnetism (1.1) contains a calculation of the radiation pressure at the earth's surface, there is little more on the mechanical effects of light. It was Poynting who quantified the momentum and energy flux associated with an electromagnetic field (1.2). In modern terms, the momentum per unit volume associated with an electromagnetic wave is given by $\varepsilon_0 \mathbf{E} \times \mathbf{B}$. The angular momentum density is, naturally enough, the cross product of this with position, that is $\mathbf{r} \times \varepsilon_0(\mathbf{E} \times \mathbf{B})$ (1.3).

Poynting reasoned that circularly polarised light must carry angular momentum (1.4, **Paper 1.1**). His argument proceeded by analogy with the wave motion associated with a line of dots marked on a rotating cylindrical shaft. His calculation showed that $E\lambda/2\pi$ is the angular momentum transmitted through a plane in unit time, per unit area, where E is Poynting's notation for the energy per unit volume and λ is the wavelength. When the energy of each photon crossing the surface is associated with $\hbar\omega$, we obtain the result that circularly polarised photons each carry $\hbar$ units of angular momentum.

Poynting's paper concludes with a proposal for measuring the angular momentum associated with circularly polarised light. His idea was that circularly polarised light passing through a large number of suspended quarter-wave plates, and so becoming linearly polarised, should give up all its angular momentum and so induce a rotation in the suspension. He concludes, however, that "my present experience of light-forces does not give me much hope that the effect could be detected". The effect was detected, however, about twenty years after Poynting's death by Beth (1.5, **Paper 1.2**) who used a single quarter wave plate, together with a mirror which sent the light back through the plate enhancing the torque on the suspension. He showed that the same quantitative result is obtained for the classical torque as for that which arises from the assumption that each photon carried an angular momentum $\hbar$.

Careful examination of $\varepsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ shows that polarisation does not account for all of the angular momentum that can be carried by the electromagnetic field. The part associated with polarisation is known as spin, but in addition there is also an orbital contribution. Until recently, however, the discussion of spin and orbital angular momenta for light was largely restricted to textbooks and related to non-specific forms for the electric field. Current research activity in this area originated with the realisation that physically realisable light beams, familiar from the paraxial optics of laser theory, can carry a well-defined quantity of orbital angular momentum for each photon (1.6, **Paper 2.1**). An extensive review of orbital angular momentum was published in 1999 (1.7). This was followed, in 2002, by the publication of a special issue of *Journal of Optics B* devoted to atoms and angular momentum of light. The introduction to that special issue provides a brief overview of the current status of the field and is reprinted here as the natural introduction to this book (1.8, **Paper 1.3**).

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The Wave Motion of a Revolving Shaft, and a Suggestion as to the Angular Momentum in a Beam of Circularly Polarised Light

By JH POYNTING, ScD, FRS
(Received June 2,—Read June 24, 1909)

When a shaft of circular section is revolving uniformly, and is transmitting power uniformly, a row of particles originally in a line parallel to the axis will lie in a spiral of constant pitch, and the position of the shaft at any instant may be described by the position of this spiral.

Let us suppose that the power is transmitted from left to right, and that as viewed from the left the revolution is clockwise. Then the spiral is a left-handed screw. Let it be on the surface, and there make an angle ε with the axis. Let the radius of the shaft be a, and let one turn of the spiral have length λ along the axis. We may term λ the wave-length of the spiral. We have $\tan\varepsilon = 2\pi a/\lambda$. If the orientation of the section at the origin at time t is given by $\theta = 2\pi \mathrm{N}t$, where N is the number of revolutions per second, the orientation of the section at x is given by

$$\theta = 2\pi \mathrm{N}t - \frac{x}{a}\tan\varepsilon = \frac{2\pi}{\lambda}(\mathrm{N}\lambda t - x), \tag{1}$$

which means movement of orientation from left to right with velocity $\mathrm{N}\lambda$.

The equation of motion for twist waves on a shaft of circular section is

$$\frac{d^2\theta}{dt^2} = \mathrm{U}_n{}^2\frac{d^2\theta}{dx^2}, \tag{2}$$

where $\mathrm{U}_n{}^2$ = modulus of rigidity/density $= n/\rho$.

Though (1) satisfies (2), it can hardly be termed a solution for $d^2\theta/dt^2$, and $d^2\theta/dx^2$ in (1) are both zero. But we may adapt a solution of (2) to fit (1) if we assume certain conditions in (1).

The periodic value

$$\theta = \Theta\sin\frac{2\pi}{l}(\mathrm{U}_n t - x)$$

satisfies (2), and is a wave motion with velocity U_n and wave-length l. Make l so great that for any time or for any distance under observation $U_n t/l$ and x/l are so small that the angle may be put for the sine. Then

$$\theta = \Theta \frac{2\pi}{l}(U_n t - x). \tag{3}$$

This is uniform rotation. It means that we only deal with the part of the wave near a node, and that we make the wave-length l so great that for a long distance the "displacement curve" obtained by plotting θ against t coincides with the tangent at the node. We must distinguish, of course, between the wave-length l of the periodic motion and the wave-length λ of the spiral.

We can only make (1) coincide with (3) by putting

$$\Theta/l = 1/\lambda \quad \text{and} \quad N\lambda = U_n.$$

Then it follows that for a given value of N, the impressed speed of uniform rotation, there is only one value of λ or one value of ε for which the motion may be regarded as part of a natural wave system, transmitted by the elastic forces of the material with velocity $= \sqrt{(n/\rho)}$. There is therefore only one "natural" rate of transmission of energy.

The value of ε is given by

$$\tan\varepsilon = 2\pi a/\lambda = 2\pi a N/N\lambda = 2\pi a N/U_n = 2\pi a N\sqrt{(\rho/n)}.$$

Suppose, for instance, that a steel shaft with radius $a = 2$ cm, density $\rho = 7.8$, and rigidity $n = 10^{12}$ is making $N = 10$ revs. per sec. We may put $\tan\varepsilon = \varepsilon$, since it is very small. The shaft is twisted through 2π in length λ or through $2\pi/\lambda$ per centimetre, and the torque across a section is

$$G = \tfrac{1}{2} n\pi a^4 2\pi/\lambda = n\pi^2 a^4 N\sqrt{(\rho/n)},$$

since

$$\lambda = \frac{U_n}{N} = \frac{1}{N}\sqrt{\frac{n}{\rho}}.$$

The energy transmitted per second is

$$2\pi NG = 2\pi^3 a^4 N^2 \sqrt{(n\rho)}.$$

Putting 1 H.P. $= 746 \times 10^7$ ergs per second, this gives about 38 H.P.

But a shaft revolving with given speed N can transmit any power, subject to the limitation that the strain is not too great for the material. When the power is not that "naturally" transmitted, we must regard the waves as "forced." The velocity of transmission is no longer U_n, and forces will have to be applied from outside in addition to the internal elastic forces to give the new velocity.

Let H be the couple applied per unit length from outside. Then the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = {U_n}^2 \frac{d^2\theta}{dx^2} + \frac{2H}{\pi a^4},$$

where $\frac{1}{2}\pi a^4$ is the moment of inertia of the cross section. Assuming that the condition travels on with velocity U unchanged in form,

$$\frac{d\theta}{dt} = -\mathrm{U}\frac{d\theta}{dx} \quad \text{and} \quad \mathrm{H} = \tfrac{1}{2}\pi a^4(\mathrm{U}_n{}^2)\frac{d^2\theta}{dx^2},$$

or H has only to be applied where $d^2\theta/dx^2$ has value, that is where the twist is changing.

The following adaptation of Rankine's tube method of obtaining wave velocities* gives these results in a more direct manner. Suppose that the shaft is indefinitely extended both ways. Any twist disturbance may be propagated unchanged in form with any velocity we choose to assign, if we apply from outside the distribution of torque which, added to the torque due to strain, will make the change in twist required by the given wave motion travelling at the assigned speed.

Let the velocity of propagation be U from left to right, and let the displacement at any section be θ, positive if clockwise when seen from the left. The twist per unit length is

$$\frac{d\theta}{dx} = -\frac{1}{\mathrm{U}}\frac{d\theta}{dt} = -\frac{\dot{\theta}}{\mathrm{U}}.$$

The torque across a section from left to right in clockwise direction is

$$-\tfrac{1}{2}n\pi a^4\frac{d\theta}{dx} = \frac{n\pi a^4}{2\mathrm{U}}\,.\,\dot{\theta}.$$

Let the shaft be moved from right to left with velocity U; then the disturbance is fixed in space, and if we imagine two fixed planes drawn perpendicular to the axis, one, A, at a point where the disturbance is θ and the other, B, outside the wave system, where there is no disturbance, the condition between A and B remains constant, except that the matter undergoing that condition is changing. Hence the total angular momentum between A and B is constant. But no angular momentum enters at B, since the shaft is there untwisted and has merely linear motion. At A, then, there must be on the whole no transfer of angular momentum from right to left. Now, angular momentum is transferred in three ways:—

1. By the carriage by rotating matter. The angular momentum per unit length is $\frac{1}{2}\rho\pi a^4\dot{\theta}$, and since length U per second passes out at A, it carries out $\frac{1}{2}\rho\pi a^4\dot{\theta}\mathrm{U}$.

2. By the torque exerted by matter on the right of A on matter on the left of A. This takes out $-n\pi a^4\dot{\theta}/2\mathrm{U}$.

3. By the stream of angular momentum by which we may represent the forces applied from outside to make the velocity U instead of U_n.

If H is the couple applied per unit length, we may regard it as due to the flow of angular momentum L along the shaft from left to right, such that $\mathrm{H} = -d\mathrm{L}/dx$. There is then angular momentum L flowing out per second from right to left. Since the total flow due to (1), (2), and (3) is zero,

$$\tfrac{1}{2}\rho\pi a^4\dot{\theta}\mathrm{U} - n\pi a^4\dot{\theta}/2\mathrm{U} - \mathrm{L} = 0,$$

and

$$\mathrm{L} = \frac{\pi a^4\dot{\theta}}{2}\left(\rho\mathrm{U} - \frac{n}{\mathrm{U}}\right) = \frac{\rho\pi a^4\dot{\theta}}{2\mathrm{U}}(\mathrm{U}_n{}^2) = -\frac{\rho\pi a^4}{2}\frac{d\theta}{dx}(\mathrm{U}^2 - \mathrm{U}_n{}^2),$$

* 'Phil. Trans.,' 1870, p. 277.

and

$$H = -\frac{dL}{dx} = \frac{\rho\pi a^4}{2}\frac{d^2\theta}{dx^2}(U_n{}^2).$$

If $H = 0$, either $U^2 = U_n{}^2$ when the velocity has its "natural value," or $d^2\theta/dx^2 = 0$, and the shaft is revolving with uniform twist in the part considered.

Now put on to the system a velocity U from left to right. The motion of the shaft parallel to its axis is reduced to zero, and the disturbance and the system H will travel on from left to right with velocity U. A "forced" velocity does not imply *transfer* of physical conditions by the material with that velocity. We can only regard the conditions as reproduced at successive points by the aid of external forces. We may illustrate this point by considering the incidence of a wave against a surface. If the angle of incidence is i and the velocity of the wave is V, the line of contact moves over the surface with velocity $v = V/\sin i$, which may have any value from V to infinity. The velocity v is not that of transmission by the material of the surface, but merely the velocity of a condition impressed on the surface from outside.

Probably in all cases of transmission with forced velocity, and certainly in the case here considered, the velocity depends upon the wave-length, and there is dispersion.

With a shaft revolving N times per second $U = N\lambda$, and it is interesting to note that the group velocity $U - \lambda dU/d\lambda$ is zero. It is not at once evident what the group velocity signifies in the case of uniform rotation. In ordinary cases it is the velocity of travel of the "beat" pattern, formed by two trains of slightly different frequencies. The complete "beat" pattern is contained between two successive points of agreement of phase of the two trains. In our case of superposition of two strain spirals with constant speed of rotation, points of agreement of phase are points of intersection of the two spirals. At such points the phases are the same, or one has gained on the other by 2π. Evidently as the shaft revolves these points remain in the same cross-section, and the group velocity is zero.

With deep water waves the group velocity is half the wave velocity, and the energy flow is half that required for the onward march of the waves.* The energy flow thus suffices for the onward march of the group, and the case suggests a simple relation between energy flow and group velocity.

But the simplicity is special to unforced trains of waves. Obviously, it does not hold when there are auxiliary working forces adding or subtracting energy along the waves. For the revolving shaft the simple relation would give us no energy flow, whereas the strain existing in the shaft implies transmission of energy at a rate given as follows.

The twist per unit length is $d\theta/dx$, and therefore the torque across a section is $-\frac{1}{2}n\pi a^4\, d\theta/dx$, or $\frac{1}{2}n\pi a^4\dot{\theta}/U$, since $d\theta/dx = -\dot{\theta}/U$. The rate of working or of energy flow across the section is $\frac{1}{2}n\pi a^4\dot{\theta}^2/U$.

The relation of this to the strain and kinetic energy in the shaft is easily found. The strain energy per unit length being $\frac{1}{2}$ (couple $\times$ twist per unit length) is $\frac{1}{4}n\pi a^4(d\theta/dx)^2$, which is $\frac{1}{4}n\pi a^4\dot{\theta}^2/U^2$. The kinetic energy per unit length is $\frac{1}{2}\rho\pi a^4\dot{\theta}^2$, or, putting $\rho = n/U_n{}^2$, is $\frac{1}{4}n\pi a^4\dot{\theta}^2/U_n{}^2$.

In the case of natural velocity, for which no working forces along the shaft are needed, when $U = U_n = \sqrt{(n/\rho)}$, the kinetic energy is equal to the strain energy at every point and the energy transmitted across a section per second is that contained in length U_n.

*O. Reynolds, 'Nature,' August 23, 1877; Lord Rayleigh, 'Theory of Sound,' vol. 1, p. 477.