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Dynamics of Flexible Multibody Systems

Rigid Finite Element Method

Springer

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1

Introduction

Modelling dynamics of multibody systems has been a subject of interest in research centres for many years. Not only machines and mechanisms themselves but also many of their constituent parts are multibody systems; therefore modelling these is essential in design. Computer programmes using these models must enable us to take into account complex phenomena connected with flexibility of links and friction and clearance in joints. The difficulty in describing such problems lies in the fact that global phenomena such as motion of a mechanism or machine, which last a few seconds, have to be considered simultaneously with local phenomena, such as contact of links in joints or vibrations of flexible links, the duration of which is tenths or hundredths of seconds. This causes problems both at the stage of model formulation and during integration of equations of motion of multibody systems.

One of the most challenging problems in modelling multibody systems is flexibility of links. The occurrence of large base motion (for example rotation of a crane body, motion of a vehicle, translation of manipulator arms) can cause vibrations of flexible links. There is also an opposite effect: vibrations of flexible links can disturb desired base motion. This is especially important when positioning is a concern. In order to compensate flexibility of links, drive systems have to be equipped with additional control systems.

For many years commercial packages such as MSC.Adams or Dads have been used for modelling multibody systems. They allow not only kinematic analysis of complex mechanisms to be carried out, but also calculations of the dynamics of complex multibody systems, and they enable flexibility of links to be taken into account by means of special interfaces. To this end, additional models of flexible links have to be formulated, usually employing the finite element method (for example in Ansys or MSC.Nastran); then modal analysis necessary for the reduction of generalised coordinates has to be carried out; and finally a simplified model of a flexible link obtained in such a way has to be transferred to the package for dynamic analysis of multibody systems. Commercial packages are used mainly in large research and computational centres because they require considerable training in the methodology and software. Moreover, it is necessary to know the appropriate, often very detailed data for calculations. Thus the use of those commercial packages is limited and many research centres are still looking for simple yet effective methods for analysing the dynamics of multibody systems.

One such method is the rigid finite element method, which is the subject of this book. It can serve both as an initial analysis before more traditional and complex methods are used and as an independent method. Its advantages are numerous: first, it is simple (the basic idea of the method is a division of flexible links into rigid elements connected with spring-damping elements); secondly, it adopts a uniform approach to describe rigid and flexible links; thirdly, it is numerically effective; fourthly, it can be applied to analyse both small and large deformations; finally, it can be widely and successfully used in industrial practice. Therefore the authors believe that many engineers and researchers will benefit from acquaintance with the rigid finite element method.

The method was formulated at the Technical University of Gdansk and its foundations were first described by Kruszewski et al. (1975). The idea of the method is to discretise flexible links into rigid elements containing inertial features of bodies; these rigid elements are connected by massless and non-dimensional spring-damping elements. In Kruszewski et al. (1975) the mathematical models and all information necessary for computer implementation of the method are presented. The models and their application in practice are limited to analysis of deflections and vibrations in a given position, which means that only systems with a stable configuration have been considered. Wittbrodt (1983) presented a generalisation of the method of rigid and flexible elements for planar systems with changing configuration, and its application in dynamic analysis of structures was described by Kruszewski et al. (1984). The method has been applied in Poland for the dynamic analysis of mechanisms, machine tools, cranes, ship drive systems and even to vibration analysis of hulls.

Wojciech (1984) presented a modification of the method which enabled large deflections of flexible links of planar linkage mechanisms with changing configuration to be analysed. An approach in which the system analysed is divided into subsystems, and flexible links are discretised by means of the rigid finite element method, is applied in both monographs (Wittbrodt, 1983; Wojciech, 1984). Equations of the dynamics of subsystems taking into account reactions in subsystem connections have been formulated on the basis of Lagrange equations and then the subsystems have been connected by means of constraint equations. Such an approach is standard when absolute coordinates are used (Gronowicz, 2003).

The classical rigid finite element method as well as its modified version were generalised for spatial systems with changing configuration in Wojciech (1990); Adamiec-Wójcik (1992), Wittbrodt and Wojciech (1995) and Adamiec-Wójcik (2003). In these the rigid finite element method was combined for the first time with the method of homogenous transformations. This method, widely used in robotics (Paul, 1981; Craig, 1988), enables us to represent the transformation of coordinates, for both translation and rotation of a rigid body, by means of one matrix operation. Adamiec-Wójcik (2003) also presents a general algorithm for formulation of equations of motion of multibody systems using joint coordinates and homogenous transformations.

This book presents a new, different formulation of the rigid finite element method. It is assumed that, for both the classical and modified formulations, homogenous transformations will be consistently used together with joint coordinates for the kinematic description of multibody systems. Joint coordinates enable us to reduce considerably the number of generalised coordinates of the system as compared to methods using absolute coordinates. The models and methods presented allow large deformations of flexible links to be considered. Simplified versions of models (called linear), which can be used when deflections of links are small, are also discussed. The models formulated give a unified approach both in cases when open and closed kinematic chains are considered as well as when the system consists of either only rigid links or when rigid and flexible links alternate. We think that this is one of most important features of the method and the description of multibody dynamics.

We assume that the reader knows theoretical mechanics at the level of a mechanical engineering graduate and is able to use some elements of analytical mechanics, especially techniques concerned with derivation of equations of motion using the Lagrange equations. As for mathematics, we expect the reader to be competent in dealing with matrix calculations and differential calculus.

In Chap. 2 the basics of transformations of coordinates and homogenous transformations are presented. In addition the equations of motion of rigid multibody systems are formulated using joint coordinates and homogenous transformations. The equations of motion of a new link attached to an existing kinematic chain are formulated and it is shown how they modify the equations of preceding links. The equations formulated in this chapter are then used throughout the following chapters.

The formulation of the classical rigid finite element method is presented in Chap. 3. The equations of motion of a flexible link divided into rigid elements with six degrees of freedom (three translations and three rotations) are derived. The energy of spring deformation and the dissipation of energy in spring-damping elements are calculated. A linear model with simpler formulae, which is useful for analysis of small vibrations, is also discussed. At the end of the chapter the methods and formulae for calculations of the parameters of both rigid (rfe) and spring-damping elements (sde) are given.

Chapter 4 deals with the modification of the rigid finite element method used to discretise beam-like links with bending and torsional flexibility. Nonlinear and linear models for analysis of large and small vibrations, respectively, are discussed. In the modified formulation of the method each rfe has only three degrees of freedom in relative motion, which are rotation angles. Thus, in relation to the classical formulation, the number of degrees of freedom is considerably smaller.

The problems of numerical calculations are discussed in Chap. 5. At the beginning we present the application of the methods described in Chaps. 3 and 4 (both linear and non-linear formulation) to analysis of the free and forced vibrations of a beam. The reader can follow detailed formulations of various models. Computer simulations show the influence of the model on the results of calculations. Problems concerning the integration of equations of motion of systems discretised using the rigid finite element method are also considered, with special attention to the integration of systems of stiff differential equations.

Chapter 6 is concerned with verification of the method. The results of numerical simulations obtained by means of the rigid finite element method are compared with those obtained by other authors who used different methods, and with results of experimental measurements. The method has been verified both for small and large deformations. An example of vibration analysis of a viscoelastic beam shows how the rigid finite element method can be used to analyse large deflections of whippy beams when complex physical relationships describe material features of flexible links. This chapter demonstrates that the rigid finite element method in both formulations gives results compatible with those published by other authors and with those obtained from experimental measurements.

Practical applications of the method in dynamic analysis of machines and mechanisms are given in Chap. 7. They concern dynamic analysis of a crane, the telescopic rapier in textile machine, and the A-frame of a ship. The chapter shows not only how to proceed with a particular machine but also the many applications of the method.

We would like to thank Krzysztof Augustynek and Andrzej Urbas for their considerable editorial help and our colleagues, co-authors of publications, whose research results we used in this book.

2

Homogenous Transformations

Displacement of a body from one position to another requires two operations: translation and rotation. In classical mechanics general motion of a body can be treated as a combination of translation and rotation about a fixed point. In robotics joint coordinates and homogenous transformations are generally used for description of rigid body motion (Craig, 1988). Joint coordinates enable us to describe the motion of a system of rigid bodies, which form open or closed kinematic chains, by using the least number of generalised coordinates. This leads to a reduction in the dimension of equations of motion describing the dynamics of multibody systems as compared to absolute coordinates, which are used more often. However, the equations are more complex and their derivation requires a specific approach, which will be described in this chapter.

Homogenous transformations allow us to present two operations (translation and rotation) in the form of one complex operation. The consequence is that the transformation of coordinates from one system to another can be expressed by means of only one multiplication of a transformation matrix by a position vector.

2.1 Transformation of Coordinates and Homogenous Transformations

In order to describe the position and orientation of a body in space, coordinate systems (called "frames of reference" by some authors) are defined and the rules of coordinate transformations are set out. Mathematical relations are formulated so that coordinates of a point in any coordinate system can be defined if the coordinates of this point in a given coordinate system and the parameters defining the reciprocal relation of the two systems are known.

Let us assume that two coordinate systems $\{A\}$ and $\{B\}$ (Fig. 2.1) are given. The method of describing the relation of those systems is as follows.

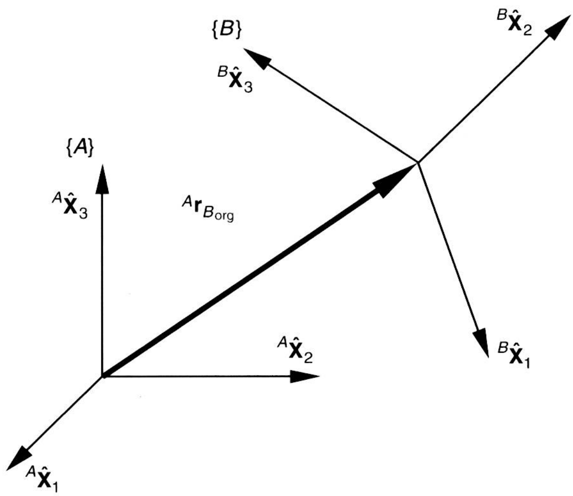

Fig. 2.1. Coordinate systems $\{A\}, \{B\}$. ${}^A\mathbf{r}_{B_{\mathrm{org}}}$ is the vector defining the coordinates of the origin of system $\{B\}$ with respect to $\{A\}$. ${}^A\hat{\mathbf{X}}_1, {}^A\hat{\mathbf{X}}_2, {}^A\hat{\mathbf{X}}_3, {}^B\hat{\mathbf{X}}_1, {}^B\hat{\mathbf{X}}_2, {}^B\hat{\mathbf{X}}_3$ are the unit vectors of coordinate systems $\{A\}$ and $\{B\}$, respectively

The position of the origin of coordinate system $\{B\}$ with respect to $\{A\}$ is defined by the components of the vector:

$$ {}^A\mathbf{r}_{B_{\mathrm{org}}} = \begin{bmatrix} {}^A x_{B,1} \\ {}^A x_{B,2} \\ {}^A x_{B,3} \end{bmatrix}, \tag{2.1} $$

while the orientation of $\{B\}$ in $\{A\}$ is defined by the elements of the rotation matrix:

$$ {}^A_B\mathbf{R} = \begin{bmatrix} {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 & {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 & {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 \\ {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 & {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 & {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 \\ {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 & {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 & {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 \end{bmatrix} = \begin{bmatrix} {}^B_A r_{11} & {}^B_A r_{12} & {}^B_A r_{13} \\ {}^B_A r_{21} & {}^B_A r_{22} & {}^B_A r_{23} \\ {}^B_A r_{31} & {}^B_A r_{32} & {}^B_A r_{33} \end{bmatrix}, \tag{2.2} $$

where $\hat{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{B}}$ is a scalar product of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$.

Matrix ${}^A_B\mathbf{R}$ is a matrix of direction cosines of the axes of coordinate system $\{B\}$ with respect to system $\{A\}$. Its columns and rows are orthonormal vectors, which results in the following relation:

$$ {}^A_B\mathbf{R} = {}^B_A\mathbf{R}^{-1} = {}^B_A\mathbf{R}^{\mathrm{T}}. \tag{2.3} $$

This means that the inverse matrix to the rotation matrix is its transpose. It is important to note that the position of the body to which coordinate system

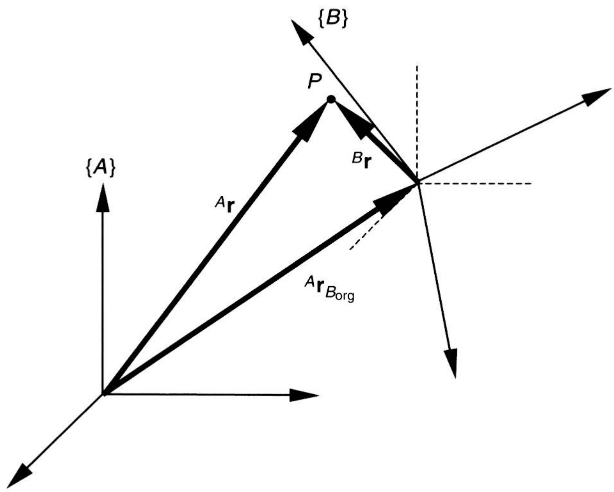

Fig. 2.2. Transformation of coordinates

$\{B\}$ is assigned is defined by vector $^A\mathbf{r}_{B_{\text{org}}}$, and $^A_B\mathbf{R}$ describes its orientation. If we know the position and orientation of $\{B\}$ with respect to $\{A\}$, then, according to Fig. 2.2, the following can be written:

$$^A\mathbf{r} = {}^A_B\mathbf{R}\ {}^B\mathbf{r} + {}^A\mathbf{r}_{B_{\text{org}}}, \tag{2.4}$$

where $^B\mathbf{r}$ is the vector of coordinates of point P with respect to $\{B\}$ and $^A\mathbf{r}$ is the vector of coordinates of point P with respect to $\{A\}$.

Among the nine components of rotation matrix $^A_B\mathbf{R}$ only three are independent because six relations resulting from the orthonormality of the matrix can be defined, which are as follows. If matrix $^A_B\mathbf{R}$ is written in the following form:

$$^A_B\mathbf{R} = \left[\, {}^A_B\hat{\mathbf{X}}_1 \ \ {}^A_B\hat{\mathbf{X}}_2 \ \ {}^A_B\hat{\mathbf{X}}_3 \,\right], \tag{2.5}$$

where

$$^A_B\hat{\mathbf{X}}_1 = \begin{bmatrix} {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 \\ {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 \\ {}^B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 \end{bmatrix}, \quad {}^A_B\hat{\mathbf{X}}_2 = \begin{bmatrix} {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 \\ {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 \\ {}^B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 \end{bmatrix}, \quad {}^A_B\hat{\mathbf{X}}_3 = \begin{bmatrix} {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_1 \\ {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_2 \\ {}^B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A\hat{\mathbf{X}}_3 \end{bmatrix},$$

the following takes place:

$$^A_B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_1 = 1, \qquad {}^A_B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_2 = 1, \qquad {}^A_B\hat{\mathbf{X}}_3^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_3 = 1, \tag{2.6a}$$

$$^A_B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_2 = 0, \qquad {}^A_B\hat{\mathbf{X}}_1^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_3 = 0, \qquad {}^A_B\hat{\mathbf{X}}_2^{\mathrm{T}}\,{}^A_B\hat{\mathbf{X}}_3 = 0. \tag{2.6b}$$

This means that the reciprocal location of the axes of systems $\{B\}$ and $\{A\}$ can be uniquely described by means of three parameters. There are many methods of choosing those parameters (Blajer, 1998; Jurewič, 1984; Shabana, 1998). Here we discuss only one, in which those parameters are called Euler rotation angles ZYX (Blajer, 1998).

Let us consider in detail the case when the origins and axes of coordinate systems $\{B\}$ and $\{A\}$ coincide (Fig. 2.3a), and the problem is to define the rotations of system $\{B\}$ about its axes, which leads to the situation as in Fig. 2.3b.

In order to convert the system from position (a) to (b), we will proceed as follows:

- First, coordinate system $\{B\}$ is rotated by angle φ_3 about axis ${}^A\hat{\mathbf{X}}_3 = {}^B\hat{\mathbf{X}}_3$ and coordinate system $\{B\}''$ is obtained.
- Secondly, the coordinate system obtained is rotated by angle φ_2 about axis ${}^{B''}\hat{\mathbf{X}}_2$, which results in coordinate system $\{B\}'$.
- Finally, the system obtained is rotated by angle φ_1 about axis ${}^{B'}\hat{\mathbf{X}}_1$ to achieve the position as in Fig. 2.3b.

This procedure is shown in Fig. 2.4.

In order to transform the body from position (b) to (a) (Fig. 2.3), one should proceed in the reverse order, which means the following should be performed:

- Rotate the system by $-\varphi_1$ about axis ${}^B\hat{\mathbf{X}}_1$
- Rotate the system by $-\varphi_2$ about axis ${}^{B'}\hat{\mathbf{X}}_2$
- Rotate the system by $-\varphi_3$ about axis ${}^{B''}\hat{\mathbf{X}}_3$

The rotation matrix parameterised by means of Euler angles ZYX can be written in the form:

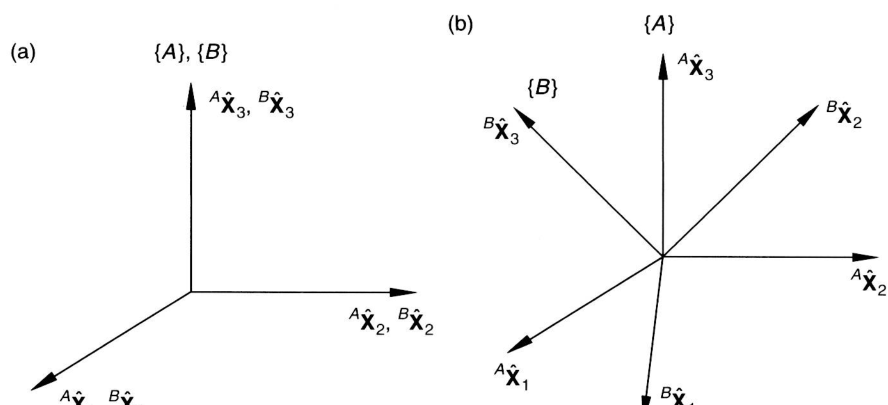

Fig. 2.3. Initial (**a**) and final (**b**) location of coordinate systems $\{A\}$ and $\{B\}$

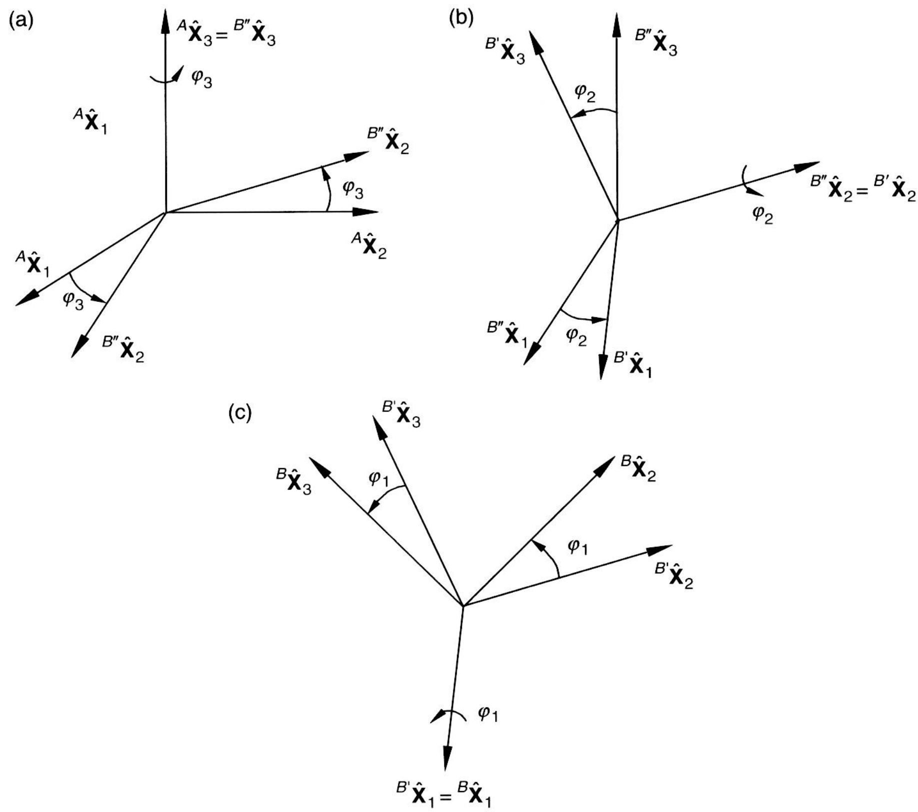

Fig. 2.4. Euler angles ZYX

$$
\begin{aligned}
{}^{A}_{B}\mathbf{R} &= {}^{A}_{B''}\mathbf{R}\; {}^{B''}_{B'}\mathbf{R}\; {}^{B'}_{B}\mathbf{R} = \mathbf{R}_3(\varphi_3)\,\mathbf{R}_2(\varphi_2)\,\mathbf{R}_1(\varphi_1) = \\
&= \begin{bmatrix} c\varphi_3 & -s\varphi_3 & 0 \\ s\varphi_3 & c\varphi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\varphi_2 & 0 & s\varphi_2 \\ 0 & 1 & 0 \\ -s\varphi_2 & 0 & c\varphi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi_1 & -s\varphi_1 \\ 0 & s\varphi_1 & c\varphi_1 \end{bmatrix} \\
&= \begin{bmatrix} c\varphi_3 c\varphi_2 & c\varphi_3 s\varphi_2 s\varphi_1 - s\varphi_3 c\varphi_1 & c\varphi_3 s\varphi_2 c\varphi_1 + s\varphi_3 s\varphi_1 \\ s\varphi_3 c\varphi_2 & s\varphi_3 s\varphi_2 s\varphi_1 + c\varphi_3 c\varphi_1 & s\varphi_3 s\varphi_2 c\varphi_1 - c\varphi_3 s\varphi_1 \\ -s\varphi_2 & c\varphi_2 s\varphi_1 & c\varphi_2 c\varphi_1 \end{bmatrix},
\end{aligned}
\tag{2.7}
$$

where $\mathrm{s}\varphi_i = \sin\varphi_i$, $\mathrm{c}\varphi_i = \cos\varphi_i$.

Angles $\varphi_3, \varphi_2, \varphi_1$ are called yaw, pitch and roll angles, respectively.

Matrix ${}^{A}_{B}\mathbf{R}$ defined by (2.7) can also be interpreted as the rotation operator. If the coordinates of point P with respect to coordinate system $\{B\}$

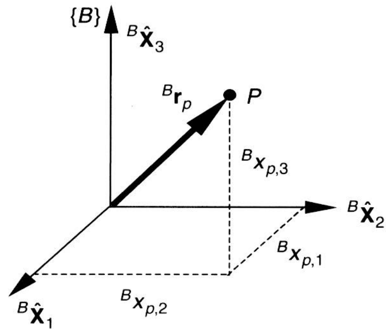

Fig. 2.5. Point P and its coordinates

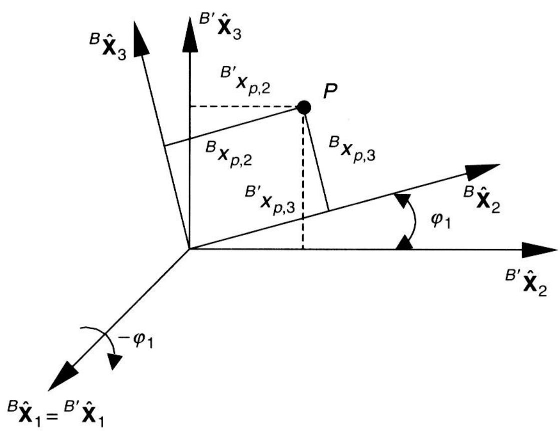

Fig. 2.6. Rotation by $-\varphi_1$

define vector $^B\mathbf{r}_P$ (Fig. 2.5), then the coordinates of this point with respect to $\{A\}$ can be calculated by carrying out successive rotations described in (2.7) as follows:

1. Rotation by $-\varphi_1$
   Coordinate systems $\{B\}$ and $\{B'\}$ obtained as a result of rotation by $-\varphi_1$ about axis $^B\hat{\mathbf{X}}_1$ are shown in Fig. 2.6.

   Coordinates of point P with respect to coordinate system $\{B'\}$ are as follows:

$$^{B'}x_{P,1} = {}^Bx_{P,1}, \tag{2.8a}$$

$$^{B'}x_{P,2} = {}^Bx_{P,2}c\varphi_1 - {}^Bx_{P,3}s\varphi_1, \tag{2.8b}$$

$$^{B'}x_{P,3} = {}^Bx_{P,2}s\varphi_1 + {}^Bx_{P,3}c\varphi_1, \tag{2.8c}$$

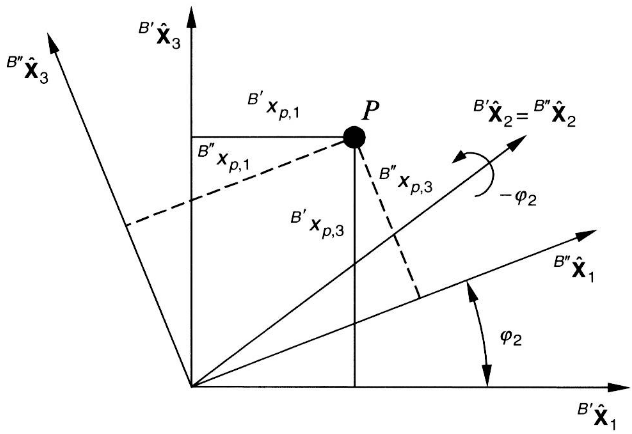

Fig. 2.7. Rotation by $-\varphi_2$

which can be written as:

$$^{B'}\mathbf{r}_P = \mathbf{R}_1(\varphi_1)^B\mathbf{r}_P. \tag{2.9}$$

2. Rotation by $-\varphi_2$
   Figure 2.7 presents coordinate systems $\{B'\}$ and $\{B''\}$ obtained by having rotated $\{B'\}$ by angle $-\varphi_2$ about axis $^{B'}\hat{\mathbf{X}}_2 = {}^{B''}\hat{\mathbf{X}}_2$.
   Coordinates of point P with respect to coordinate system $\{B''\}$ are defined by formulae:

$$^{B''}x_{P,1} = {}^{B'}x_{P,1}c\varphi_2 + {}^{B'}x_{P,3}s\varphi_2, \tag{2.10a}$$

$$^{B''}x_{P,2} = {}^{B'}x_{P,2}, \tag{2.10b}$$

$$^{B''}x_{P,3} = -{}^{B'}x_{P,1}s\varphi_2 + {}^{B'}x_{P,3}c\varphi_2, \tag{2.10c}$$

   or in the vector form:

$$^{B''}\mathbf{r}_P = \mathbf{R}_2(\varphi_2)^{B'}\mathbf{r}_P = \mathbf{R}_2(\varphi_2)\mathbf{R}_1(\varphi_1)^B\mathbf{r}_P. \tag{2.11}$$

3. Rotation by angle $-\varphi_3$
   Coordinate systems $\{B''\}$ and $\{A\}$ are shown in Fig. 2.8. Coordinate system $\{A\}$ is the result of rotation of $\{B''\}$ about axis $^{B''}\hat{\mathbf{X}}_3 = {}^{A}\hat{\mathbf{X}}_3$ by angle $-\varphi_3$.

   Coordinates of point P with respect to coordinate system $\{A\}$ are now as follows:

$$^{A}x_{P,1} = {}^{B''}x_{P,1}c\varphi_3 - {}^{B''}x_{P,2}s\varphi_3, \tag{2.12a}$$

$$^{A}x_{P,2} = {}^{B''}x_{P,1}s\varphi_3 + {}^{B''}x_{P,2}c\varphi_3, \tag{2.12b}$$

$$^{A}x_{P,3} = {}^{B''}x_{P,3} \tag{2.12c}$$