Differential and Integral Equations and Their Applications

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GROUP-THEORETIC METHODS IN MECHANICS AND APPLIED MATHEMATICS

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FOREWORD

Group theory, which in contemporary science has become a powerful tool for calculus, dates from the middle of the 19th century. The basic notions of group theory were clearly stated by Evariste Galois in connection with the solution of algebraic equations in radicals (1832). Somewhat later, in his fundamental work *Treatise on Permutations*, Camille Jordan not only developed the ideas of Galois, but indicated many other applications of group theory and obtained significant results in the theory itself (1870).

At first the groups studied were finite and discrete. The notion of the continuous group was introduced by Sophus Lie, a pupil of Jordan's, who tried to apply the methods of Galois to the problem of integrating differential equations in quadrature (1891). Lie understood continuous groups as transformation groups expressed by means of analytic (or infinitely differentiable) functions. Later such groups were called Lie groups, while the term *continuous group* has remained, but is now used for abstract groups.

Neither Sophus Lie nor his followers found any constructive methods for integrating differential equations in quadrature, and for a long time the theory was developed to solve purely algebraic problems, regardless of the objective that had initiated its appearance.

The first, and extremely illustrative, applications of Lie group theory pertained to relativistic mechanics (1905) and quantum mechanics (1924).

The group-theoretic analysis of the foundations of relativistic mechanics, carried out by Henri Poincaré, was a brilliant instance of the implementation of Felix Klein's Erlangen program (1872): a meaningful physical theory should be expressed only in terms of invariants of the appropriate group. Poincaré, having postulated that Lorentz transformations provide the connection between different inertial systems, established the group-theoretic character of these transformations, computed the invariants of this group and, using these invariants, found the law of motion of a relativistic particle (the analogue of Newton's second law).

It is also to Poincaré that we owe the first application of group-theoretic ideas to analytic mechanics. He found the generalization of the Lagrange equations by using the operator basis for the transitive group acting on the configuration space of the system (1901).

A special role in mechanics and physics is played by the group SO(3) of orientation-preserving orthogonal transformations, which is a subgroup of the Galilean group and, in addition, the configuration manifold of a rigid body in space with one fixed point. The parametrization of the kinematics of rotations, performed by Hamilton and Klein, based on group variables, allowed this parametrization to be simplified considerably as compared with the parametrization proposed by Euler in 1776. Thus, unlike the Euler equations, the kinematic equations of Poisson are linear. The new kinematics of rigid bodies later found successful applications in the navigation of moving objects.

It is interesting to note that Hamilton, when he "discovered" quaternions in 1843, immediately found applications for them to the kinematics of rigid bodies, long before the notions of the continuous group and of the covering spaces of groups appeared. This shows that not only do group-theoretic ideas serve as the basis for constructing realistic models in mechanics and physics, but the converse is also true: the construction of these models exhibits such a basis.

It is no accident that solutions of autonomous differential equations form a group, whereas those of nonautonomous ones do not. The presence of time on the right-hand side in explicit form indicates that the idealization involved in this model is quite crude. Another example in this field is the equivalence, discovered by J. J. Moreau in 1959, between the dynamics of an ideal liquid in a cavity and the dynamics of an infinite-dimensional rigid body.

It is easy to see that the applications of group theory to mechanics and physics described above were used to create models, derive differential equations, and analyze their properties, and so go far beyond Sophus Lie's narrower goal—integration of differential equations in quadrature. Actually, Lie himself undertook the group-theoretic analysis of partial differential equations (he found the symmetry group of the heat equation), although in such equations the presence of symmetries does not mean that the order of equations can be lowered. However, Lie apparently did not notice that the group invariance of partial differential equations makes it possible to decrease the number of independent variables, to construct particular solutions, and to classify boundary conditions.

In 1950, George Birkhoff pointed out the usefulness of group-theoretic analysis for the equations of hydrodynamics. These ideas were widely developed in the work of L. Ovsiannikov and his pupils.

As to the original goal (the integration of differential equations in quadrature), the first results after Lie were obtained with a certain delay. In 1918, Emmy Noether established the connection between the symmetries of Hamiltonian actions and the first integrals of the corresponding Lagrange equations.

Later on, many authors derived integrability (or the decrease of order) of Hamiltonian systems from the existence of symmetries for the Hamilton equations.

There are only two or three examples of completely integrable systems possessing solvable symmetry groups of the appropriate order, and they are all classical problems whose solutions by other methods were known long before group-theoretic methods appeared.

The only constructive applications of this theory were given by groups of linear transformations, whose existence as symmetries is usually easy to observe, and this makes it possible to lower the order of the system. However, on the one hand, this only requires a small part of the theory and, on the other, it is completely covered by dimensional and similarity considerations.

Here we should note that the problem of exact integration of differential equations, which was regarded as the main problem in the 19th century, is no longer considered so important today. Often the decrease of order by the use of first integrals spoils the system so much that, if complete integrability is not achieved, the procedure is useless. Sometimes it is advantageous to raise the order, whenever this improves the analytical properties of the system, making it more convenient for investigation, e.g., by using a computer.

Not so long ago a new domain of application of group-theoretic methods, in which they turned out to be extremely efficient, arose. This is the formalization of various procedures for finding approximate solutions of differential equations. Well-known asymptotic methods, e.g., the Krylov–Bogolyubov method, Poincaré's method of normal forms, the multiscale method, and others, widely used in celestial mechanics and in the theory of oscillations, have a significant drawback: as the number of approximation steps grows, the amount of computations increases catastrophically, so that the feasibility of high-order approximations lags behind the demand for them.

Still, the procedures involved in such calculations consist of various operations, and if one can discern an algebraic structure in the set of operations, or organize them so that they obey such a structure, then the existing algorithms can be significantly simplified.

Thus the following main domains of application of group-theoretic methods in contemporary science have emerged:

- Methods of exact integration of differential equations.
- Construction of paradigms, or models, possessing some group action.
- Various methods of qualitative analysis involving ideas of group theory in one way or the other.
- Group-theoretic formalizations of approximate solution methods for integrating differential equations.

We have listed these domains roughly in the same order as the appearance of their main results.

Here, when we speak of the applied character of various results, we should clarify what is meant by "applied." Thus, if as our starting point we take, say, the theory of continuous groups as presented by Pontryagin, then the group-theoretic analysis of differential equations as described by Ovsiannikov or Olver is an applied field. Nevertheless, its fundamental role in the theory of differential equations itself is indisputable.

It seems natural to call "applied" any result obtained by group-theoretic analysis if it pertains to a field that lies outside group theory or differential equations (e.g., mechanics or physics).

What are the goals of the present book, how does it differ from other monographs on group-theoretic methods in differential equations and their applications to physics?

First of all, it is obvious that the dominating topics in the literature devoted to applications of groups to physics are quantum mechanics, crystallography, and nuclear physics. The authors of the present monograph would like to shift the emphasis towards mechanics and applied mathematics: classical mechanics, relativistic mechanics, theory of oscillations, perturbation theory, and concrete problems of a purely mechanical type, e.g., problems involving dry friction and the like.

Nevertheless, in this connection we hardly touch on continuum mechanics, since this topic is presented in the literature nearly as fully as quantum mechanics or crystallography.

Secondly, the authors intend to present constructive methods only, i.e., only those where there is no place for statements of the type "if something or other exists, then this or that can be found," or if such a statement appears, then it is immediately followed by an explicit description of the premise.

The contents of the book may be roughly divided into three parts. The first is an introduction to Lie groups sufficient for understanding the specific features of the application of group-theoretic ideas in contemporary mechanics and applied mathematics.

The second part contains a large variety of mechanics problems to which group-theoretic methods can be successfully applied. This includes the analysis of the foundations of classical and relativistic mechanics, perturbation theory for configuration spaces of resonance systems, the kinematics of rigid bodies, elements of Hamiltonian mechanics, several examples of the integration of the equations of motion with the help of symmetry groups.

In the third part we present a group-theoretic formalization of asymptotic methods of applied mathematics and illustrate it by its application to problems of oscillation theory.

The book is intended for research scientists, engineers, and graduate and undergraduate students specializing in applied mathematics, mechanics and physics. Individual chapters may be used as the basis for specialized graduate courses in universities and colleges.

The only prerequisites are the standard introductory courses in advanced calculus and in differential equations.

We would like to express our deep gratitude to Alexei Zhurov who translated this book into English and made a lot of useful comments. We also appreciate the help of Alexei Sossinsky who translated the Foreword.

Dmitry M. Klimov
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Professor Zhuravlev is an author of over 150 scientific publications, four monographs, and a textbook "Fundamentals of Theoretical Mechanics."

CONTENTS

Chapter 1

Basic Notions of Lie Group Theory

1.1. Notion of Group

Let the symbol G denote a set of elements of arbitrary nature (e.g., a set of numbers, functions, or some objects of geometric nature, etc.).

The set G is said to be a *group* if:

(1) An operation is defined over the set G which assigns a unique element $C \in G$ to any two elements $A \in G$ and $B \in G$ taken in a certain order. This operation is called multiplication (or composition) and written symbolically as

$$A \circ B = C.$$

In general, this operation is noncommutative, i.e., $A \circ B \neq B \circ A$, which defines the order of taking the elements in the product.

(2) There exists an element $E \in G$ such that

$$A \circ E = E \circ A = A$$

for any $A \in G$. This element is called the *identity element* or *unit* of the group.

(3) For any $A \in G$ there exists an element A^{-1} such that

$$A \circ A^{-1} = A^{-1} \circ A = E.$$

This element is called the *inverse* of A.

(4) The associativity of multiplication holds, i.e.,

$$A \circ (B \circ C) = (A \circ B) \circ C$$

for any A, B, and C of G.

The above four conditions define the notion of an abstract group, i.e., a group in which the nature of its constituent elements is of no importance.

Condition (1) is referred to as the condition of closure of the set G under the operation $\circ$ thus defined on this set.

If there exists a subset $H \subset G$ closed under the same operation, then such a subset is called a *subgroup*.

If the operation introduced in the group G is commutative, i.e., $A \circ B = B \circ A$ for any elements A and B of G, then the group is called *commutative*, or *abelian*.

Let us demonstrate that in any group the unit is unique. We proceed by reductio ad absurdum. Suppose there exists an $E_1 \neq E$ such that

$$A \circ E_1 = E_1 \circ A = A.$$

Multiply this equation by A^{-1}. We have

$$A^{-1} \circ A \circ E_1 = A^{-1} \circ A \quad \text{and} \quad E_1 \circ A \circ A^{-1} = A \circ A^{-1}.$$

Whence, with reference to conditions (3) and (4), we have $E_1 = E$.

The uniqueness of the inverse elements can be proved in much the same simple manner.

Exercise. Show that in the definition of group it suffices to require the existence of a *right* unit and a right inverse, i.e., $A \circ E = A$ and $A \circ A^{-1} = E$, since this will imply the existence of a left unit and its identity with the right unit, as well as the existence of a left inverse and its identity with the right one.

Examples of groups.

1. The set of all real numbers. The group operation is addition. The role of the unit is played by zero, and the inverse of an element is the negative of the element. The set of all rational numbers is a subgroup. Other subgroups include the set of all integers and the set of all even numbers. The set of all odd numbers is not a subgroup.

2. The set of reals is a group under the operation of arithmetical multiplication, provided that the number zero is excluded.

3. Any finite number of elements with the operation defined by the Cayley table (an example of 5 elements is given):

$\circ$	A	B	C	D	E
A	B	C	D	E	A
B	C	D	E	A	B
C	D	E	A	B	C
D	E	A	B	C	D
E	A	B	C	D	E

For example, $C \circ D = B$. The role of the unit is played by E.

4. The set of points lying on a circle. The operation is defined as follows. Let the position of a point A relative to some reference point be defined by the angle φ_A and let the position of a point B be defined by the angle φ_B. To the points A and B the operation assigns the point C whose position is defined by the angle $\varphi_C = \varphi_A + \varphi_B$.

5. The set of $n \times n$ matrices with determinants other than zero. The group operation is matrix multiplication.

Below are some examples of sets which are not groups under the operations introduced over them.

1. The set of integers with multiplication operation. Although the product of two integers is an integer and there exists a unit, none element has its inverse, except for the number one.

2. The set of vectors in the three-dimensional space with the operation of outer product. Except for the first condition in the definition of a group, no other conditions are satisfied.

The groups in examples 1 through 4 are commutative, or abelian, groups. Example 5 presents a noncommutative group.

1.2. Lie Group. Examples

One may easily see that in examples 1, 2, 4, and 5 (see Section 1.1) it is possible to introduce, independently of the group axioms, a notion of closeness between any two elements for the corresponding sets of elements. By virtue of this, the group operations turn out to be continuous functions, which makes it possible to treat such groups in two aspects, from the viewpoint of algebra and from the viewpoint of analysis.

This unification turns out to be quite fruitful. This is what the theory of Lie groups utilizes rather essentially.

Presently, by the term "Lie group" a wider object is understood compared with that introduced by Lie himself. In what follows, we consider this wider object.

By G we mean a set of transformations of an n-dimension real arithmetic space into itself,

$$q' = Q(q, a) \qquad (q \in \mathbb{R}^n),$$

on which an operation satisfying the group axioms is defined. The set of transformations is numbered by a parameter a. If a is real, then the set of transformations is said to be one-parameter. If a is a k-vector, $a \in \mathbb{R}^k$, the set of transformations is k-parameter. In this case, all $a_1, \ldots, a_k$ must be essential, that is, irreducible to fewer parameters by a transformation of the parameters.

The operation introduced over the set G is the composition of two transformations. For example, suppose the transformation $q \to q'$ with a fixed value of a is followed by a transformation $q' \to q''$ with a different value of a, denoted b:

$$q'' = Q(q', b).$$

If the composition

$$q'' = Q\big(Q(q, a), b\big)$$

defining the transformation $q \to q''$ is a transformation from the same set (corresponding to some other value c of the parameter), i.e.,

$$q'' = Q\big(Q(q, a), b\big) = Q(q, c),$$

then this means that an operation is defined on the set of transformations in question (the composition of two transformations from the set does not go beyond the limits of the set).

The transformations are usually assumed to be defined on some open set from $\mathbb{R}^n$ and in a sufficiently small neighborhood of the point a. This assumption is made to avoid discussions about the domain of definition of the functions $q' = Q(q, a)$ in both the variable q and the parameter a.

Thereby the function $q' = Q(q, a)$ determines a local family of local transformations.

DEFINITION. *A set of transformations $q' = Q(q, a)$ is called a local Lie group of transformations (from now on, a Lie group) if:*

1. *the composition of any two transformations from this set is again a transformation from the same set (i.e., the set of transformations is closed with respect to the composition);*
2. *the identity transformation, which plays the role of the group unit, belongs to the set of transformations in question;*
3. *for any transformation from the set there exists its inverse, which belongs to the same set, so that the composition of the two transformations yields the identity transformation;*
4. *the function $q' = Q(q, a)$ is analytic in q and a in some open set of variation of q and in some neighborhood of the unit element, e, for the variable a.*

Note that the requirement that the operation be associative, which is used in the abstract definition of a group, is unnecessary here, since the composition of transformations obviously satisfies this property.

Group operations. The composition of the transformations $q' = Q(q, a)$ and $q'' = Q(q', b)$ determines the transformation $q'' = Q(q, c)$ in which the parameter c is related to a and b by

$$c = \gamma(a, b).$$

It is this function, analytic in a neighborhood of the unit ($a = e$ and $b = e$), which represents an expression for the group operation.

By using the notion of group operation, one can rewrite the definition of a Lie group more concisely as follows.

A set of transformations $q \to q' : q' = Q(q, a)$ is called a Lie group if:
(1) $Q\big(Q(q, a), b\big) = Q\big(q, \gamma(q, a)\big) \quad \forall\, a, b;$
(2) $\exists\, a = e: \quad \forall\, b \quad \gamma(e, b) = \gamma(b, e) = b;$
(3) $\forall\, a \quad \exists\, b = a^{-1}: \quad \gamma(a, a^{-1}) = \gamma(a^{-1}, a) = e;$ and
(4) all the functions involved in the definition are analytic.

Most important examples of Lie groups.

1. *Group of translations*: $q' = q + a$. Here the dimensionality of the parameter a is the same as that of the variable q.

2. *Group of extensions*: $q' = a_i q_i$ $(i = 1, \dots, n)$. If all $a_i = a$, where a is a scalar quantity, then this group is called a *similarity group*.

3. *Group of orthogonal transformations*: $q'_i = \sum_{j=1}^n a_{ij} q_j$ $(i = 1, \dots, n)$, where $A = \{a_{ij}\}$ is an orthogonal matrix, $A^{\mathrm{T}} = A^{-1}$. This group is conventionally denoted $\mathrm{O}(n)$. The most important subgroup of this group is the subset of transformations with $\det A = 1$ (in general, $\det A = \pm 1$). It is conventional to denote this subgroup by $\mathrm{SO}(n)$ and call it a *group of rotations*.

4. *Group of linear transformations*: $q'_i = \sum_{j=1}^n a_{ij} q_j$ $(i = 1, \dots, n)$ with $\det A \neq 0$. This group is designated as $\mathrm{GL}(n)$, which stands for a *general linear group* in $\mathbb{R}^n$. If we require additionally that $\det A = 1$, then we obtain a volume-preserving subgroup of transformations. This subgroup is called a *special linear group*, or a *unimodular group*, and denoted by $\mathrm{SL}(n)$.

5. *Group of motions*: $q' = a + Aq$, where Aq stands for $\sum_{j=1}^n a_{ij} q_j$ $(i = 1, \dots, n)$ for brevity, and A is an orthogonal matrix with $\det A = 1$. In particular, this group contains subgroups such as the group of translations and the group of rotations.

6. *Affine group*: $q' = a + Aq$, where $\det A \neq 0$. The group of motions is a subgroup of the affine group.

7. *Projective group*:

$$q'_i = \frac{\sum_{j=1}^n a_{ij} q_j + b_i}{\sum_{j=1}^n a_j q_j + b} \quad \text{with} \quad \det\begin{pmatrix} a_{ij} & b_i \\ a_j & b \end{pmatrix} \neq 0 \qquad (i, j = 1, \dots, n).$$

In the matrix under the determinant sign, the matrix $A = \{a_{ij}\}$ is extended by a column of b_i and a row of a_j, with the last diagonal entry being b.

8. *Volume-preserving group of transformations*:

$$\det\left\{\frac{\partial q'_i}{\partial q_j}\right\} \equiv 1 \qquad (i, j = 1, \dots, n).$$

Below are two groups which are of particular importance in mechanics and physics.

9. *Galilean group*:

$$t' = t + t_0, \quad r' = r + r_0 + vt + Ar \qquad (A^{\mathrm{T}} = A^{-1}),$$

where notation customary in mechanics is used. The Galilean group involves ten parameters: t_0, three components of the translation vector r_0 of the origin, three components of the velocity v, and three independent parameters of the rotation matrix A.

10. *Lorentz group*:

$$t' = \frac{t - vx}{\sqrt{1-v^2}}, \quad x' = \frac{x - vt}{\sqrt{1-v^2}},$$

where the x-axis of the fixed reference frame is directed along the velocity v at which the origin of the primed reference frame moves.

Let us find the group operation of the Lorentz group. We have

$$t' = \frac{t - v_1 x}{\sqrt{1-v_1^2}}, \quad x' = \frac{x - v_1 t}{\sqrt{1-v_1^2}};$$

$$t'' = \frac{t' - v_2 x'}{\sqrt{1-v_2^2}} = \frac{t - (v_1+v_2)x/(1+v_1v_2)}{\sqrt{1-[(v_1+v_2)/(1+v_1v_2)]^2}},$$

$$x'' = \frac{x' - v_2 t'}{\sqrt{1-v_2^2}} = \frac{x - (v_1+v_2)t/(1+v_1v_2)}{\sqrt{1-[(v_1+v_2)/(1+v_1v_2)]^2}}.$$

Thus the group operation has the form

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

It represents the law of addition of velocities in relativistic mechanics.

1.3. Group Generator. Lie Algebra

Let the group parameter a be scalar. We transform the parameter in accordance with the formula $a \to \mu : a = e + \mu$. Then the value $\mu = 0$ corresponds to the identity transformation, i.e., the unit of the group.

Let μ be the parameter occurring in the equation of the group, $q' = Q(q, \mu)$. Expand this expression in a series in powers of μ about the point $\mu = 0$:

$$q' = q + \mu\eta(q) + \cdots .$$

The linear part of the group represented by the first two terms of this expansion is called the *germ of the group*. Let a scalar function $F(q)$ be specified in the space of the variables q. The group transformation $q \to q'$ takes the function $F(q)$ to the form

$$F(q) \to F(q') = F\big(q + \mu\eta(q) + \cdots\big) = F(q) + \mu\eta(q)\frac{dF}{dq} + \cdots .$$

The part of the increment of $F(q')$ linear in μ is given by

$$\Delta F(q') = \mu\eta(q)\frac{dF}{dq} \equiv \mu \sum_{i=1}^{n} \eta_i(q)\frac{\partial F}{\partial q_i} \equiv \mu U F,$$

where U is a linear differential operator of first order,

$$U = \eta_1(q)\frac{\partial}{\partial q_1} + \cdots + \eta_n(q)\frac{\partial}{\partial q_n},$$

This operator is referred to as the *infinitesimal generator of the group*, also called the generator or operator of the group.

If the group is multiparameter, then the above procedure can be carried out with respect to each of the parameters. Thus, a multiparameter group has as many infinitesimal generators as the number of independent parameters.

Below we present expressions of the infinitesimal generators corresponding to the Lie groups listed in the above examples.

1. Group of translations:

$$U_i = \frac{\partial}{\partial q_i} \qquad (i = 1, \ldots, n).$$

2. Group of extensions:

$$U_i = q_i \frac{\partial}{\partial q_i} \qquad (i = 1, \ldots, n);$$
$$U = q_1 \frac{\partial}{\partial q_1} + \cdots + q_n \frac{\partial}{\partial q_n} \qquad \text{(similarity group)}.$$

3. Group of rotations. Consider a small neighborhood of the identity matrix E in the set of orthogonal matrices:

$$A = E + \mu N,$$

where μ is a small scalar parameter. Neglecting the terms of the second order of smallness, we find from the condition of orthogonality of the matrix A,

$$(E + \mu N)^{\mathrm{T}}(E + \mu N) = E,$$

that $N^{\mathrm{T}} = -N$. Thus, a small variation of an orthogonal matrix is a skew-symmetric matrix which is what determines $\frac{1}{2}n(n-1)$ independent parameters of an orthogonal group: $\mu_{ij} = \mu n_{ij}$, where $\{n_{ij}\} = N$ and $n_{ij} = -n_{ji}$. The germ of an orthogonal group has the form

$$q_i' = q_i + \sum_{j=1}^{n} \mu_{ij} q_j;$$

to each parameter μ_{ij} there corresponds the generator

$$U_{ij} = q_j \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial q_j} \qquad (i > j = 1, \ldots, n).$$

4. Group of linear transformations. The variation of the nondegenerate matrix A in a neighborhood of the identity matrix is given by $A = E + M$, where $M = \{\mu_{ij}\}$ is a matrix with small independent entries μ_{ij}. The number of independent generators is equal to n^2. These are

$$U_{ij} = q_j \frac{\partial}{\partial q_i} \qquad (i, j = 1 \ldots, n).$$

5. The generators of the group of motions involve the generators of the translation group and those of the rotation group.

6. The generators of the affine group involve the generators of the groups of translations and linear transformations.

7. Projective group. Introduce notation for the parameters μ so that they vanish for the identity transformation:

$$\{a_{ij}\} = E + \{\mu_{ij}\}, \quad b_i = \mu_i, \quad a_j = \mu_j, \quad b = 1 + \mu, \qquad (i, j = 1, \ldots, n).$$