STUDIES IN MATHEMATICS AND ITS APPLICATIONS

30

OPERATOR THEORY AND NUMERICAL METHODS

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Preface

Recent developments in numerical computations are amazing. A lot of huge projects in applied and theoretical sciences are becoming successful by them, while similar things are happening even in the level of personal computers. Under such a situation, theoretical studies on numerical schemes are fruitful and highly needed.

The purpose of the present book is to provide some of them, particularly for schemes to solve partial differential equations. In 1991, we published an article on the finite element method applied to evolutionary problems from Elsevier Publishers (Fujita and Suzuki [148]). This book follows basically that way of description. We study various schemes from the operator theoretical point of view. Many parts are devoted to the finite element method, of which history is described in Oden [306]. We deal with elliptic and then time dependent problems in use of the semigroup theory and so forth. Some other schemes and problems are also discussed, with the later development taken also into account. We are led to believe that any scheme used practically has significant and tight structures mathematically and the converse is also true.

Besides some corrections and supplementary descriptions, we have added the following topics: (1) L^p estimate of the Ritz operator associated with the finite element approximation (2) asymptotic expansions and Richardson's extrapolation for the finite element solution (3) Trotter-Lee's product formula for holomorphic semigroups (4) mixed finite element method (5) Nehari's iterative method for non-stable solutions of elliptic problems (6) finite element approximation of nonlinear semigroups and applications (7) boundary element method for elliptic problems (8) charge simulation method for elliptic problems (9) domain decomposition method for elliptic problems.

To make the description consistent, the domain in consideration is mostly supposed to be a convex polygon, and piecewise linear trial functions are adopted unless otherwise stated. The other cases are described in the notes at the end of chapters.

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Contents

Chapter 1

Elliptic Boundary Value Problems and FEM

The present chapter is concerned with the finite element method (FEM) applied to elliptic boundary value problems. For this topic, we have several monographs such as Strang and Fix [359], Ciarlet [83], Raviart and Thomas [326], Johnson [193], Ciarlet and Lions [85], Szabó and Babušuka [372], and Brenner and Scott [60]. Here, we describe it in the framework of operator theory, picking up approximate operators of the schemes. This way is natural and efficient, particularly in dealing with time dependent problems, because the finite element method is regarded as a discretization of the underlying variational structure.

1.1 Elliptic Boundary Value Problems

To fix the idea, let $\Omega \subset \mathbb{R}^2$ be a convex polygon, and consider the Poisson equation

$$-\Delta u = f \qquad \text{in } \Omega \tag{1.1}$$

with Dirichlet condition

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.2}$$

This boundary value problem has the *weak form*, described under the following notations.

1° $L^p(\Omega)$ denotes the set of p-integrable functions for $p \in [1, \infty)$ and that of essentially bounded functions for $p = \infty$. Unless otherwise stated we assume functions to be real-valued and Banach spaces real.

2° $W^{m,p}(\Omega)$ denotes the Sobolev space, the set of measurable functions with their distributional derivatives up to m-th order belonging to $L^p(\Omega)$, where $m = 0, 1, 2, \cdots$.

3° $C_0^\infty(\Omega)$ denotes the set of infinitely many times differentiable functions having compact supports contained in Ω.

4° $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

5° $H^m(\Omega)$ and $H_0^m(\Omega)$ stand for $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$, respectively. We set $|v|_m = |v|_{m,\Omega} = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^2(\Omega)}$.

6° (,) and $\mathcal{A}(\ ,\ )$ denote the L^2 inner product and the Dirichlet form, respectively:

$$(f,g) = \int_\Omega f(x)g(x)\ dx,$$
$$\mathcal{A}(u,v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx.$$

7° The trace operator is denoted by

$$\gamma = \ \cdot\ |_{\partial\Omega} : H^1(\Omega) \ \rightarrow\ H^{1/2}(\partial\Omega).$$

It holds that $\mathrm{Ker}(\gamma) = H_0^1(\Omega)$.

Then, (1.1) with (1.2) is reduced to the problem in $V = H_0^1(\Omega)$: find $u \in V$ satisfying

$$\mathcal{A}(u,v) = (f,v) \tag{1.3}$$

for any $v \in V$. Supposing that u is smooth, we have by Green's formula that

$$\mathcal{A}(u,v) = \int_\Omega \nabla u\cdot\nabla v\ dx = -\int_\Omega (\Delta u)v\ dx = (f,v),$$

since $v \in H_0^1(\Omega)$. Equality (1.1) follows from arbitrariness of v.

Henceforth, various generic constants are denoted indifferently by C. If it depends on some parameters, say $\alpha, \beta, \cdots$, we may write it as $C_{\alpha,\beta,\cdots}$.

Poincaré's inequality is indicated as

$$\|v\|_{L^2(\Omega)} \le C_\Omega \|\nabla v\|_{L^2(\Omega)} \qquad (v\in V) \tag{1.4}$$

so that $\mathcal{A}(\ ,\ )$ provides an inner product to the Hilbert space $V = H_0^1(\Omega)$. Let V' be the dual space of V. Then, regarding $f \in L^2(\Omega)$ as an element of V' through

$$\langle f, v\rangle_{V',V} = (f,v)$$

for $v \in V$, we can verify the unique solvability of (1.3) from Riesz' representation theorem. Because Ω is convex, *regularity* of the *weak solution* follows; $u \in V$ belongs to $H^2(\Omega)$ if $f \in L^2(\Omega)$. Therefore, the above calculation is justified and u becomes a *strong solution*.

We proceed to the case of $V = H^1(\Omega)$ in (1.3). Assuming that u is smooth, we obtain similarly that

$$\mathcal{A}(u,v) = \int_\Omega \nabla u\cdot\nabla v\ dx = \int_{\partial\Omega}\frac{\partial u}{\partial n}v\ dS - \int_\Omega (\Delta u)v\ dx = (f,v).$$

Here, $\partial/\partial n$ denotes the differentiation along the outer unit normal vector $n = (n_1, n_2)$ on $\partial\Omega$ and dS denotes the surface element of $\partial\Omega$. Taking $v \in H_0^1(\Omega)$, we have

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}v\ dS = 0$$

so that equality (1.1) follows from arbitrariness of $v \in H_0^1(\Omega)$. This implies

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS = 0$$

for any $v \in H^1(\Omega)$ and hence

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

follows.

In other words, Neumann problem

$$-\Delta u = f \qquad \text{in } \Omega \tag{1.5}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{1.6}$$

is reduced to variational problem (1.3) for $V = H^1(\Omega)$.

Since Poincaré's inequality (1.4) dose not hold for $V = H^1(\Omega)$, this time $\mathcal{A}(\ ,\ )$ does not provide an inner product to $H^1(\Omega)$. Actually, problem (1.5) with (1.6) is not uniquely solvable. For instance, any constant function $u \equiv c \in \mathbb{R}$ satisfies (1.5) with (1.6) for $f = 0$.

Above considerations are generalized in the following way. Let $a_{ij}(x) = a_{ji}(x), b_j(x)$ and $c(x)$ be real-valued smooth functions on $\overline{\Omega}$ and suppose that the uniform ellipticity

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \delta_1 |\xi|^2 \tag{1.7}$$

holds for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x \in \overline{\Omega}$ with a constant $\delta_1 > 0$. Given

$$\mathcal{L}(x, D) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{j=1}^{2} b_j(x) \frac{\partial}{\partial x_j} + c(x),$$

we take the boundary value problem

$$\mathcal{L}(x, D)u = f \qquad \text{in } \Omega \tag{1.8}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.9}$$

Then it is reduced to problem (1.3), where $V = H_0^1(\Omega)$ and

$$\mathcal{A}(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{j=1}^{2} b_j(x) \frac{\partial u}{\partial x_j} v + c(x)uv \right\} dx.$$

Similarly, if a smooth function $\sigma(\xi)$ on $\partial\Omega$ is given and

$$\frac{\partial}{\partial n_{\mathcal{L}}} = \sum_{i,j=1}^{2} n_i(x) a_{ij}(x) \frac{\partial}{\partial x_j}$$

denotes the outer co-normal differentiation associated with $\mathcal{L}$, boundary value problem (1.8) with

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + \sigma u = 0 \qquad \text{on } \partial\Omega \tag{1.10}$$

is reduced to variational problem (1.3) with $V = H^1(\Omega)$ and

$$\mathcal{A}(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{j=1}^{2} b_j(x) \frac{\partial u}{\partial x_j} v + c(x)uv \right\} dx + \int_{\partial\Omega} \sigma u \cdot v \, dS \tag{1.11}$$

for $u, v \in V$.

Those bilinear forms are bounded in the sense that

$$|\mathcal{A}(u,v)| \leq C \|u\|_V \|v\|_V \qquad (u, v \in V). \tag{1.12}$$

Given $\epsilon > 0$, the trace operator $\gamma = \cdot|_{\partial\Omega}$ admits a constant $C_\epsilon > 0$ satisfying

$$\|\gamma v\|_{L^2(\partial\Omega)} \leq \epsilon \|v\|_{H^1(\Omega)} + C_\epsilon \|v\|_{L^2(\Omega)} \tag{1.13}$$

for any $v \in H^1(\Omega)$. Therefore, for $\delta \in (0, \delta_1)$ and $\lambda \in \mathbb{R}$, it holds that

$$\mathcal{A}(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_X^2 \qquad (v \in V) \tag{1.14}$$

by (1.7). Here and henceforth, we set $X = L^2(\Omega)$.

In the case that

$$b_j(x) \equiv 0, \qquad c(x) \geq 0, \qquad \text{and} \qquad V = H_0^1(\Omega),$$

or

$$b_j(x) \equiv 0, \qquad c(x) > 0, \qquad \sigma(\xi) \geq 0, \qquad \text{and} \qquad V = H^1(\Omega),$$

we can take $\lambda = 0$ in (1.14). Then, problem (1.3) is uniquely solvable by Lax-Milgram's theorem.

In use of the dual space V' of V, boundedness (1.12) of $\mathcal{A}(\ ,\ )$ implies the well-definedness of the bounded linear operator $A: V \to V'$ through the relation

$$\mathcal{A}(u,v) = \langle Au, v \rangle_{V',V}$$

for $u, v \in V$. On the other hand, identifying X' with X by Riesz' representation theorem provides a triple of Hilbert spaces $V \subset X \subset V'$ with continuous and dense inclusions. Let

$$D(A) = \{u \in V \mid Au \in X\}.$$

We shall write the restriction of $A: V \to V'$ to $D(A)$ by the same symbol A. It is regarded as an operator in X, and (1.3) is written as an abstract equation in X if $f \in X = L^2(\Omega)$:

$$Au = f$$

The elliptic regularity is expressed as

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

for boundary condition (1.9) and

$$D(A) = \left\{ v \in H^2(\Omega) \,\middle|\, \frac{\partial v}{\partial n_{\mathcal{L}}} + \sigma v = 0 \quad \text{on } \partial\Omega \right\} \tag{1.15}$$

for boundary condition (1.10). Those relations hold if $\partial\Omega$ is sufficiently smooth for instance. We get a strong solution $u \in H^2(\Omega)$ of (1.8) with (1.9) or (1.10), whenever $\lambda = 0$ holds in (1.14).

The operator A in X arising in such a way from the bilinear form $\mathcal{A}(\ ,\ )$ on $V \times V$ is called *m-sectorial*. To describe the meaning of this terminology, let us suppose $\lambda = 0$ in (1.14) for simplicity, and specify the constant C in (1.12) as C_1. We can take natural complex extensions of the Banach space X and the operator A. This means that the functions in X are extended to be complex-valued with the inner product

$$(f, g) = \int_\Omega f(x)\overline{g(x)}\, dx$$

and $Af = Af_1 + \imath Af_2$ if $f = f_1 + \imath f_2$ with f_1 and f_2 being real-valued, where $\imath = \sqrt{-1}$. Denote them by the same symbols X and A. Then, inequality (1.12) keeps to hold, while (1.14) is replaced by

$$\operatorname{Re} \mathcal{A}(v, v) \geq \delta \left\|v\right\|_V^2 \qquad (v \in V).$$

A sector is given as

$$\Sigma_\theta = \{ z \in \mathbb{C} \mid 0 \leq |\arg\ z| \leq \theta \},$$

where $\theta \in (0, \pi/2)$. If $\cos\theta = \delta/C_1$, an elementary calculation gives that the numerical range $\nu(\mathcal{A})$ of $\mathcal{A}$ is included in Σ_θ, where

$$\nu(\mathcal{A}) \equiv \{ \mathcal{A}(u, u) \mid u \in V, \quad \|u\|_X = 1 \}.$$

A fundamental property of the numerical range says $\sigma(A) \subset \nu(\mathcal{A})$. The relation $\nu(\mathcal{A}) \subset \Sigma_\theta$ implies that $\mathbb{C} \setminus \Sigma_\theta \subset \rho(A)$, where $\sigma(A)$ and $\rho(A)$ denotes the spectrum and the resolvent set of A, respectively. More precisely, if we take $\theta_1 \in (\theta, \pi/2)$ and $z \in \mathbb{C}$ in $\theta_1 \leq |\arg\ z| \leq \pi$, then we have

$$\left\|(zI - A)^{-1}\right\|_{X,X} \leq \frac{1}{\sin(\theta_1 - \theta) \cdot |z|}. \tag{1.16}$$

In particular, $-A$ generates a holomorphic semigroup $\left\{e^{-tA}\right\}_{t\geq 0}$ in X, of which details are discussed in the next chapter.

Inequality (1.16) is proven in the following way. Let

$$d(z) = \text{dist}\ (z, \partial\Sigma_\theta) \equiv \inf\ \{ |z - \zeta| \mid \zeta \in \Sigma_\theta \}.$$

Then, because $z \in \Sigma_{\theta_1}$ we have

$$d(z) \geq |z| \sin(\theta_1 - \theta) \tag{1.17}$$

and also

$$\left| z - \frac{(Au, u)}{\|u\|^2} \right| \geq \text{dist } (z, \nu(A)) \geq \text{dist } (z, \Sigma_\theta) = d(z).$$

This implies

$$|((zI - A)u, u)| \geq d(z)\|u\|^2,$$

or

$$\|(zI - A)u\| \geq d(z)\|u\| \tag{1.18}$$

for $u \in D(A)$. In particular, Ker $(zI - A^*)$ = Ker $(zI - A) = \{0\}$ follows and hence Ran $(zI - A) = X$ holds by the closed range theorem. Inequality (1.16) now follows from (1.18) and (1.17).

The following notation is commonly used in the operator theory: A densely defined closed linear operator A with the domain and the range in a Banach space X is said to be of *type* (θ, M) for $\theta \in (0, \pi)$ and $M \geq 1$, if $\mathbb{C} \setminus \Sigma_\theta \subset \rho(A)$,

$$\|(zI - A)^{-1}\| \leq \frac{M}{|z|}$$

for $z < 0$, and

$$\|(zI - A)^{-1}\| \leq \frac{M_\epsilon}{|z|}$$

for $\theta + \epsilon \leq |\arg z| \leq \pi$ with $\epsilon > 0$, where M_ϵ is a positive constant. In use of this terminology, we can say that any m-sectorial operator is of type (θ, M) for some $\theta \in (0, \pi/2)$.

The adjoint form $\mathcal{A}^*$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^*(u, v) = \mathcal{A}(v, u) \qquad (u, v \in V).$$

If $\mathcal{A} = \mathcal{A}^*$, we say that it is *symmetric*. In this case it holds that $D(A^{1/2}) = V$ and

$$\delta^{1/2} \|v\|_V \leq \left\|A^{1/2} v\right\|_X \leq C_1^{1/2} \|v\|_V , \tag{1.19}$$

where $A^{1/2}$ is the square root of A defined through its spectral decomposition.

1.2 Ritz-Galerkin Method

Considerations of the previous section are summarized as follows. Let $V \subset X \subset V'$ be a triple of real Hilbert spaces, with the first inclusion continuous and dense so that the second inclusion follows from identifying X' with X by Riesz' representation theorem. Let $\mathcal{A}(\ ,\ )$ be a bilinear form on $V \times V$ and assume that it is bounded:

$$|\mathcal{A}(u,v)| \leq C \|u\|_V \|v\|_V \qquad (u, v \in V) \tag{1.20}$$

and is *strongly coercive*, which means that (1.14) hold for $\lambda = 0$:

$$\mathcal{A}(v,v) \geq \delta \|v\|_V^2 \qquad (v \in V). \tag{1.21}$$

Given $f \in V'$, the problem in consideration is formulated in an abstract manner so as to find $u \in V$ satisfying

$$\mathcal{A}(u,v) = \langle f, v \rangle_{V',V} \tag{1.22}$$

for any $v \in V$. Then it is uniquely solvable by Lax-Milgram's theorem.

Ritz-Galerkin method approximates (1.22) in the following way: Prepare a family of finite dimensional subspaces $\{V_h\}_{h>0}$ of V approximating the latter as $h \downarrow 0$. Then, we take the problem to find $u_h \in V_h$ satisfying

$$\mathcal{A}(u_h, \chi_h) = \langle f, \chi_h \rangle_{V',V} \tag{1.23}$$

for any $\chi_h \in V_h$.

Unique solvability of (1.23) follows from the same reasoning as for (1.22). Let X_h be the space V_h equipped with the topology induced from X, and $P_h : X \to X_h$ the orthogonal projection. The linear operator $A_h : X_h \to X_h$ is defined through

$$\mathcal{A}(u_h, \chi_h) = (A_h u_h, \chi_h)$$

for $u_h, \chi_h \in X_h$. If $f \in X$, problem (1.23) is equivalent to the equation

$$A_h u_h = P_h f$$

in the finite dimensional space X_h.

Stability and error estimate of the approximate solution are verified as follows. Let $R_h : V \to V_h$ be *Ritz operator* defined through

$$\mathcal{A}(R_h u, \chi_h) = \mathcal{A}(u, \chi_h)$$

for $\chi_h \in V_h$ and $u \in V$. Its well-definedness follows from Lax-Milgram's theorem similarly to the unique solvability of (1.23). If $u \in V$ is the solution of (1.22), then the approximate solution u_h of (1.23) is nothing but $R_h u$. This gives the relation

$$R_h A^{-1} = A_h^{-1} P_h. \tag{1.24}$$

If $\mathcal{A}(\ ,\ )$ is symmetric, the operator $R_h : V \to V_h$ is nothing but the orthogonal projection with respect to the inner product $\mathcal{A}(\ ,\ )$. Therefore, we have

$$\mathcal{A}(R_h u, R_h u) \le \mathcal{A}(u, u)$$

and

$$\mathcal{A}(R_h u - u, R_h u - u) \le \mathcal{A}(\chi_h - u, \chi_h - u)$$

for any $\chi_h \in V_h$. Then boundedness (1.20) and strong coerciveness (1.21) of $\mathcal{A}(\ ,\ )$ imply the stability

$$\|R_h u\|_V \le C \|u\|_V \tag{1.25}$$

and the error estimate

$$\|R_h u - u\|_V \le C \inf_{\chi_h \in V_h} \|\chi_h - u\|_V . \tag{1.26}$$

Those relations hold even in the general case of $\mathcal{A}^* \ne \mathcal{A}$. In fact, from (1.21) and (1.22) we have

$$\delta \|R_h u\|_V^2 \le \mathcal{A}(R_h u, R_h u) = \mathcal{A}(u, R_h u) \le C \|R_h u\|_V \cdot \|u\|_V$$

and

$$\begin{aligned}\delta \|R_h u - u\|_V^2 &\le \mathcal{A}(R_h u - u, R_h u - u) = \mathcal{A}(R_h u - u, \chi_h - u) \\ &\le C \|R_h u - u\|_V \cdot \|\chi_h - u\|_V\end{aligned}$$

for any $\chi_h \in V_h$. Hence (1.25) and (1.26) follow in turn.

Operator theoretical features of A_h in X_h are rather similar to those of the continuous version, A in X. If the complex extensions are taken as in the continuous case, A_h in X_h is m-sectorial of type (θ, M) for some $\theta \in (0, \pi/2)$ and $M \ge 1$ independent of h.

If $X = L^2(\Omega)$ and $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$, and $\mathcal{A}(\ ,\ )$ is associated with the elliptic boundary value problem described in §1.1, this uniform structure of $\sigma(A_h)$ is refined in the following way.

Theorem 1.1. *The spectrum $\sigma(A_h)$ of A_h is contained in a parabolic region in the complex plane independent of $h > 0$. Consequently, each $\theta \in (0, \pi/2)$ admits constants $C > 0$ and $M \ge 1$ independent of h, with $A_h - C$ of type (θ, M) in X_h.*

Proof: Split $\mathcal{A}(\ ,\ )$ as

$$\mathcal{A}(u, v) = \mathcal{A}^0(u, v) + (Bu, v), \tag{1.27}$$

where

$$\mathcal{A}^0(u, v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \int_{\partial\Omega} \sigma u v\, dS$$

and

$$Bu = \sum_{j=1}^{2} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u.$$

Here, $\mathcal{A}^0$ is a strongly coercive symmetric form on $V \times V$, and B is regarded as a bounded operator $B : V \to X$. Let A_h^0 be the self-adjoint operator in X_h associated with $\mathcal{A}^0|_{V_h \times V_h}$.

The inequality

$$\|(zI_h - A_h^0)^{-1}\|_{X_h, X_h} \le \frac{1}{|\mathrm{Im}\ z|} \tag{1.28}$$

is a consequence of a property of self-adjoint operators, where $z \in \mathbb{C} \setminus \mathbb{R}$ and I_h denotes the identity operator on X_h. This implies

$$\begin{aligned} \left\| A_h^0 \left(zI_h - A_h^0\right)^{-1} \chi_h \right\|_X &\le \|\chi_h\|_X + |z| \cdot \left\| \left(zI_h - A_h^0\right)^{-1} \chi_h \right\|_X \\ &\le \left(1 + \frac{|z|}{|\mathrm{Im}\ z|}\right) \|\chi_h\|_X \le \frac{2\,|z|}{|\mathrm{Im}\ z|} \|\chi_h\|_X \end{aligned}$$

for $\chi_h \in X_h$. Heinz' inequality now gives that

$$\left\| \left(A_h^0\right)^{1/2} \left(zI_h - A_{0h}\right)^{-1} \right\|_{X_h, X_h} \le \frac{\sqrt{2}\,|z|^{1/2}}{|\mathrm{Im}\ z|}.$$

On the other hand, the relation

$$A_h = A_h^0 + B_h$$

holds with $B_h = P_h B|_{V_h}$ so that $B_h : V_h \to X_h$ is uniformly bounded:

$$\|B_h\|_{V_h, X_h} \le C_2,$$

where $C_2 > 0$ is a constant.

We have

$$\left\| \left(A_h^0\right)^{1/2} \chi_h \right\|_X^2 = \left(A_h^0 \chi_h, \chi_h\right) = \mathcal{A}^0(\chi_h, \chi_h) \ge \delta \|\chi_h\|_V^2$$

for $\chi_h \in X_h$. This implies

$$\left\| \left(A_h^0\right)^{-1/2} \right\|_{V_h, X_h} \le C_3 = \delta^{-1/2}.$$

similarly to (1.19). Given $\chi_h \in X_h$, we have

$$\begin{aligned} \|(zI_h - A_h^0)^{-1} B_h \chi_h\|_V &\le C_2 \left\| \left(A_h^0\right)^{1/2} \left(zI_h - A_h^0\right)^{-1} B_h \chi_h \right\|_X \\ &\le \frac{\sqrt{2} C_2 C_3\,|z|^{1/2}}{|\mathrm{Im}\ z|} \|\chi_h\|_V = \frac{M\,|z|^{1/2}}{|\mathrm{Im}\ z|} \|\chi_h\|_V. \end{aligned}$$