SEQUENCY THEORY

Foundations and Applications

Advances in Electronics and Electron Physics

Supplement 9

HENNING F. HARMUTH

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HENNING F. HARMUTH

DEPARTMENT OF ELECTRICAL ENGINEERING
THE CATHOLIC UNIVERSITY OF AMERICA
WASHINGTON, D.C.

1977

CADEMIC PRESS New York San Francisco London

Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 63-12814

ISBN 0-12-014569-3

PRINTED IN THE UNITED STATES OF AMERICA

Foreword

In publishing this supplementary volume to *Advances in Electronics and Electron Physics*, it is useful to give some background. A few years ago we became aware of the importance of Walsh functions for telecommunications. Our early contact with Professor Harmuth resulted in a critical review published in the 36th volume (1974) of our regular series. Two years later a second review on a related subject followed and, at the time of writing this Foreword, it is being printed in the 41st volume of the regular series. Other discussions followed, and it is a great pleasure to present here his monograph on "Sequency Theory: Foundations and Applications."

Dr. Harmuth's Preface traces very briefly the early history of sequency theory. The initial disappointments of its proponents, at the hand of its opponents, are not surprising. Many of us, who tried to introduce a new concept or a new process, are only too familiar with resistance to new ideas. The important aspect is that good ideas survive and bring forth such results as are reported here.

We hope that this monograph will find as much favor with the readers as our earlier ones.

L. MARTON
C. MARTON

Preface

Sequency theory started about a decade ago. An adverse climate for scientific research prevailed during most of its infancy, yet the theory was advanced to the practical level in television image processing, in the generation of moving images by means of sound waves underwater, and in radar. These three lines of practical development will be emphasized. The theoretical developments have become too varied to be covered in one book; hence, the developments discussed are those which appeared to be most stimulating or most controversial, depending on one's point of view.

The difficulties that have to be overcome by a new theory and the rapid progress of sequency theory are best illustrated by the advancement in the area of nonsinusoidal electromagnetic waves. At the first scientific meeting on sequency theory in May 1968 at the Research Institute of the Deutsche Bundespost in Darmstadt, West Germany, the very notion of using nonsinusoidal electromagnetic waves was roundly denounced, but in January 1976 the first useful application in radar was demonstrated by J. Chapman. The trail from ridicule to realization was blazed in eight years. The lively discussion of May 1968 was preserved on magnetic tape by H. Hübner of the Deutsche Bundespost as a lasting record of the emotions aroused by a new idea.

The bibliographies of K. Beauchamp and J. Bramhall, augmented by the listing of very recent publications in this volume, give credit to the scientific contributors to sequency theory. Credit for administrative and financial support of sequency theory in general and the author in particular is due to the following organizations and persons:

the Electromagnetic Compatibility Group of The Institute of Electrical and Electronics Engineers, especially its officers N. Ahmed, W. E. Cory, H. Randall, G. R. Redinbo, H. M. Schlicke, R. B. Schulz, R. M. Showers, L. W. Thomas, and J. C. Toler;

various groups of the US Navy and its officers C. A. Bass, A. Berman, M. A. Blizard, A. Cecelski, R. Golding, R. D. Matulka, A. E. Showalter, J. Trimble, and R. W. Zeek;

the US Air Force Office of Scientific Research and its officers J. W. Gregory and J. H. Rosenbloom;

the US Air Force Office of Scientific Research and its officers F. Rohde and J. Hannigan;

personal thanks for continued support are due to R. Meister of The Catholic University of America, and to V. Anderson, T. Caldwell, and J. Hess of Undersea Research Corporation; to J. W. Bayless of A. A. Collins Inc., and to H. Bergmann of the Deutsche Bundesmarine.

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* Equations are numbered consecutively within each of Sections 1.1.1 to 4.7.8. Reference to an equation in a different section is made by writing the number of the section in front of the number of the equation, e.g., Eq. (211-6) for Eq. (6) in Section 2.1.1.

Illustrations and tables are numbered consecutively within each section, with the number of the section given first, e.g., Fig. 243-4, Table 313-2.

References are characterized by the name of the author(s), the year of publication, and a lowercase latin letter if more than one reference by the same author(s) is listed for that year.

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Introduction

The Dogma of the Circle

The Greek philosopher Plato is credited with the introduction of the dogma of the circle. It was expressed by Claudius Ptolemy as follows:[1] "We believe that the object which the astronomer must strive to achieve is this: to demonstrate that all the phenomena in the sky are produced by uniform and circular motions." The astronomical observations of the Greeks were accurate enough to show that the planets were not moving on circles or on surfaces of spheres, regardless of whether one assumed the Earth or the Sun as the center of motion. Eudoxus, a disciple of Plato, used a superposition of rotating spheres in an attempt to reconcile the observed data with the dogma of the circle. Four spheres were needed for each one of the five known planets,[2] three each for the Sun and the Moon, and one for the fixed stars. These 27 spheres proved unsatisfactory. Aristotle reduced the discrepancies between theory and observation by using 54 spheres.

Claudius Ptolemy replaced the spheres by circles. The five planets, the Sun, and the Moon moved around the Earth on primary circles called *deferents*. Superimposed on each deferent was a secondary circle, called *epicycle*, as shown in Fig. 0-1. Another epicycle was superimposed on the first epicycle, and so on. In modern language, we would say that the orbits were represented by a superposition of circles. Ptolemy used 36 circles to

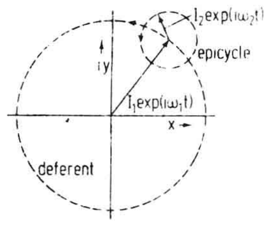

FIG. 0-1. Superposition of circles in astronomy and in electrical communications.

[1] A detailed account of the dogma of the circle in astronomy is given by Koestler (1959). German translations of the most pertinent Greek and Latin texts are collected in a book by Zinner (1951).

[2] Mercury, Venus, Mars, Jupiter, and Saturn are visible to the unaided eye.

represent the orbits of the Sun, the Moon, and the five planets. This was not quite enough to fit the observed data, and better representations using more circles were subsequently worked out. Nicholas Copernicus moved the center of motion from the vicinity of the Earth to the vicinity of the Sun, but he retained the representation of orbits by a superposition of circles. The orbit of Mercury required eleven circles, Venus and Earth nine circles each, the Moon four circles, and the remaining planets five circles each. This adds up to 48 circles.

Johannes Kepler put an end to the superposition of circles in 1609, when he showed in his book "Astronomia Nova" that elliptical orbits fitted the observed data better and simpler.

It is generally believed that Kepler ended the dogma of the circle, but this is not so. The circle disappeared from astronomy, but it reappeared in other fields of science in disguise. In electrical engineering and a good part of physics, we meet the old circle under the new name of exponential function $e^{i\omega t}$ or unit circle in the complex plane. Anyone with the usual background of electrical communications will interpret Fig. 0-1 not as a superposition of a deferent and an epicycle but as a superposition of two sinusoidal oscillations $I_1 \exp(i\omega_1 t)$ and $I_2 \exp(i\omega_2 t)$ using complex notation.[1] Indeed, Fig. 0-1 is a standard illustration for single sideband modulation of a sinusoidal carrier by a sinusoidal signal. Speaking more generally, the superposition of circles by Ptolemy and Copernicus became the Fourier series expansion in complex notation.

The expression *character group of the topologic group of real numbers* does not seem to have anything to do with the circle, but its mathematical notation $\{e^{ixy}\}$ reveals the truth. This character group implies the topology of the continuum for space–time, which in turn permits the use of differential calculus for functions of space and time. Considering the universal use of differential calculus in physics, one cannot help but suspect that the circle influences physics today as much as it once influenced astronomy.

Finding and studying hidden remnants of the dogma of the circle is the purpose of sequency theory.

Let us observe that the deferents and epicycles of Ptolemy represented orbits well enough to get Vasco da Gama to India, Christopher Columbus to America, and one of Ferdinand Magellan's ships around the world. It was a sufficiently good theory for many practical purposes, but its finer details always indicated that something was not quite right. Similarly, the exponential function or the sine–cosine functions in communications have proved to be

[1] $I \exp(i\omega_1 t)$ and $I_2 \exp(i\omega_2 t)$ are called vectors in the older literature (Cherry, 1949; Cuccia, 1952) and phasors in the newer literature (Van Valkenburg, 1964; Taub and Schilling, 1971).

perfectly good for many applications, but it is generally known that something is not quite right. All real signals have an infinite frequency bandwidth, filters yield an output voltage before an input voltage is applied, etc. We use experience and common sense to correct for such deficiencies of the theory; but a correct theory would not need corrections, and the known need for corrections may be like the tip of the iceberg. Turning to the character group of the topologic group of real numbers, there can be no doubt about the success of differential calculus in physics; but it is unsatisfactory to talk about what is happening at a point x and at another point $x + dx$, if we cannot make measurements at two points having a distance dx from each other.

The Circle and the Circular Functions in Communications

The unit circle in the complex plane, $e^{i\omega t} = \cos \omega t + i \sin \omega t$, and its decomposition into circular functions play a dominant role in electrical communications and physics. Whenever one uses the term frequency, one refers implicitly to these functions. Let us see how this dominant role came about and where its limitations are.

During the 19th century, the most important functions for communications were the block pulses shown in Fig. 0-2. Voltage and current pulses could be generated by mechanical switches, amplified by relays, and detected by a variety of magnetomechanical devices. Sine–cosine functions and the exponential function were well known and so was Fourier analysis, although

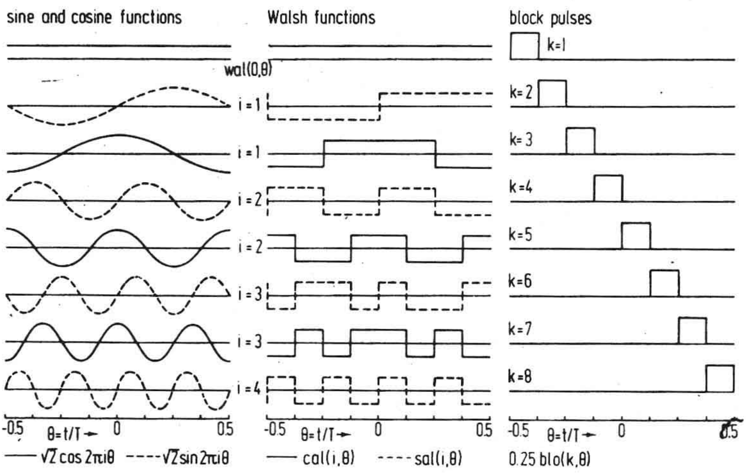

FIG. 0-2. Sine–cosine functions, Walsh functions, and block pulses.

in a somewhat rudimentary form. Almost no practical use could be made of this knowledge with the technology available at that time. Heinrich Hertz used the exponential function to obtain his famous solution of Maxwell's equations for dipole radiation, but he was never able to produce sinusoidal waves. His experiments were done with what we would call colored noise today (Hertz, 1889). Alexander Graham Bell tried to develop telegraphy multiplex equipment using sinusoidal functions, but failed because he could not produce voltages with sinusoidal time variation. His voltages were square waves, while his receivers resonated with sine waves. Two results of this work were the introduction of the word *sinusoidal* into communications engineering and the discovery of voice transmission by electricity. Bell's telegraphy transmitter decomposed voice into square waves and the receiver recomposed it from the square waves. Hence, the decomposition of voice into square waves predates the decomposition into sinusoidal waves by many decades (Bell, 1876; Marland, 1964).

Telegraphy equipment using orthogonal sine–cosine functions according to Bell's concept was successfully developed more than seventy years later under names like Kineplex, Rectiplex, and Digiplex.

The first practical use of sinusoidal functions came toward the end of the 19th century with the development of capacitors in a useful form. Capacitors in the form of metallic spheres and Leyden jars had existed for a long time, but their capacitance was small and their physical structure inconvenient. The implementation of inductances through the use of coils had been known long before. Practical resonant circuits for the separation of sinusoidal electromagnetic waves with different frequencies could thus be built around the turn of the century. Low-pass and band-pass filters using coils and capacitors were introduced in 1915, and a large new field for the application of sinusoidal functions was opened. Speaking more generally, the usefulness of sinusoidal functions in communications is intimately related to the availability of linear, time-invariant circuit components in a practical form.

On the theoretical level, the use of sinusoidal functions is strongly favored by differential calculus, and our concept of the topology of space–time derived from it. This theoretical basis is discussed in some detail in the sections of this book devoted to physics.

The first indication that a theory of communications based on sine–cosine functions would eventually prove unsatisfactory comes from the importance of linear, time-invariant circuit components and circuits for these functions. One cannot transmit information if everything is (time) invariant. The telegrapher's key, the microphone, and the amplitude modulator are linear but time-*variable* devices. Making them time *invariant* by not operating the key, not speaking into the microphone, or not feeding a time-variable modulating voltage into the modulator puts an end to the transmission of information. The requirement for time variability for information transmission holds quite generally. An atom with all electrons in certain quantum states transmits no information. A photon is emitted by a change of quantum state, and this time variation provides information about the energy difference of the initial and final state.

Sine–cosine functions are obtained as the eigenfunctions of systems described by linear differential equations with constant coefficients. Hence, sine–cosine functions are most convenient as long as one may ignore the time variability. Increasing sophistication forces one to use equations (not necessarily differential equations) with variable coefficients; their eigenfunctions are no longer sinusoidal functions.

We have so far discussed *time* variability. *Space* variability is a straightforward extension. Sinusoidal functions have slanted our thinking heavily toward time signals. As an example, consider a tunable generator for sinusoidal functions. All the commercially available ones are generators for time-variable sinusoidal functions. Indeed, it is not only impossible to buy generators for space-variable sinusoidal functions but it is rather difficult even to imagine a generator that can be tuned, e.g., from 20 to 20,000 cycles per meter. Most textbooks do not mention space signals. Publications on filters are almost exclusively concerned with time signals.

A simple example of a space signal with two variables is a black-and-white photograph that has various shades of gray as a function of x and y in cartesian coordinates or of r and φ in polar coordinates. A television signal is a function of two space variables and the time variable. The ubiquity of TV signals makes it safe to conclude that most transmitted information does not consist of functions of the time variable only. Beyond TV, most of the information received by us comes through the eyes and not the ears.

Why then do we hear so little about space signals and filters for space signals? One reason is that the concept of time invariance, meaning that something has always been as it is now and will always remain so, is acceptable to our thinking although we know it is unrealistic. Space invariance, on the other hand, is so unrealistic that we cannot accept it. A television image clearly has a left and a right edge, a top and a bottom, while the finite extension in time is much less obvious. There are $30 \times 3600 = 108{,}000$ images as a sequence of time per hour according to the U.S. standard, but none of the more widely used TV systems has more than some five or six hundred space points in the x and y direction. Hence, a theory of space filters must begin with space-variable filters and cannot consider space-invariant filters as a starting point. A second reason for not hearing much about filters for space signals is that filters for time signals are overwhelmingly implemented by inductances and capacitances, but this technology is not applicable to filters for space signals.

We may answer at this point a question that has often been raised: Why should one use nonsinusoidal functions when sinusoidal functions have proved to be so good for theoretical investigations and practical applications? The answer is that there are certain uses for which sinusoidal functions are good. These are the uses that have been developed during the last eighty years. There are other uses for which sinusoidal functions are not good and which therefore have not been developed and are not found in our textbooks. Spatial electric filters were not derived from sinusoidal functions. Television scanners based on sine–cosine functions were never developed. Sinusoidal electromagnetic waves cannot be used to discriminate between a reflector and a scatterer, or between a conducting and a nonconducting scatterer. Both effects are of great interest in radar, but their very existence escaped our attention as long as we were restricted to thinking in terms of sinusoidal waves. Multipath transmission is known to lead to signal cancellation due to interference fading, but only for sinusoidal and other *polarity-symmetric* waves. Several more effects of electromagnetic waves have been found that are so obscure for sinusoidal waves that they were never noticed.

Let us turn to the third basic reason for going beyond sine–cosine functions: the convergence of Fourier series and Fourier transform. Any practical signal can be approximated by the Fourier series or the transform in the sense of a vanishing mean-square error. Mean-square convergence implies that the energy of a signal is the same as that of a superposition of sine–cosine functions approximating it. This preservation of energy is certainly necessary, but it is not sufficient for the transmission of information. For an explanation of this statement refer to Fig. 0-2. The Walsh functions shown there are characterized by the location of the zero crossings or sign changes. The constant sections between the zero crossings can always be filled in; they convey no information. A Fourier series or transform of these Walsh functions converges everywhere except at the zero crossings. The divergence at these "jumps" is so well known that it received its own name, *Gibbs phenomenon.* Hence, we must conclude that the Fourier series converges everywhere, except where it is needed. Let us go one step further and consider a current flowing in a Hertzian dipole. The electric and magnetic field strengths in the far zone are proportionate to the first derivative of the current. If the current is represented by a series expansion, we cannot differentiate term by term to obtain the field strengths, since convergence of a series does not imply convergence of the differentiated series. To obtain some idea about the number of solutions of the wave equation or Maxwell's equations that cannot be represented with uniform convergence by a Fourier series or transform, let us note that for each solution with uniform convergence there are infinitely many solutions without uniform convergence.

It is worthwhile returning to astronomy at this point. An elliptical orbit can be represented by a sum of circles without any problems of convergence or differentiability. The concept of epicycles was not wrong—it was only unnecessarily complicated. The simplification of the representation of the orbits of the planets by means of ellipses with the Sun in one focal point led in due course to the theory of gravitation. There is hardly a better example to show the importance of simplicity. Even if a series expansion is used correctly, it may obscure features that a simpler representation would reveal.

A look at the practical side of the convergence problem shows that circuit design is well ahead of theory. The typical on–off type switching functions preferred by semiconductor circuits do not permit an approximation of the transients due to the Gibbs phenomenon, and the transients are the important parts of the switching functions. But this is no problem, since nobody builds pulse generators that contain many amplitude-, frequency-, and phase-stable sinusoidal oscillators in order to produce two-valued pulses according to the Fourier series. On the contrary, it is general practice to synthesize stable sinusoidal oscillations by means of block pulses generated by digital circuits.[1]

Basic Mathematical Concepts

To see in which way one may profitably generalize our theory of communications based on sine–cosine functions, let us consider Fig. 0-2 again. Block pulses, which were the historically first important system of functions, are shown on the right. The sine–cosine functions plus the constant function used in the Fourier series are shown on the left. One may readily see why we have an extensive theory based on sine–cosine functions, but not one based on block pulses. The block pulses differ by a time shift only. In other words, they contain one free parameter, which we call *delay*. The periodically continued sine–cosine functions contain the parameter delay too, which is called *phase* for these particular functions, but in addition they contain the parameter frequency. In essence, sine–cosine functions of different frequency have a different shape, while the block pulses all have the same shape. For a satisfying, more general theory, one will thus have to look for nonsinusoidal functions that have at least as many parameters as the sinusoidal functions. Since sine–cosine functions are a particular system of orthogonal functions, one may replace them by general systems of orthogonal functions.

The term *orthogonal* is defined as follows: Two functions $f(j, \theta)$ and $f(k, \theta)$ with the variable θ and the parameters j and k are called orthogonal in the interval $-\frac{1}{2} < \theta < \frac{1}{2}$ if the integral $\int_{-1/2}^{1/2} f(j, \theta) f(k, \theta)\, d\theta$ is zero for $j \neq k$. They are called orthogonal and normalized or orthonormal if the integral equals 1 for $j = k$.

[1] An even better synthesis by means of Walsh functions was reported by Kitai (1975b).