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Design and Analysis of Plates and Shells

edited by
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PART I—DESIGN PROCEDURES

FOREWORD

The ASME Pressure Vessel and Piping Division has as one of its primary objectives the dissemination of information on current research in the design, analysis, fabrication and testing of pressure vessels. This is accomplished through direct exchanges at the annual PVP Conference and Exhibition as well as through publications. This special publication represents the written version of presentations given at a two session symposium, "Design Procedures for Cylindrical Pressure Vessels", which was held at the 1986 ASME Joint Pressure Vessel and Piping Computer Engineering Divisions Conference and Exhibition held in Chicago from July 20-24. Session chairmen were G. E. O. Widera (the developer) and S. E. Moore. It was the intent of the symposium to present some of the current research work in pressure vessel design which employs either a shell theory or finite element method type of approach.

In the lead-off paper, Farr and Harvey present the history of code formulas for cylindrical vessels under internal pressure. The impossibility of coming up with a single uniform rule is demonstrated. Widera discusses the assumptions inherent in various cylindrical shell theories and illustrates their application via the problem of the line load along a generator. In the paper by Galletly the results of buckling tests on steel cylinders under combined axial compression and external pressure are given. Best overall agreement was obtained using the quadratic interaction equation and the Det Norske Veritas rules.

The effect of attachments or penetrations on pressure vessels has been a problem of long standing interest. Mirza and Gupgupoglu carry out a stress analysis of a cylindrical shell having square or rectangular lugs. Their FEM model uses doubly curved shell elements. It is shown that the maximum stress is located at the upper end of the lug-vessel interface. In his paper, Brooks determines the stresses in the neighborhood of a rigid rectangular attachment to a cylindrical shell with simply supported ends. Shallow shell theory is employed in the analysis. Steele has developed a computer code, FAST2, for analyzing nozzle intersections in thin cylindrical shells. The code utilizes the Very Large Finite Element Method in which each segment of a complex shell structure is treated as a single element. In the present paper, good agreement is shown between the code results and those from both thin and thick models. Following is a paper by Natarajan et al. in which a 3D FEM model for analyzing cylinder-to-cylinder intersections is developed and the stress distribution near the intersection is then studied. The effect of boundary conditions on these stress distributions has also been analyzed. The final paper in this group presents an investigation by Mokhtarian and Endicott in which the sensitivity of the flexibility of cylinder-cylinder intersections to the assumed boundary conditions is considered. It is shown that this sensitivity exists for short cylinders and that it is more pronounced for larger penetrations.

In the next paper Raju discusses the generation of stress indices for out-of-plane moments on laterals. It is shown that this moment governs from a peak stress and/or fatigue point of view. Jawad et al. present results which indicate that a theoretical predicition of the behavior of tube-to-tubesheet joints is feasible for various materials, methods of attachment and details of construction. In the final paper, Dalton and Sabbaghian discuss the use of multilayer vessels in high pressure applications, which dates back to the 1930's. They obtain the prestress in wrapped vessels. Then using the power function creep low, equations describing the individual interface pressures, the creep stress distribution and the safety factor as a function of time are determined.

In closing the editor wishes to indicate his indebtedness to the authors who so generously donated their talents and time in the preparation of the individual papers.

G. E. O. Widera

WHY SO MANY DIFFERENT ASME CODE FORMULAS FOR A CYLINDRICAL VESSEL

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At first thought, it certainly would seem that the ASME Boiler and Pressure Vessel Codes would contain only one formula, rule, or criteria that would apply to the safe construction of a simple right cylinder subjected to internal pressure. Not so! The ASME Codes for boiler and pressure vessels did not always proceed in a preplanned and orderly manner. In fact, it took a somewhat haphazard one, starting with its conception which was a direct result of the radically increasing number of disastrous boiler explosions in the 1800's and early 1900's as the United States pursued its own industrial revolution. Figure 1 shows this situation which cusped shortly after the turn of the century at which time its seriousness drew the attention of several of the more highly industrialized states, as well as the engineering community, to produce safety rules. Of these efforts, that of the American Society of Mechanical Engineers (ASME) has had the most pronounced and comprehensive effect on assuring the quality and safety of boilers and pressure vessels through their publication of new construction safety codes.

SIGNIFICANT CODE CYLINDRICAL VESSEL FORMULA CHRONOLOGICAL CHANGES

A. Boiler and Pressure Vessel Code, Section I, Power Boilers

In 1914, the first ASME Boiler Code was published. One of the goals in preparing it was to provide a simple formula that, when employed with a limited number of materials and a few arbitrary workmanship controls, provided a safe vessel. So it was that the first code formula came about. It was:

$$P = \frac{TS \times t \times E}{R \times FS} \qquad (1)$$

The following nomenclature is used throughout:

P = maximum allowable working pressure, lb. per sq in.
TS = ultimate tensile strength of material, lb per sq in.
YS = yield strength of material, lb per sq in.
t = minimum thickness of shell plate, in.
E = efficiency of longitudinal joint or ligaments between tube holes (whichever is least)
R = inside radius of the shell, in.
R_o = outside radius of the shell, in.
Z = $(R_o/R)^2$
FS = factor of safety (ratio of ultimate tensile strength of the material to the allowable stress) = 5

This formula simply gives the permissible pressure based on an allowable average membrane hoop stress in the vessel wall thickness. There was nothing wrong with this approach. Pressures and temperatures were low, and shells were thin compared to their diameter. Boiler and pressure vessel shops were numerous and few had engineers in their employ; hence, some degree of sophistication had to be sacrificed for the safety that accrues from simplicity.

Tubes were handled on a "stated" basis with the maximum allowable working pressure given for each tube size and gage (thickness); and for both internal pressure as encountered in water-tube boilers, and external pressure as encountered in fire-tube boilers, Fig. 2. The allowable pressures were established experimentally. Since pressures were not high and materials were limited to lap-welded or seamless mild carbon steel, this approach worked quite well. In fact, it is rather interesting to note that these allowable external tube pressures established experimentally over 70 years ago were again extensively reinvestigated by the Pressure Vessel Research Committee, WRC Bulletin 284 The External Collapse Tests of Tubes, by

E. Tschoepe and J. R. Maison, April 1983, and found to be completely reliable. This tube design section from the 1914 edition of the ASME Boiler Code is shown in Fig. 2 to illustrate the simplicity of this approach.

This tabulation of permissible tube pressures and the foregoing membrane hoop stress formula for drums was essentially the entire design requirement of the first ASME Boiler Code. But the accomplishments of this code were remarkable. Boiler explosions decreased rapidly as the code use spread. For the next decade or so, there was little change in this code formula for a cylindrical vessel. There was little need for change. Vessels were of low pressure and mostly of riveted construction (seldom exceeding 285 psi) with thin walls; and most of all the safety record continued to be most impressive.

It is fair to say that in an effort to provide a simple approach to all product forms (vessels, tubes, pipes) it was inevitable that slightly different formulas for the same straight geometric cylinder, subjected to internal pressure, dependent upon their diameter and thickness would arise. For instance, tubes and pipes are of small diameter, the fabrication process is one of forming from the outside, with measurement and tolerances established from this surface. Also, they are frequently used in firetube boilers or heat exchangers in which the applied pressure is external. Hence, formulas for tubes and pipes endeavored to simplify and cope with this situation by using the outside diameter as the pressure boundary surface and prime parameter in satisfying static equilibrium ($\Sigma F = 0$). Vessels, on the other hand, while still straight right cylinders and subjected to internal pressure, are relatively large in diameter, are fabricated by rolling or pressing, and are measurement and tolerance controlled from the inside surface. Hence, the basic pressure contact surface is likely to be only the inside diameter of the vessel, and accordingly, formulas for vessels use the inside diameter for establishing static equilibrium. Code formulas must be both accurate and simple, which is somewhat of a paradox because as we endeavor to use our increasing storehouse of knowledge to gain accuracy, we necessarily lose simplicity. So, the code faced, and continues to face, the battle of preserving construction safety for all vessels made by all fabricators.

In the late twenties and early thirties, the welding process of vessel fabrication came on the scene. This made possible a quantum jump in pressure because it eliminated the low structural efficiency of the riveted joint. This was widely utilized by industry as it strove to increase operating efficiencies by the use of higher pressures and temperatures, all of which meant thick walled vessels. Recognizing that the hoop stress variation through a thick wall cylindrical vessel can be pronounced, the code endeavored to establish at what ratio of outside-to-inside diameter a "pseudo average stress" formula could be considered applicable; and beyond which the true thick-walled Lame' formula must be adopted. Here again, the underlying code approach was to keep it simple.

The Code recognized this trend to higher pressures; and in 1940, while retaining the original membrane hoop stress formula for thicknesses less than ten percent of the inside radius, required thicker ones to comply with the exact Lame' formula, Fig. 3. Shortly thereafter, 1943, it appreciated that for such a small thickness the difference between the average stress and the exact maximum stress at the inside wall surface is nil, and selected a more judicious breakpoint as one-half the inside radius. It provided a simple linear formula for the maximum allowable working pressure that also closely approximates that given by the more exact Lame' formula for thicknesses below this breakpoint, Eq. 2, while requiring the Lame' formula for thicknesses exceeding one-half the inside radius, Eq. 3. This latter formula for thicknesses greater than one-half the inside radius has remained throughout the code evolution:

$$P = \frac{SEt}{R + 0.6t} \qquad (2)$$

$$P = SE\,\frac{Z-1}{Z+1} \qquad (3)$$

In 1952, Section I made another effort to fine-tune the accuracy of its formulas for thicknesses less than one-half the inside radius and adopted two formulas within this range; namely, one for thickness below a half inch, and one for thickness above a half inch, Fig. 3. The thought was to make provision for the reduction of wall thickness by threading that was frequently used in vessel fabrication in this period.

In 1959, a further effort was made to broaden these formulas to embody the effect of temperature on creep relaxation of the hoop stress gradient through the wall thickness, and incorporated a material temperature coefficient "y" for wall thicknesses greater than one-half inch. The resulting formula became:

$$P = \frac{SE\,(t - 0.1)}{R + (1-y)(t-0.1)} \qquad (4)$$

where y has the value shown below:

	TEMPERATURE, F							
	900 and Below	950	1000	1050	1100	1150	1200	1250 and Above
Ferritic	0.4	0.5	0.7	0.7	0.7	0.7	0.7	0.7
Austenitic	0.4	0.4	0.4	0.4	0.5	0.7	0.7	0.7
Alloy 800	0.4	0.4	0.4	0.4	0.4	0.4	0.5	0.7
800H	0.4	0.4	0.4	0.4	0.4	0.4	0.5	0.7
825	0.4	0.4	0.4	...	...	...	...	...

Finally, in 1971, the insignificance of differentiating in these formulas for thicknesses below or above one-half inch (but below one-half the inside radius) was reappraised and the above formula for a cylindrical vessel under internal pressure was altered to its present simpler form:

$$P = \frac{SE\,(t-C)}{R + (1-y)(t-C)} \qquad (5)$$

where C is an allowance for threading and structural stability. For uniform thickness cylinders, such as drums, C = 0 and for temperatures below 900°F, y = 0.4 which gives the usual formula for boiler drums as

$$P = \frac{SEt}{R + 0.6t} \qquad (6)$$

These are the code formulas in Section I for right cylindrical vessels under internal pressure that prevail today.

B. Unfired Pressure Vessel Code, Section VIII

The success of the ASME Boiler Code, Section I, drew attention to the associated multitude of "unfired" vessels, so-to-speak, that were not directly associated with or in contact with a combustion furnace; but nonetheless were subjected to pressure and temperature from a contained media. The need was eminent as industry sought to use the universal catalysis of high pressure and temperature to increase operating efficiency, and the new welding processes and high strength materials provided a means to achieve this goal. Thus was born in 1925 the ASME Unfired Pressure Vessel Code, Section VIII, later to be renamed "Pressure Vessels."

Section VIII also adopted the thin-wall membrane circumferential or hoop stress formula for setting the thickness or the maximum allowable working pressure, Fig. 4. The formula was simple and accurate when the vessel diameter/thickness is above 20. This formula was used in Section VIII until the 1943 Edition. At that time, the formula was modified to more accurately determine results for thicker walls due to higher temperatures and pressures. That modified formula is still used in the 1986 Edition of Section VIII, Division 1. Of course, there are formulas still in the Code for very thick vessels (Lame' formula) and for setting thickness or maximum allowable working pressure based on the outside radius.

Although the hoop stress due to internal pressure is twice the axial stress due to internal pressure, in tall vertically-supported vessels, there may be axial stresses from other loadings. In 1957, a longitudinal or axial stress formula was added to Division 1. Thickness of the vessel may be set by this stress when the loadings are from both internal pressure and other loadings such as dead loading and wind or earthquake loading.

With the 1968 Edition of Section VIII, two divisions of the section were established. The old unfired pressure vessel code became Section VIII, Division 1, Pressure Vessels; and a new section was added, Section VIII, Division 2. Alternative Rules for Pressure Vessels, Fig. 5. The new division was based on a factor of safety of three on tensile strength and was limited to vessels operating in the temperature range where tensile and yield strength sets allowable stresses. In addition, the basic stress theory was different. Instead of the Maximum Stress Theory, the new division was based on the Maximum Shear Stress Theory. In addition, the new division required that calculations be made to establish the adequacy of components. Consequently, the formula for circumferential or hoop stress was only one for setting the initial thickness of a shell and was based on a modified thin-wall formula. This is similar to but not exactly the same as the Division 1 hoop stress formula. The new division also contained a different formula for determining the thickness or maximum allowable working pressure for thick-wall vessels. It was a version of the Lame' formula.

Again, longitudinal or axial stress formulas were approximate. When all loadings were taken into account, the actual stresses are to be determined and compared to the maximum allowable stress intensity. If the resulting stress is compressive due to a combination of the internal pressure and the other loadings, the compressive buckling stresses are to be considered.

THE OUTLOOK FOR CHANGE

Will the ASME Code ever reach the point where it will have a single uniform approach or formula or rule for the construction requirement of a cylindrical vessel subjected to internal pressure? Hardly, - - - and for the very simple reason that it must continue to fulfill its mandate to provide rules for the safe construction of all kinds of vessels, types of manufacturers and kinds of materials. There is no single all inclusive panacea. It must make available suitable methods and means for use and implementation by the small fabricator or "garage-shop" builder of low pressure heating boilers, vessels or tanks who does not have a professional engineering staff at his disposal. This segment makes up by far the major number of vessels built each year. Yet, the very same ASME Code body must also provide for the safe construction of the most advanced sophisticated vessels employing exotic materials in hostile environments and subjected to ultra-high pressures; and constructed by large companies well staffed with professional engineers backed up by extensive computer facilities and research organizations. This code approach is illustrated by Fig. 6 for the ASME Pressure Vessel Code Section VIII wherein the three divisions provide for a degree of trade-off for knowledge in the areas of stress analysis, materials and fabrication techniques. In this manner, the code "fits the tools to the job." The "tools" are knowledge and the "job" is pressure vessel safety. In all probability, there will be changes in the code requirements for cylindrical vessels subjected to pressure, but they will be directed toward the "purpose of service" approach. An illustration of such a future change is that now in the formative stage and is tentatively known as Division 3 of the ASME Pressure Vessel Code, Section VIII. While Section VIII, Division 1 is the oldest code which establishes a minimum vessel-wall thickness and requires no stress analysis, Section VIII, Division 3 is at the opposite end of the spectrum and requires a total stress analysis of the entire vessel under all operating service conditions. This is called "design by analysis" and simply means demonstration by analytical or experimental stress and strain analysis that all parts of the vessel are in compliance with code established limits. The latter ASME Code is not yet complete, but it is indicative of the code approach to safe vessel construction by letting the "requirements" fit the "purpose."

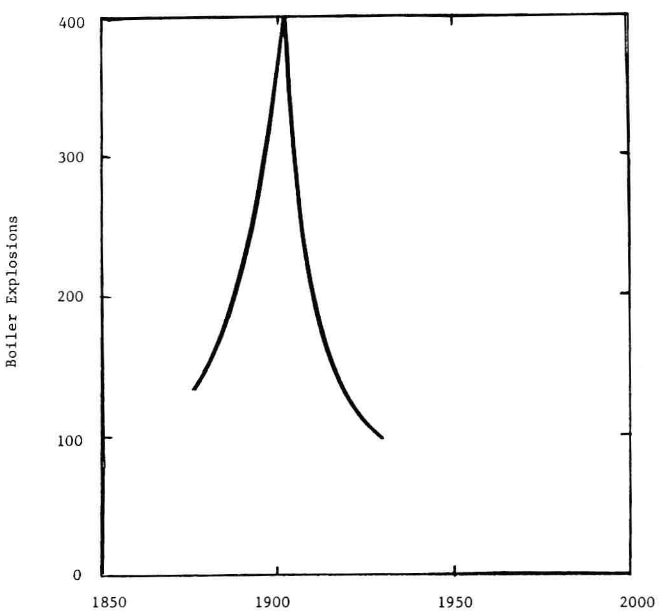

FIG. 1 Boiler Explosions in the USA, 1879-1930

"21 *Tubes for Water-Tube Boilers.* The minimum thicknesses of tubes used in water-tube boilers measured by Birmingham wire gage, for maximum allowable working pressures not exceeding 165 lb. per sq. in., shall be as follows:

Diameters less than 3 in.	No. 12 B.W.G.
Diameter 3 in. or over, but less than 4 in.	No. 11 B.W.G.
Diameter 4 in. or over, but less than 5 in.	No. 10 B.W.G.
Diameter 5 in.	No. 9 B.W.G.

The above thicknesses shall be increased for maximum allowable working pressures higher than 165 lb. per sq. in. as follows:

Over 165 lb. but not exceeding 235 lb.	1 gage
Over 235 lb. but not exceeding 285 lb.	2 gages
Over 285 lb. but not exceeding 400 lb.	3 gages

Tubes over 4-in. diameter shall not be used for maximum allowable working pressures above 285 lb. per sq. in.

22 *Tubes for Fire-Tube Boilers.* The minimum thicknesses of tubes used in fire tube boilers measured by Birmingham wire gage, for maximum allowable working pressures not exceeding 175 lb. per sq. in., shall be as follows:

Diameters less than 2½ in.	No. 13 B.W.G.
Diameter 2½ in. or over, but less than 3¼ in.	No. 12 B.W.G.
Diameter 3¼ in. or over, but less than 4 in.	No. 11 B.W.G.
Diameter 4 in. or over, but less than 5 in.	No. 10 B.W.G.
Diameter 5 in.	No. 9 B.W.G.

For higher maximum allowable working pressures than given above the thicknesses shall be increased one gage."

Fig. 2 Tabulation of Permissible Tube Pressures for Water-Tube and Fire-Tube Boilers from the First ASME Boiler Code, 1914

CYLINDRICAL SHELL FORMULAS, SECTION I

Code Year	Thin-Wall Formula	Thick-Wall Formula
1914	$P = \frac{TS}{FS} \frac{t\ E}{R}$	none
1940	$P = \frac{TS}{FS} \frac{t\ E}{R}$	$P = \frac{TS\ E}{FS} \frac{Z-1}{Z+1}$ (for thickness over 10% of R)
1943	$P = \frac{SEt}{R + 0.6t}$	$P = SE \frac{Z-1}{Z+1}$ (for thickness over one-half R)
1952	$P = \frac{0.8\ SEt}{R + 0.6t}$ (for thickness less than 1/2")	
	$P = \frac{SE(t - 0.1)}{R + 0.6(t - 0.1)}$ (for thickness over 1/2")	$P = SE \frac{Z-1}{Z+1}$
1959	$P = \frac{0.8\ SEt}{R + 0.6t}$ (for thickness less than 1/2")	
	$P = \frac{SE\ (t - 0.1)}{R + (1 - y)(t - 0.1)}$ (for thickness over 1/2" and "y" is material temperature correction coefficient)	$P = SE \frac{Z-1}{Z+1}$
1971-present	$P = \frac{SE\ (t - C)}{R + (1 - y)(t - C)}$	$P = SE \frac{Z-1}{Z+1}$

Fig. 3 - History of Cylindrical Shell Formula in ASME Code, Section I

CYLINDRICAL SHELL FORMULAS, SECTION VIII, DIVISION 1

Code Year	Circumferential Stress	Longitudinal Stress
1925	$t = \frac{PR}{SE}$ and $P = \frac{SEt}{R}$	none
1943	$t = \frac{PR}{SE - 0.6P}$ $P = \frac{SEt}{R + 0.6t}$	none
1957-present	$t = \frac{PR}{SE - 0.6P}$ (Note 1) $P = \frac{SEt}{R + 0.6t}$ (Note 1)	$t = \frac{PR}{2SE + 0.4P}$ (Note 2) $P = \frac{2SEt}{R - 0.4t}$ (Note 2)

Note 1: Limited to a thickness not to exceed one-half of the inside radius and a pressure not to exceed 0.385 SE. Beyond these limits, the formulas given below apply.

Note 2: Limited to a thickness not to exceed one-half of the inside radius and a pressure not to exceed 1.25 SE. Beyond these limits, the formulas given below apply.

Circumferential or Hoop Stress	Longitudinal or Axial Stress
$t = R(Z^{\frac{1}{2}} - 1)$ where $Z = \left(\frac{SE+P}{SE-P}\right)$	$t = R(Z^{\frac{1}{2}} - 1)$ where $Z = \left(\frac{P}{SE}+1\right)$
$P = SE\,\frac{Z - 1}{Z + 1}$ where $Z = \left(\frac{R+t}{R}\right)^2$	$P = SE(Z - 1)$ where $Z = \left(\frac{R+t}{R}\right)^2$

Fig. 4 - History of Cylindrical Shell Formulas, Section VIII, Div. 1

CYLINDRICAL SHELL FORMULAS, SECTION VIII, DIVISION 2

Code Year	Circumferential Stress	Longitudinal Stress
1968-present	$t = \frac{PR}{S - 0.5P}$ If P > 0.4S, the following may be used: $\ell n\ \frac{(R + t)}{R} = \frac{P}{S}$ where ℓn is the natural log	$t = \frac{0.5PR + F}{S - 0.5P}$ (Note 1)

Note 1: Use if F is positive and exceeds 0.5 PR. If F is negative, the condition of buckling shall be considered.

Fig. 5 - History of Cylindrical Shell Formula, Section VIII, Div. 2

DESIGN BASIS

ASME PRESSURE VESSEL CODE, SECTION VIII

DIVISION 1	DIVISION 2	DIVISION 3 (Tentative)
DESIGN BY:	DESIGN BY:	DESIGN BY:
Cookbook Formulas (No stress Analysis)	Minimum Thickness Formulas and Limited Analysis	Total Stress Analysis
With	With	With
Minimum material, fabrication and examination requirements	Extensive material, fabrication and examination requirements	Elaborate material, fabrication and examination requirements
Based on	Based on	Based on
(a) Elastic stress through wall thickness, and	(a) Elastic stress through wall thickness, and	(a) Full plastic yield stress through wall thickness, and
(b) Maximum stress theory of failure	(b) Maximum shear stress theory of failure	(b) Maximum shear stress theory of failure
$P = \frac{SEt}{R + 0.6t}$	$P = \frac{St}{R + 0.5t}$	$P = \frac{YS\ \ln (R_o/R)}{1.75}$

Fig. 6 - Trends in Code Construction Requirements

VALIDITY OF VARIOUS SHELL THEORIES APPLICABLE IN THE DESIGN AND ANALYSIS OF CYLINDRICAL PRESSURE VESSELS

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Abstract

The present paper discusses the inherent assumptions and range of applicability of various simplified shell theories which are employed in the linear analysis of their pressure vessels. Via the problem of the line load along a generator, it is shown how to obtain the complete solution through a proper superposition of the solutions from the simplified theories.

1. Introduction

Pressure vessel analysts and designers have long made use of shell theory to solve their problems. Various types of shell theories have been proposed in the literature. These theories are in wide use, but because of their approximate character they often lead to failures which could have been predicted only on the basis of more refined theories. However, even the so-called refined theories are themselves derived from the exact three-dimensional elasticity equations on the basis of artificial assumptions whose validity is often open to question. Moreover, in the process of making these assumptions certain contradictions occur which have resulted in shell theories of great variety and complexity.

The reduction of the three-dimensional equations of elasticity to an equivalent set of linear two-dimensional equations is based on the assumption that the shell be thin and that the displacements be small compared to the thickness. The classical derivation of thin shell equations incorporates hypotheses, such as those of Kirchoff, which lead to a priori assumptions regarding the spatial distribution of displacements and stresses over the thickness of the shell. Another method of deriving shell equations, one which is free from a priori assumptions, is that of the asymptotic integration of the elasticity equations. The method incorporates the use of the boundary layer technique to furnish, depending on the choice of characteristic length scales, different sequences of systems of differential equations. Subsequent integration over the shell thickness and application of the surface boundary conditions yields the desired two-dimensional shell equations. The lowest order system so obtained represents the simplest appropriate shell equations. The higher order systems systematically incorporate thickness corrections associated with the effects of transverse shear and normal stress.

Many authors (see [1-8], for example) have employed the method of asymptotic integration to derive various shell theories. Relatively little, though, has been written [9-14] about the application of these theories to the solution of actual shell problems, such as those involving pressure vessels. It is the aim of this paper to demonstrate the assumptions underlying the asymptotic theories and to illustrate their application with reference to the particular problem of a fixed-ended cylinder subjected to a constant line load along a generator. The starting point of the analysis is a set of very general shell equations developed by Goldenveizer. The loading and solution state is represented in the form of a Fourier series in the circumferential direction. Depending on the number of circumferential waves considered, various simplifications of the general shell equations can be carried out on the basis of the results for the asymptotic theories. The complete solution is obtained by a superposition of the solutions of the simplified systems of equations.

2. Formulation

In what is to follow, a closed cylindrical shell of middle surface radius a, constant thickness 2h and length L is assumed. A point on the middle surface will be specified by the coordinates ξ and θ (see Fig.1). Here, ξ is the dimensionless arc-length of the generator and θ the central angle measured from the initial generator. According to Goldenveizer [9], the most general equations governing the unsymmetric deformation of a thin cylindrical shell are given by

$$\left[\frac{\partial^2}{\partial\xi^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\theta^2}\right]u + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial\xi\partial\theta} - \nu\,\frac{\partial w}{\partial\xi} + a^2{}_x\bar{p} = 0 \quad (1)$$

$$\left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial\xi\partial\theta} + \{\left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2} + \lambda^2[2(1-\nu)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}]\}v$$

$$+\{-\frac{\partial}{\partial\theta} + \lambda^2[(2-\nu)\frac{\partial^3}{\partial\xi^2\partial\theta} + \frac{\partial^3}{\partial\theta^3}]\}w + a^2\,\bar{p}_\theta = 0 \quad (2)$$

$$-\nu\frac{\partial u}{\partial\xi} + \{-\frac{\partial}{\partial\theta} + \lambda^2[(2-\nu)\frac{\partial^3}{\partial\theta\partial\xi^2} + \frac{\partial^3}{\partial\theta^3}]\}v + [1+\lambda^2\Delta^2\Delta^2]w - a^2\bar{p}_z = 0 \quad (3)$$

where u, v, w are the components of displacement in the axial, circumferential and radial directions, respectively; p_x, p_θ, p_z the components of surface loading; ν Poisson's ratio; E the modulus of elasticity; and

$$\Delta^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}, \quad \xi = \frac{x}{a}$$

$$\bar{p}_x = \frac{1-\nu^2}{2Eh}p_x, \quad \bar{p}_\theta = \frac{1-\nu^2}{2Eh}p_\theta, \quad \bar{p}_z = \frac{1-\nu^2}{2Eh}p_z \quad (4)$$

$$\lambda^2 = \frac{h^2}{3a^2}$$

Equations (1-3) are determined on the basis of the following sets of equations:

Equilibrium:

$$\frac{\partial N_x}{\partial\xi} - \frac{\partial N_{\theta x}}{\partial\theta} + ap_x = 0, \quad \frac{\partial M_{\theta x}}{\partial\xi} - \frac{\partial M_\theta}{\partial\theta} + aQ_\theta = 0$$

$$\frac{\partial N_{x\theta}}{\partial\xi} - \frac{\partial N_\theta}{\partial\theta} - Q_\theta + ap_\theta = 0, \quad \frac{\partial M_x}{\partial\xi} + \frac{\partial M_{\theta x}}{\partial\theta} - aQ_x = 0 \quad (5)$$

$$N_\theta + \frac{\partial Q_x}{\partial\xi} + \frac{\partial Q_\theta}{\partial\theta} + ap_z = 0, \quad N_{x\theta} + N_{\theta x} + \frac{M_{\theta x}}{a} = 0$$

Constitutive relations:

$$N_x = \frac{2Eh}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\theta), \quad M_x = -\frac{2\varepsilon h^3}{3(1-\nu^2)}(X_x + \nu X_\theta)$$

$$N_\theta = \frac{2Eh}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_x), \quad M_\theta = -\frac{2\varepsilon h^3}{3(1-\nu^2)}(X_\theta + \nu X_x) \quad (6)$$

$$N_{x\theta} = \frac{2Eh}{1+\nu}\left(\frac{\gamma}{2} + \frac{h^2}{3}\frac{X}{a}\right), \quad M_{x\theta} = \frac{2\varepsilon h^3}{3(1+\nu)}X$$

$$N_{\theta x} = -\frac{2\varepsilon h}{1+\nu}\left(\frac{\gamma}{2}\right), \quad M_{\theta x} = -M_{x\theta}$$

Geometry of deformation:

$$\varepsilon_x = \frac{1}{a}\frac{\partial u}{\partial\xi}, \quad \varepsilon_\theta = \frac{1}{a}\left(\frac{\partial v}{\partial\theta} - w\right), \quad \gamma - \frac{1}{a}\left(\frac{\partial v}{\partial\xi} + \frac{\partial u}{\partial\theta}\right) \quad (7)$$

$$X_x = \frac{1}{a^2}\frac{\partial^2 w}{\partial\xi^2}, \quad X_\theta = \frac{1}{a^2}\frac{1}{\partial\theta}\left(\frac{\partial w}{\partial\theta} + \nu\right), \quad X = \frac{1}{a^2}\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\theta} + \nu\right)$$

Here, N, M and Q represent the force, moment and transverse shear force stress resultants, and ε, γ and X represent the normal strain, shear strain and changes in curvature, respectively.

The solution of Eqs. (1-3) for a specific problem is carried out using four boundary conditions on each of two transverse edges $\xi=0$ and $\xi=L/a$ as well as eight periodicity conditions which stipulate that static and geometric quantities in longitudinal sections must return to their original values after a complete turn. Particular integrals to these equations can often be obtained either via the use of approximate methods or by employing membrane theory.

Let us consider the case where the loading and support cause a symmetrical state of stress relative to the initial generator $\theta=0$. Then for a closed cylindrical shell, it is convenient to seek integrals of the homogeneous part of Eqs. (1-3) in the form of a trignometric series in terms of the variable θ,

$$u = \sum_{m=0}^{\infty} Ae^{k\xi}\cos m\theta, \text{ etc.} \quad (8)$$

Substitution into Eqs. (1-3) yields an eight degree characteristic equation for the determination of the k's corresponding to each m,

$$k^8 - 4m^2k^6 + 6m^4k^4 - (8-2\nu^2)m^2k^4 + (1-\nu^2)\lambda^{-2}k^4$$

$$-4m^2(m^2-1)^2k^2 + m^4(m^2-1)^2 = 0 \quad (9)$$

This formula as well as the other calcuations which are required, are generally too cumbersome to be used in the analysis of pratical problems. One therefore seeks simplifications obtained by omitting insignificant terms and which thus result in approximate theories.

2.1 Simplified Bending Theory

The simplified bending theory or theory of edge effects rests on the following hypotheses:

Geometry of deformation:

It is assumed that the dominant effect is that of the normal displacement. Thus, the expressions for ε_θ, x_θ and X can be simplified as follows:

$$\varepsilon_\theta = -\frac{W}{a}, \quad x_\theta = \frac{1}{a^2}\frac{\partial^2 w}{\partial\theta^2}, \quad x = \frac{1}{a^2}\frac{\partial^2 w}{\partial\xi\partial\theta} \quad (10)$$

Constitutive relations:

The expression for $N_{x\theta}$, N_x, M_x, M_θ can be replaced by:

$$N_{x\theta} = \frac{2Eh}{1+\nu}\left(\frac{\gamma}{2}\right), \quad \varepsilon_x + \nu\varepsilon_\theta = 0$$

$$M_x = \frac{-2Eh^3}{3(1-\nu^2)}X_x, \quad M\theta = \frac{-2Eh^3\nu}{3(1-\nu^2)}X_x \quad (11)$$

Equilibrium:

In the fourth equilibrium equation only M_θ and Q_θ need to be retained while in all others M_θ, $M_{x\theta}$, $M_{\theta x}$, and Q_θ can be set equal to zero.

This theory has built into it the fact the characteristic length for changes in the axial direction is $O[(ah)^{1/2}]$ and O(a) in the circumferential. It thus holds for $|k| \simeq \lambda^{-1/2}$, $m << \lambda^{-1/2}$.

2.2 Semi-Membrane Theory

The semi-membrane theory or theory of the basic state of stress is also associated with a characterestic circumferential length scale O(a) while its axial one is $O[a(a/h)^{1/2}]$. This implies $m << \lambda^{-1/2}$, $k \simeq \lambda^{1/2} m^2$. It can be obtained on the basis of the following assumptions:

Geometry of deformation:

The general expressions for ε_θ and γ can be replaced by

$$\frac{\partial v}{\partial \theta} - w = 0, \quad \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \xi} = 0 \qquad (12)$$

Constitutive relations:

The relations for $N_{x\theta}$, N_θ, M_x, M_θ can be simplified as follows:

$$N_{x\theta} = \frac{2Eh}{1-\nu}\left(\frac{\gamma}{2}\right), \quad \varepsilon_\theta + \nu\varepsilon_x = 0$$

$$M_x = -\frac{2Eh^3}{3(1-\nu^2)}\nu X_\theta, \quad M_\theta = -\frac{2Eh^3}{3(1-\nu^2)} X_\theta \qquad (13)$$

Equilibrium:

All equilibrium equations except the fifth can use $M_x = M_{x\theta} = M_{\theta x} = Q_x = 0$.

2.3 Simplified Donnell Theory

The simplified Donnell theory corresponds to axial and circumferential length scales $O[(ah)^{1/2}]$. It holds for $m \geqslant \lambda^{-1/2}$, with k determined from the relation

$$(k^2-m^2)^4 + \left(\frac{1-\nu^2}{\lambda^2}\right)k^4 = 0 \qquad (14)$$

The theory can be obtained from the original cylindrical shell equations through use of the following simplifications:

Geometry of deformation:

The changes of curvature expressions contain only the normal displacement,

$$X_x = \frac{1}{a^2}\frac{\partial^2 w}{\partial \xi^2}, \quad X_\theta = \frac{1}{a^2}\frac{\partial^2 w}{\partial \theta^2}, \quad X = \frac{1}{a^2}\frac{\partial^2 w}{\partial \xi \partial \theta} \qquad (15)$$

Constitutive realtions:

The relation for the membrane shear force can be taken in the form:

$$N_{x\theta} = -N_{\theta x} = \frac{2Eh}{1+\nu}\left(\frac{\gamma}{2}\right) \qquad (16)$$

Equilibrium:

In the second equilibrium equation we can set $Q_\theta = 0$ while in the sixth $M_{\theta x} = 0$ can be assumed.

It is noted that the simplified Donnell theory is often used for both small m and large m. This however gives acceptable results only when one is dealing with shells which are not too long.

2.4 Plate Bending Theory

For every large m, $m >> \lambda^{-1/2}$ and k determined from

$$(k^2-m^2)^4 = 0 \qquad (17)$$

the cylindrical shell equations become two uncoupled sets: those corresponding to plate bending theory and those to plane stress theory. They can be obtained if, in addition to the assumptions of Section 2.3, the following ones are made:

Geometry of deformation:

The expression for ε_θ is taken as

$$\varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (18)$$

Equilibrium:

The third equilibrium equation becomes

$$\frac{\partial Q_x}{\partial \xi} + \frac{\partial Q_\theta}{\partial \theta} + a p_z = 0 \qquad (19)$$

2.5 Membrane Theory

It is recalled that membrane theory gives an adequate description of shell behavior only at sufficient distances from lines of distortion of the stress state. Examples of such lines are the: edges of a shell, lines along which there occur discontinuities in the external load components and certain of their derivaties, lines along which the middle surface of a shell is discontinuous or the curvature if the middle surface changes abruptly, and lines along which the rigidity of a shell or its thickness undergo sudden changes.

The characteristic length scales inherent in membrane theory are both O(a). The region of applicability of membrane theory can be shown [9] to be given by

$$\frac{h^2 m^4 (m^2-1)^2 L^4}{3(4!)a^6} << 1 \qquad (20)$$

For a shell with no surface loading and fixed radius, thickness, and m (number m here characterizes the edge loading), the accuracy of membrane thoery is seen to be proportional to the fourth power of the length of the shell. It thus is necessary for the shell to be relatively short in order for it to be able to be analyzed by membrane theory. The exceptions are the cases of m=0 and m=1 for which a statically non-self-equilibriated loading exists on transverse sections. The length in these cases can be as large as one wishes as membrane shells here behave like beams of hollow cross-section.