Quantitative Micro-economics

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Preface

This text on Micro-economics, I expect will fill a 'lack' felt by me while teaching this course to postgraduate students. The 'lack' to which I refer is based on my hypothesis that there is every possibility to upgrade and modernise the material in the micro-economics course which is presently taught in our universities. Since the text assumes the knowledge of mathematics, though not necessarily a great deal, it might be argued that the text will be of great use only to those students and teachers who are well acquainted with mathematics. This concept, in my opinion, holds little merit. If the postgraduate student with an average intelligence in micro-economics course is not a mathematical sophisticate, neither is he a mathematical illiterate. There is no reason why he could not be stimulated to develop appreciation for the use of mathematics that helps to convey an understanding of economic concepts and phenomena in a more intelligible fashion. Moreover, if a teacher can achieve this, the student never gets bored with what he is offered in the class. The present text aspires to encourage the student to go considerably beyond the ideas encountered in his Bachelor's course by making the economic theory aesthetically pleasing, more accessible and applicative in nature.

This book is the 'end product' of numerous drafts and revisions of the 'class notes' prepared by me while teaching postgraduate students for several years. As such my principal acknowledgement is to the students. They have always saved me from obscurantism and abstractionism and incited me to fresh thinking through their insatiate inquisitive nature which at times is suppressed by erstwhile mentors who know not what they are doing.

One may not claim professional glory by translating the original work into simple text and compact mathematics. Yet, I surely hope that the text should provide significant ground for acceptance of my null hypothesis.

G. M. K. MADNANI

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1

Introduction

The basic problem with which economics is concerned is that of *economising* — economising on the scarce resources. Because of the scarcity of resources, choice making becomes an inescapable necessity. A rational choice has to be made for the achievement of specific ends amongst many competing ends within the limitation of resource scarcity. All economic problems arise from the fact that resources are scarce relative to the human wants for the goods and services made available with these resources. Therefore, every economic system — capitalist, socialist or mixed — is faced with the problem of allocating the scarce resources among their competing alternative uses. It must decide:

1) which goods and services should be made available,
2) the manner in which these goods and services will be produced,
3) the manner in which these goods and services will be distributed, and
4) the provisions which need to be made (from the goods produced) for the future growth of the system.

As long as the insatiable desire of materialistic possessions and longings remains in human nature; as long as man retains the capacity to be envious and discontented, every economic system will have to perform these functions. However, the manner in which the functions are carried out depends on the institutional framework of the system.

1.1 Positive and Normative Thinking

To carry out the above functions, some public policy decisions are to be made. Policy decisions are compounded of two parts: (1) a scientific judgement as to the possible consequences of the alternative policy recommendations for the existing working of the economic system, and (2) a value judgement as to the desirability of the one amongst these alternative policy recommendations.

Positive economics deals with the former, while *Normative economics* is concerned with both. To put it more simply, the task of positive economics is to understand the functioning of the economic system *as it in fact exists*; to understand the mechanism that determines the relative prices, output levels and the distribution of the output. Instead, one may be more interested in only knowing how whatever is produced should not (or should be) produced and how whatever is produced could be distributed on a more equitable basis. This is the field of normative economics which sounds more logical and interesting to humanitarians. Their possible disappointment with (positivenists) economists is that sciences (including economics) are concerned with explaining *what is*, not with what ought to be. All sciences are positive disciplines; they are involved only with explaining *things as they are*. But such a fundamental nature of science(s) is not at variance with the notion that scientists should use their sciences to help mankind. In fact positive scientific theories can help a lot in achieving the normative goals of societies at large. The only thing is that sciences cannot determine such normative conceptions. They are to be exogeneously determined but on the basis of full knowledge of "the things as they are". Logically, therefore, positive theory precedes the determination of normative goals. Without perfect knowledge of "how things are or how they exist", one cannot make an attempt to propose realistically the desirability of "what it ought to be". The positive theory acts as a guide to normative theory.

Here is an example: positive economic theory helps to explain how income in the economic system is in fact distributed amongst the factors of production. A normative theory of income distribution seeks to explain how income can be distributed so as to achieve a well-defined norm which the system considers to be more just. But it is also very essential to know the positive theory in order to make the necessary changes to conform to the normative theory (which is in variance with the positive theory). It is, in fact, the existence of a difference between the two theories that indicates the desirability for change.

1.2 Role of Theory

The objective of all sciences is to understand the workings of a complex reality whether this reality be in the physical universe or in the institutions of a society. Yet, as in all sciences, a book on micro-economics obviously must be theoretical. The theories that are developed to explain the behaviour of consumer, producer, and market are merely abstract formulations apparently divorced from reality. Often, therefore, many students specially those untutored in the methodology of sciences get impatient with such a theoretical analysis and question its usefulness; at times with the common stereotyped remark, "that's all right in theory but not in practice."

Is such a criticism valid? Why could not then someone, to this day, venture a micro-economic book about practice ever since the publication of Wealth of Nations"? Even David Ricardo, despite having both his feet planted firmly in the world of business and politics, used a theoretical approach in his work. The reason for practical man's antipathy to theory is due to hasty theorisation on an insufficient basis of fact. But 'fact' and 'theory' are not opposed to each other, they are complementary. It is 'theory' that makes the 'fact' understandable. Facts by themselves are dumb.

Facts (obtained through data-gathering) are important at two stages in the process of achieving an understanding reality.

Facts suggest the relationships between the crucial variables that may be fundamental in understanding phenomena of interest. These obtained suggested relations help to formulate a *hypothesis*. We go a step further and draw the relevant *implications* of this hypothesis. The combination of the hypothesis and its implications is usually called a *model*. The model, so built through deductive logic, may either be of informal or formal mathematical nature. A model is merely a caricature of reality, it does not attempt to represent reality in every detail, for then we would have reality itself and not a model.

Suppose a single-production-process firm wants to determine the demand curve for its product. It collects facts—only certain facts (in case it bothered to try and collect all possible facts, it would have to spend an unlimited period of time on the job) and then tries to examine if these collected facts suggest any relation between themselves to understand the fluctuations in the demand for the commodity (Q) produced by the firm. Say, it picks out (hypotheses) price (P) and income of the consumer (Y) as two important variables that affect the demand for Q: $Q=f(P, Y)$. Then, utilising the collected facts (in the form of data) the equation:

$$Q = 1.25\, P^{-0.5}\, Y^{2.30}$$

or,

$$Q = 2.50 - 1.5\, P + 20.25\, Y$$

illustrates an actual model. The model undoubtedly overlooks many facts or factors that may influence the demand for Q in reality. But all the same it implies that there exists an inverse relationship between price and quantity and a positive relation between the income of the consumer and the quantity demanded (disregarding all other facts involved). Thus the model (in the given two forms) provides an idealised representation that captures the heart of reality. But at times, in an attempt to strip away the unessential complexity and lay bare the fundamental variables (P and Y in above case), there is always the danger of neglecting some crucially important facts in the formulated hypothesis. Through deductive logic, one can formulate many different hypotheses leading to many models to analyse an economic event. The question, therefore, arises as to what characteristics a model should have to make it *superior* to an alternative one designed with different variables.

This leads to the second stage in which facts play a necessary role in determining *a good model*.

A good model may be judged on several criteria. They are: its predictive power, the consistency and realism of assumptions, its generality and its simplicity. Of course for examining the validity of the model, the facts used at this stage must be different from those used to suggest the hypothesis (in the first stage). Also, repeated successful confrontations are needed before a model can be certified to be good and hence be declared a model of theory a law.

It is, therefore, wrong to hold theories in contempt with criticisms such as "that's all right in theory but not in fact." It should, however, be clear from the foregoing discussion that no theory can be all right that cannot hold up against facts. A good theory stems from facts, that is, reality itself. Theories are simply the unavoidable arrangements and interpretations of facts to understand reality; yet theories cannot be the truth because they are reflections of reality.

1.3 Role of Mathematics

In the present day economic theories are put in any of the following three different languages: verbal logic, mathematics and graphs. The advantage of all the three languages has been taken while expressing theories in this book.

Theories when described in mathematical language gain not only notational simplicity, clarity and condensation but the more prominent virtue of mathematisation is the manipulation of theories by taking advantage of well-established mathematical operations. The purpose behind such manipulations is to obtain inferences that are unattainable without replacing verbal arguments by quantitative precision.

Mathematics, however, is not an end in itself. As long as it is wisely used, it renders invaluable help; but if over-emphasised it can obfuscate and result in "bad economics". To make good economics better and to make the theory operational and convincing, it is essential to recognise the importance of behavioural considerations in the application of quantitative techniques. Individuals do act in erratic ways. They do not, at times, necessarily behave with usually assumed rationality and exactness that mathematics tends to ascribe them.

But this does not imply that the use of mathematics in economics is "debunk". Essentially, the object is to stress that the human behaviour is far too complicated to be grasped *in toto*. Based on this sound logic, the use of mathematics should not go to incautious limits lest it becomes merely frill to literary economics to make it elegant and saleable.

2

Theory of the Household

The total demand curve of any commodity is the lateral summation of the individual demand curves. In order to fully understand the properties of the market demand curve, one must, therefore, first understand the behaviour of the individual households. The *household* is defined as any group of individuals (consumers) sharing an income so as to purchase and consume goods and services.

The problem of the household is of deciding how much of each of the available goods and services it should purchase so as to *obtain the most for the money spent*. This chapter examines the behaviour of a single such household whose object is "how to obtain best buys"; given the prices of all goods and services and given its income.

2.1 The Preference Relation: Indifference Curves

A single household is one of many households that make purchases in the market. Its purchases, therefore, may be assumed to be relatively insignificant. Each household can secure as much as it desires of any commodity without affecting the going price of the commodity. A commodity is a particular good or service obtained by a household at a specific time and at a specific location.

Suppose there is a finite number, n, of available commodities ($x_1, x_2, \ldots, x_n$) from which the household is to make purchases on the going prices ($p_1, p_2, \ldots, p_n$). The household's total expenditure will be given by

$$E = \sum_{i=1}^{n} p_i q_i$$

For the sake of covenience, each commodity is assumed to be perfectly divisible so that any non-negative quantity can be purchased. Further, the household can choose any commodity bundle comprising different combinations of the q_i. The choice of a particular commodity bundle of the household shall depend on the tastes of the household.

The preference for a particular commodity bundle may involve three different situations or *relations*:

(i) *Strict preference relation*: $\mathbf{q}' > \mathbf{q}''$
i.e., the household prefers bundle $\mathbf{q}' = (q'_1, q'_2, \ldots, q'_n)$
to bundle $\mathbf{q}'' = (q''_1, q''_2, \ldots, q''_n)$

(ii) *Indifferent relation*: $\mathbf{q}' \sim \mathbf{q}''$
i.e., the household is indifferent between the bundles $\mathbf{q}'$ and $\mathbf{q}''$.
(iii) *Weak relation*: $\mathbf{q}' \geqslant \mathbf{q}''$
i.e., the household either prefers $\mathbf{q}'$ to $\mathbf{q}''$ or is indifferent between $\mathbf{q}'$ and $\mathbf{q}''$.

The preference relations can also be understood by considering the expenditure involving these commodity bundles to the household.
Suppose that

$$\sum_{i=1}^{n} p_i q_i' \leqslant \sum_{i=1}^{n} p_i q_i'',$$

and the household chooses the $\mathbf{q}''$ combination of commodities though $\mathbf{q}''$ is more costly than $\mathbf{q}'$. One, therefore, infers that $\mathbf{q}'' > \mathbf{q}'$. A household would *reveal* indifference between the bundles ($\mathbf{q}' \sim \mathbf{q}''$) if it permits any other person to select the combination that it is to receive. All such commodity bundles to which the household is indifferent are said to lie on the same *indifferent surface*. If commodity space of n-commodities is restricted to two commodities so that,

$$\mathbf{q}' = (q'_1, q'_2), \ \mathbf{q}'' = (q''_1, q''_2) \text{ and } \mathbf{q}' \sim \mathbf{q}'',$$

the indifference surface becomes an *indifference curve* as shown in Fig. 2.1 (a).

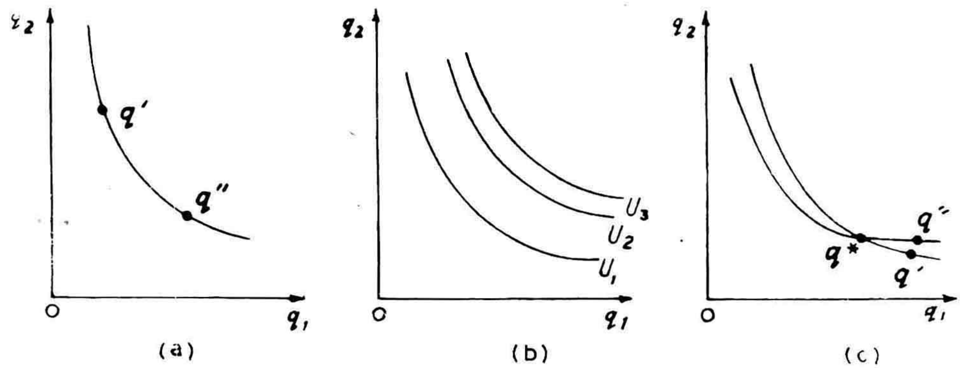

Fig. 2.1. (a), (b) and (c).

2.2 Basic Concepts and Assumptions

The points $\mathbf{q}'$ and $\mathbf{q}''$ shown on indifference curve in Fig. 2.1(a) represent two combinations of two commodities which are preferred equally by the household. All such combinations to which an household is indifferent are said to possess equal utility; if one combination is preferred to another, it is said that it has greater utility for the household at that particular line. The *ranking* or *ordering* the preferences for various bundles of commodities (on the basis of relative utility) is referred to as *ordinal preference ranking* of combinations. Two points need to be emphasised in such type of ranking: (1) there is no implication as to the degree or amount to which the combination with higher utility is preferred above the other, and (2) that interpersonal comparisons of utility are not possible. For instance, if two households assert that $\mathbf{q}'$ bundle of commodities has a higher utility for them, it can only be inferred that $\mathbf{q}' > \mathbf{q}''$ for each household. Nothing can be inferred about the extent to which each household prefers $\mathbf{q}'$ over $\mathbf{q}''$, nor can it be inferred as to which of the two households has a higher relative degree of preference for $\mathbf{q}'$ over $\mathbf{q}''$.

Further, for analytical purposes, it is essential to assume that each household is *reasonable* and *rational* in the process of ranking, that is, if a household expresses preference for $\mathbf{q}'$ over $\mathbf{q}''$ bundle of commodities, and also preference for $\mathbf{q}''$ over $\mathbf{q}'''$ bundle of commodities, then it seems very reasonable to assume that the household would also show preference for $\mathbf{q}'$ over $\mathbf{q}'''$ when asked to choose between $\mathbf{q}''$ and $\mathbf{q}'''$ bundles of commodities. This rationality in behaviour is referred to as *transitivity in preferences.* The assumption of transitivity dismisses the possibility of intersection of two indifference surfaces as shown in Fig. 2.1 (c). Nor can two surfaces touch each other. Each household's indifference space will consist of a set of indifference curves such as those in Fig. 2.1 (b). A higher indifference surface U_3 corresponds to a higher level of satisfaction than that of U_1. If the indifference curves were to intersect, as in Fig. 2.1 (c), then the point of intersection, $\mathbf{q}^*$ would lie on both curves such as $\mathbf{q}^* \sim \mathbf{q}'$ on U_1 and $\mathbf{q}^* \sim \mathbf{q}''$ on U_2. From the transitivity assumption, it can be derived that $\mathbf{q}' \sim \mathbf{q}''$ such as the two indifference surfaces U_1 and U_2 must coincide and become one and the same indifference surface but *cannot intersect.*

2.3 Utility Function

A utility function indicates the amount of utility of satisfaction a household receives from the various amounts of different bundles of commodities. Returning to the case of two goods and assuming that the household derives satisfaction to the level equivalent to utility from $\mathbf{q}'\{=(q_1', q_2')\}$ combination of the commodities, the utility function is then defined as $U=f(q_1', q_2')$. It is assumed that $f(q_1', q_2')$ is continuous and has continuous first- and second-order partial derivatives. First-order partial derivatives, called marginal utilities, should be positive to fulfil the assumption of *nonsatiation* necessary for the analysis. The following *two* points need to be emphasised regarding the utility function:

(a) The utility number or measure U assigned to the particular commodity combination ($\mathbf{q}'$ in the above case) indicates that it is preferable to all combinations with lower number and inferior to those with higher numbers. Hence the utility function is sometimes referred to as an *ordinal utility function*, the values taken by the function being *ordinal utilities.*

(b) The utility function is not unique; any *monotonic* and *strictly increasing* function of combination of commodities can serve as a utility function. In general, if $U(\mathbf{q}')\ [=f(q_1', q_2')]$ is a utility function then so is $f[U(\mathbf{q}')]$, where f is a strictly increasing function ($f'>0$).

Thus, $aU(\mathbf{q}') + b$, where a and b are constants and $a>0$ is an example of the utility function, as is $e^{U(q')}$.

2.4 Problem of the Household

The problem of a household is that of choosing a bundle of commodities and services; given the preference relation (or utility function) and the budget constraint. The budget constraint states that the total money expenditure on all goods and services cannot exceed money income. The problem of the household may now be written in the following notation form:

Maximise: $U=f(q_1, q_2, \ldots, q_n)$

Subject to: $E = \sum_{j=1}^{n} p_j q_j = (p_1 q_1 + p_2 q_2 + \ldots + p_n q_n) \leqslant I$

$q_1 \geqslant 0,\ q_2 \geqslant 0, \ldots, q_n \geqslant 0.$

This problem is one of linear programming. To transform it into a simplified form, two things are assumed:

(a) that all commodities are purchased, thereby eliminating the consideration of some commodities out of n-available commodities are not purchased by the household; and

(b) that all income is spent on the commodities, thereby eliminating the inequality from the constraint.

The above problem then reduces to that of simple optimisation problem in which an objective function (utility function) is to be maximised subject to a single (budget) constraint, an equality.

The problem may be rewritten as follows:

Maximise : $U = f(q_1, q_2, \ldots, q_n)$ (2.1)

Subject to: $E = I = \sum_{i=1}^{n} p_i q_i$ or, $I - \sum_{i=1}^{n} p_i q_i = 0$ (2.2)

$$q_1 \geqslant 0,\ q_2 \geqslant 0, \ldots, q_n \geqslant 0$$

Such problems are solved through a procedure wherein we employ *Lagrange multipliers* λ's, one for each constraint in the new objective function called *Lagrangian function.* The first-order condition (to seek a stationary point) and second-order condition (to fulfil convexity requirement) are then required to be satisfied to obtain a solution of the problem.

2.5 Order Conditions of Optimisation

(a) First-Order Conditions

From (2.1) and (2.2), frame a new Lagrangian function:

$$L = f(q_1, q_2, \ldots, q_n, \lambda)$$

The problem of household is to determine q's and λ so to maximise:

$$L = f(q_1, q_2, \ldots, q_n) - \lambda(\Sigma p_i q_i - I) \tag{2.3}$$

To maximise function L of $(n+1)$ variables, first seek a stationary point such that each of $(n+1)$ first-order partial derivatives equal zero; i.e.,

$$\frac{\partial L}{\partial q_i} = U_i - \lambda p_i = 0,\ i = 1, 2, \ldots, n; \tag{2.4a}$$

and,

$$\frac{\partial L}{\partial \lambda} = I - \Sigma p_i q_i = 0 \tag{2.4b}$$

We have denoted $\frac{\partial f}{\partial q_i}$ as U_i for the sake of notational simplicity

We shall also require $\frac{\partial^2 f}{\partial q_i^2}$ which is denoted as U_{ii} and

$$\frac{\partial^2 f}{\partial q_i q_j} = U_{ij} = \frac{\partial^2 f}{\partial q_j q_i} = U_{ji}$$

From (2.4a),

$$\lambda = \frac{U_i}{p_i} = \frac{U_j}{p_j};\ \frac{U_i}{U_j} = \frac{p_i}{p_j} \tag{2.5}$$

for all $i, j, = 1, 2, \ldots, n.$

Since, $$U_i = \frac{\partial}{\partial q_i}[f(q_1, q_2, \ldots, q_n)]$$

depicts the marginal utility of commodity *i*, equation (2.5) states that the ratio of the marginal utilities of any two goods is equal to the ratio of their prices.

First-order condition in the form of equation (2.5) can also be described in terms of *(marginal) rate of commodity substitution* (RCS) between any two commodities.

If the household is to make a movement along a given indifference space by changing the 'mix' of the $\mathbf{q}_i$ then,

$$dU = U_1dq_1 + U_2dq_2 + \ldots + U_ndq_n$$

$$= \sum_{i=1}^{n} U_idq_i = 0 \qquad (2.6)$$

But if the household is to change the 'mix' only by two commodities, *i*th and *j*th, then its movement will be regarded along an indifference curve with $U=U_0$ (a particular level of satisfaction), so that,

$$dU = 0 = U_idq_i + U_jdq_j$$

or, $$-\left(\frac{dq_j}{dq_i}\right)_{u=u_0} = \frac{U_i}{U_j} \qquad (2.6a)$$

That is, RCS of the *j*th commodity for the *i*th on any indifference surface is equal to the ratio of their marginal utilities.

Also, $$\frac{U_i}{U_j} = -\left(\frac{dq_j}{dq_i}\right)_{u=u_0} = \frac{p_i}{p_j} \qquad (2.7)$$

Equation (2.7) expresses the first-order conditions, that RCS between any two goods should be equal to their price ratio.

Analyse this equation.

What does the $\left\{-\left(\frac{dq_j}{dq_i}\right)\right\}$ depict?

$\frac{dq_j}{dq_i}$ expresses the RCS for a household, i.e., a rate at which a household could be willing to substitute the *i*th commodity for the *j*th or the *j*th for the *i*th in order to maintain a level of satisfaction (i.e., utility) equivalent to U_0. Negative $\frac{dq_i}{dq_j}$ represents the slope of an indifference curve.

What does $\left(\frac{p_i}{p_j}\right)$ depict?

Since, $I = \sum_{i=1}^{n} p_iq_i$

and if the household is to alter purchase on q_j and q_i commodities within a given income level, then; $dI = p_i dq_i + p_j dq_j = 0$

$$\text{i.e., for } I = I_0, \quad -\left(\frac{dq_j}{dq_i}\right)_{I=I_0} = \frac{p_j}{p_i} \tag{2.7a}$$

In other words, $\frac{p_i}{p_j}$ is the rate of exchange of the jth commodity for the ith commodity that can be obtained within a given income $(=I_0)$. The negative sign along with this rate represents the slope of the budget constraint.

First-order conditions in the form of equation (2.7), thus, states the equality between the slopes of indifference surface and the budget constraint. Nevertheless, this condition does not ensure that a maximum is actually reached on fulfilment of this equality. It only guarantees the attainment of the stationary point. In the case of two commodities when $I = p_1q_1 + p_2q_2$ and $U = f(q_1, q_2)$, the four situations can be described in which condition (2.7) can be said to have been fulfilled.

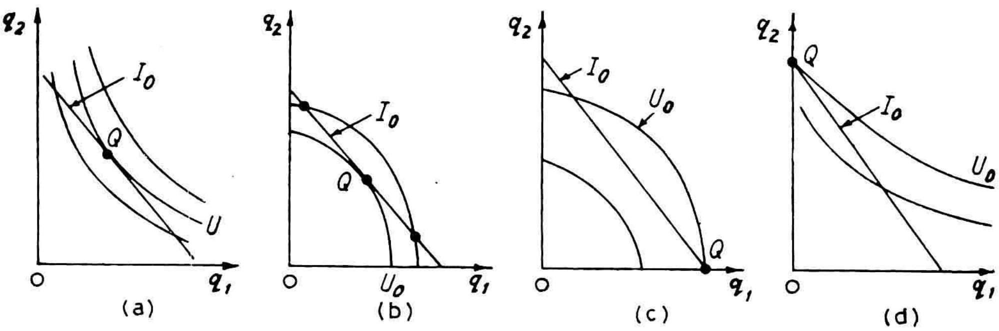

Fig. 2.2. (a), (b), (c) and (d).

To determine in which of the four diagrams, given in Fig. 2.2, the household's utility will be maximised at the stationary point Q, the second-order conditions have to be examined.

(*b*) *Second-Order Conditions*

The following second-order derivatives are obtained from the first-order conditions (2.4a and 2.4b):

$$\frac{\partial^2 L}{\partial q_i q_j}, \ \frac{\partial^2 L}{\partial q_j q_i} \text{ and } \frac{\partial^2 L}{\partial \lambda \partial q_i}$$

But,

$$\frac{\partial^2 L}{\partial q_i q_j} = \frac{\partial^2 L}{\partial q_j q_i} = \frac{\partial^2 f}{\partial q_i q_j} = \frac{\partial^2 f}{\partial q_j q_i} = U_{ij} = U_{ji}$$

and,

$$\frac{\partial^2 L}{\partial \lambda \partial q_i} = p_i \text{ for } i, j = 1, 2, \ldots\ldots, n.$$

To examine the second-order conditions, the matrix **A** has to be framed in terms of these derivatives:

$$\mathbf{A} = \begin{bmatrix} 0 & p_1 & p_2 & \ldots & p_n \\ p_1 & U_{11} & U_{12} & \ldots & U_{1n} \\ p_2 & U_{21} & U_{22} & \ldots & U_{2n} \\ \vdots \\ p_n & U_{n1} & U_{n2} & \ldots & U_{nn} \end{bmatrix}$$

$U = f(q_1, q_2, \ldots, q_n)$ will be maximised, subject to $I = \Sigma p_i q_i$, at the stationary point if $A_3 > 0$, $A_4 < 0$, $A_5 > 0, \ldots\ldots$, where $A_3, A_4, A_5 \ldots\ldots$ are the principal minors of $|\mathbf{A}|$.

In two-commodity case,

$$\mathbf{A} = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{bmatrix} \text{ so that},$$

$$A_3 = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{vmatrix} = -p_1(p_1 U_{22} - p_2 U_{12}) + p_2(p_1 U_{12} - p_2 U_{11});$$

Since $U_{12} = U_{21}$,

$$A_3 = -(p_1^2 U_{22} - 2 p_1 p_2 U_{12} + p_2^2 U_{11}) \qquad (2.8)$$

To determine whether $A_3 > 0$, equation (2.6a) must be differentiated:

$$\left(\frac{\partial q_2}{\partial q_1}\right)_{U=U_0} = -\left(\frac{U_1}{U_2}\right)$$

$$\frac{\partial^2 q_2}{\partial q_1^2} = \frac{\left[-U_{11} - U_{12}\left(\frac{\partial q_2}{\partial q_1}\right)\right] - \left[U_{21} - U_{22}\left(\frac{\partial q_2}{\partial q_1}\right)\right]\left[\left(-U_1\right)\right]}{U_2^2}$$

$$= \left\{\frac{\left[-U_{11}U_2 + U_{12}\left(\frac{U_1}{U_2}U_2\right)\right] + \left[U_{21}U_1 - U_1^2 U_{22}\left(\frac{1}{U_2}\right)\right]}{U_2^2}\right\}\left(\frac{U_2}{U_2}\right)$$

$$\frac{\partial^2 q_2}{\partial q_1^2} = \frac{1}{U_2^3}\left(-U_{11}U_2^2 + 2U_{21}U_1U_2 - U_1^2 U_{22}\right)$$

$$= -\frac{1}{U_2^3}\left(U_1^2 U_{22} - 2U_{21}U_1U_2 + U_2^2 U_{11}\right)$$

By using $\frac{U_1}{U_2} = \frac{p_1}{p_2}$ from equation (2.7)

$$\frac{\partial^2 q_2}{\partial q_1^2} = -\frac{1}{U_2}\left(\frac{p_1^2}{p_2^2} U_{22} - 2 U_{21}\frac{p_1 p_2}{p_2^2} + \frac{p_2^2}{p_2^2} U_{11}\right)$$

Since $U_{12} = U_{21}$:

$$\frac{\partial^2 q_2}{\partial q_1^2} = -\frac{1}{U_2 p_2^2}\left(p_1^2 U_{22} - 2 p_1 p_2 U_{12} + p_2^2 U_{11}\right)$$

$$\frac{\partial^2 q_2}{\partial q_1^2} = \frac{1}{U_2 p_2^2}(A_3) = \frac{A_3}{U_2 p_2^2} \text{ (from 2.8)}$$