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CONTROL THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS: CONTINUOUS AND APPROXIMATION THEORIES

II

ABSTRACT HYPERBOLIC-LIKE SYSTEMS OVER A FINITE TIME HORIZON

IRENA LASIECKA
ROBERTO TRIGGIANI

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Control Theory for Partial Differential Equations: Continuous and Approximation Theories

This is the second volume of a comprehensive and up-to-date three-volume treatment of quadratic optimal control theory for partial differential equations over a finite or infinite time horizon and related differential (integral) and algebraic Riccati equations. Both continuous theory and numerical approximation theory are included. An abstract space, operator theoretic treatment is provided, which is based on semigroup methods, and which is unifying across a few basic classes of evolution. A key feature of this treatise is the wealth of concrete multi-dimensional PDE illustrations, which fit naturally into the abstract theory, with no artificial assumptions imposed, at both the continuous and numerical level.

Throughout these volumes, emphasis is placed on unbounded control operators or on unbounded observation operators as they arise in the context of various abstract frameworks that are motivated by partial differential equations with boundary/point control. Relevant classes of PDEs include: parabolic or parabolic-like equations, hyperbolic and Petrowski-type equations (such as plate equations and the Schrödinger equation), and hybrid systems of coupled PDEs of the type that arise in modern thermo-elastic and smart material applications. Purely PDE dynamical properties are critical in motivating the various abstract settings and in applying the corresponding theories to concrete PDEs arising in mathematical physics and in other recent technological applications.

Volume II, after an introductory chapter that collects relevant abstract settings and properties of hyperbolic-like dynamics, is focused on the optimal control problem over a finite time interval for such dynamical systems. A few abstract models are considered, each motivated by a particular canonical hyperbolic dynamics. Virtually all the regularity theory needed in the illustrations is provided in detail, including second-order hyperbolic equations with Dirichlet boundary controls, plate equations (hyperbolic and not) and the Schrödinger equation under a variety of boundary controls or point controls, and structural acoustic models that couple two hyperbolic equations.

Volume I covers the abstract parabolic theory for both the finite and infinite horizon optimal control problems, as well as the corresponding min–max theory, with PDE illustrations. Recently discovered, critical dynamical properties are provided in detail, many of which appear here in print for the first time.

Volume III is in preparation.

Irena Lasiecka is Professor of Mathematics at the University of Virginia, Charlottesville. She has held positions at the Control Theory Institute of the Polish Academy of Sciences, the University of California, Los Angeles, and the University of Florida, Gainesville. She has authored or coauthored over 150 research papers and one other book in the area of linear and nonlinear PDEs. She serves on the editorial boards of *Applied Mathematics and Optimization*, *Journal of Mathematical Analysis and Applications*, and the *IEEE Transactions on Automatic Control*, among others, and she holds, or has held, numerous offices in the professional societies SIAM, IFIP, and the AMS.

Roberto Triggiani is Professor of Mathematics at the University of Virginia at Charlottesville. He has held regular and visiting academic positions at several institutions in the United States and Europe, including Iowa State University, Ames, and the University of Florida, Gainesville. He has authored or coauthored over 140 research papers and one other book in PDEs and their control theoretic properties. He currently serves on the editorial boards of *Applied Mathematics and Optimization, Abstract and Applied Analysis, Systems and Control Letters*.

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To Maria Ugenti, Janina Krzeminska, Antoni Lech,
and Giuseppe Triggiani

Preface

This three-volume treatise presents, in a unified framework, a comprehensive, in-depth, and up-to-date treatment of quadratic optimal control theory for (linear) partial differential equations (PDEs) over a finite or infinite time horizon and related differential (integral) and algebraic Riccati equations. Both continuous theory and numerical approximation theory are included. An abstract space, operator theoretic treatment is provided, which is based on semigroup methods, and which is unifying across a few basic classes of evolution.

While addressing all three volumes regarding the basic, broad-range theme covered and the philosophy of approach followed, this preface focuses mostly on Volumes I and II for specific details. Indeed, driven also by recent, new PDE models such as they arise in modern technological applications, the treatment of this work has grown far beyond the original intentions and the anticipated plan. As a result, two volumes now appear in print, with a third one in preparation. A justification for the criteria that have dictated the selection of a natural subdivision of the entire work into three volumes is given below.

This treatise is a much expanded outgrowth, at least in the ratio 1 to 10, of the authors' Springer-Verlag Lectures Notes in Control and Information Sciences, Volume 164, entitled: *Differential and Algebraic Riccati Equations with Applications to Boundary-Point Control Problems: Continuous and Approximation Theory*. TheseLecture Notes, published in 1991, contained a comprehensive account of the theories that were available at that time, along with an array of numerous illustrative PDE applications with boundary/point control. However, most technical proofs were referred to the literature. A completion of these Lectures Notes was therefore called for, which inevitably stimulated an extension of their range of coverage with the addition of both new theoretical topics, as well as new PDE models and applications of modern technological origin. These, in turn, required further still theoretical analysis.

The basic dynamics is an abstract equation $\dot{y} = Ay + Bu$, where A (free dynamic operator) is the generator of a strongly continuous (s.c.) semigroup on the Hilbert (state) space Y, and where B (control operator) is an unbounded operator with a degree of unboundedness up to the degree of unboundedness of A. Moreover, u is the control function, which runs over the class of L_2-functions in time, with values in a Hilbert space U. All the boundary/point control problems for PDEs can be modeled by this abstract equation, for specific choices of the operators A, B and of the spaces U, Y. The dynamics is further penalized by a (quadratic) functional cost, containing an observation operator R, to be minimized over a preassigned finite or infinite time horizon. The theory of this problem culminates with the analysis of the corresponding differential or algebraic Riccati (operator) equations, which arise in the (pointwise) feedback synthesis of the optimal solution pair $\{u^0, y^0\}$. This problem, which originated in the late 1950s in the context of ordinary differential equations (with A, B, R matrices of appropriate size) has long been considered a truly central issue – a "battle-horse" – in deterministic optimal control theory, and related stochastic filtering theory, of dynamical systems. In the finite dimensional context, the solution in pointwise feedback form, via Riccati equations, of both the deterministic and the stochastic versions of this problem, has been known since the 1960s, through the work of Kalman and Kalman-Bucy, respectively.

These volumes present the far-reaching, technical extension of the deterministic problem, aimed at accommodating and encompassing multidimensional PDEs with boundary/point control and/or observation, in a natural way. Thus, throughout this work, emphasis is placed on unbounded control operators and/or, possibly, on unbounded observation operators as well, as they arise in the context of various abstract frameworks that are motivated by, and ultimately directed to, PDEs with boundary/point control and observation. A key feature of the entire treatise is then a wealth throughout of concrete, multidimensional PDE illustrations, which naturally fit into the abstract theory, with no artificial assumptions imposed, at both the continuous and numerical level. Justification of the abstract models adopted rests, unequivocally, with their intrinsic ability of capturing the characterizing dynamical properties of specific, relevant classes of PDEs, which motivate them in the first place. Regarding abstract modeling, the flow runs unmistakenly from an understanding of the concrete into the proper abstract.

Naturally, to extract best possible results and tune the technical tools to the problem at hand, it is necessary to distinguish at the outset between different types of PDE classes: primarily, parabolic-like dynamics versus hyperbolic-like dynamics, with further subdistinctions in the latter class. This is due to well-known, intrinsically different dynamical properties between these two classes. As a consequence, they lead to two drastically different basic abstract models, whose defining, characterizing features set them apart. Accordingly, these two abstract models need, therefore, to be investigated by correspondingly different technical strategies and tools. As a consequence, different types of distinctive results are achieved to characterize the two classes. All this dictates that the abstract theory needs to bifurcate at the very outset into a parabolic-like model and hyperbolic-like basic models; moreover, in the latter class, a further distinction into finite and infinite time horizon is called for, to account for different, critical properties between these two cases.

Thus, Volume I contains the optimal control theory for the parabolic-like class, over both the finite and the infinite time horizon, where the s.c. semigroup generated by A is, moreover, analytic; while Volumes II and III refer to the optimal control theory for the hyperbolic-like class over a finite or, respectively, infinite time horizon. This includes hyperbolic dynamics as well as Petrowski-type PDEs such as platelike models, Schrödinger equation, etc.

As already emphasized, purely PDE dynamical properties are critical in motivating the various abstract settings, as well as in applying the corresponding theories to concrete PDEs arising in mathematical physics and in other technological endeavors. This is particularly true in the case of hyperbolic-like dynamics. Unlike the parabolic-like class, which offers a certain degree of flexibility in the choice of the abstract space setting (subject to established parabolic regularity theory), by contrast, the framework in the case of hyperbolic-like dynamics is far more rigid. It requires a preliminary knowledge of the space of optimal regularity theory – apurely PDEs problem – and thus leaves no choice. Moreover, regarding the infinite time optimal control problem, the most complete theory is achieved in the cases (which occur most often, but by no means always) where the space of optimal regularity of the solution under L_2-control coincides with the space of exact controllability (or of uniform stabilization) – in other words, where the map from the class of admissible L_2-controls to the state space is surjective at some finite time. In short: In the hyperbolic-like case, optimal regularity theory is an intrinsic, critical, essential prerequisite factor in the analysis of the corresponding optimal control problem, which rigidly depends upon it (while a margin of latitude exists in the parabolic-like case, once parabolicity has been established). Accordingly, optimal regularity theory of many hyperbolic-like dynamical equations considered in the illustrations is an intrinsic part of the present volumes. A more detailed description is given below in the synopsis of Volume II. The inclusion, on the one hand, of this massive regularity theory and, on the other hand, of new PDE dynamics such as thermo-elastic plate equations and various models of coupled PDEs arising in structural acoustics, helps explain the explosion of this subject matter into three volumes.

Throughout this work, special emphasis is paid to the following topics:

(i) Abstract operator models for boundary/point control and observation problems for PDEs.

(ii) Identification of the space of optimal regularity of the solutions, typically under L_2-controls in time, and particularly for the class of hyperbolic and Petrowski-type systems or coupled PDEs problems; it is with respect to the norm of this space that the solution is then penalized in the cost functional.

(iii) Identification of the regularity properties of the optimal pair of the optimal control problem, particularly, in the parabolic-like case over a finite or infinite interval, and in the hyperbolic-like case over a finite interval. In the hyperbolic-like case over an infinite time horizon, the optimal pair need not be better than the original L_2 regularity in time, inherited from the optimization problem.

(iv) Verification of what we call the "finite cost condition" (F.C.C.) in the infinite time horizon problem and related algebraic Riccati equations, which guarantees the existence of at least one admissible control yielding a finite cost functional. In the case of parabolic-like dynamics, the F.C.C. is most readily verified via uniform feedback stabilization, as the unstable space of the dynamics is, at most, finite dimensional. By contrast, in the case of hyperbolic-like dynamics, the F.C.C. is verified via a study of the related exact controllability problem, or of the related (generally, more challenging) uniform stabilization problem, by means of an explicit, dissipative, boundary, velocity feedback operator. Exact controllability/uniform stabilization of hyperbolic-like dynamics is a topic in its own right, intimately connected with, yet distinct from, the main thrust of the optimal control problem of the present volumes. A vast literature exists, including treatments in book form. We shall return to these topics in Volume III.

(v) Constructive variational approach to the issue of existence of a solution (Riccati operator), and possibly uniqueness, of a corresponding differential or algebraic Riccati operator equation.

(vi) Development of numerical algorithms that reproduce numerically the key properties of the continuous problems. This can be done directly in the parabolic-like case. By contrast, the hyperbolic-like (conservative) case requires that a regularization procedure be performed first, before passing to the approximation analysis.

A brief description of the contents of the first two volumes follows.

Volume I focuses on abstract parabolic systems (continuous and approximation theory), where the s.c. semigroup of the free dynamics is, moreover, analytic. Save perhaps for some possible refinements, the overall theory in this chapter and companion notes is essentially optimal. This includes both the finite (Chapter 1) and infinite horizon (Chapter 2) optimal control problems, as well as the corresponding min–max theory with nondefinite quadratic cost (Chapter 6). Here, both control operator and disturbance operator are of the same "maximal" degree of unboundedness allowed with respect to the free dynamics operator. A lengthy Chapter 3 presents many multidimensional PDE illustrations with boundary/point control and observation. They include not only traditional, classical parabolic equations such as the heat equation with Dirichlet- or Neumann-boundary control, or point control, but also second-order equations with "structural" or "high" damping, as well as thermo-elastic plate equations with no rotational inertia term. For the latter two classes, recently discovered, critical dynamical properties are proved in details. These include "parabolicity" (analyticity of the corresponding semigroup) and uniform stability. Various appendices in Chapter 3, taken cumulatively, provide a self-contained subvolume focused on thermo-elastic parabolic plate equations, whose theory has become available only over the past year or so. Chapter 4 provides a detailed numerical approximation treatment, with appropriate convergence properties (possibly, with rates of convergence) of all the quantities of interest: optimal control, optimal solution, Riccati operator, gain operator, optimal cost, etc. Finally, Chapter 5 provides detailed PDE illustrations of numerical schemes that fit into the theory of Chapter 4. Regarding the theoretical treatment, the analysis in Volume I is almost exclusively operator-theoretic and is based on singular integrals as they arise in the description of the control-solution (state) map, by virtue of the key property of analyticity of the free dynamics semigroup (generated by the operator A). As it turns out, analyticity of the free dynamics compensates, in this case, for the unboundedness of the control operator or of the disturbance operator. Indeed, such analyticity yields a controlled smoothing of the control-solution map and of its adjoint. Once applied to the optimality conditions characterizing the optimal pair, such double smoothing snowballs into a bootstrap argument, which eventually leads to higher regularity of the optimal pair (over the initial regularity inherited from the optimization problem) and – finally – to a smoothing property of the Riccati operator. As a consequence, the gain operator is bounded from the state to the control space, a distinctive, critical property of the parabolic-like class. In applications to concrete PDEs, elliptic theory and identification of domains of appropriate fractional powers with Sobolev spaces play a critical role.

Volume II considers the optimal control theory for hyperbolic or Petrowski-type PDEs over a finite time horizon. It begins with an introductory chapter (Chapter 7) that collects relevant abstract settings and abstract properties of these dynamics that are to be used in subsequent chapters. It then considers three different abstract frameworks. The abstract model of Chapter 8 is motivated by the optimal control problem for second-order hyperbolic equations with Neumann-boundary control and Dirichlet-trace observation. The abstract model of Chapter 9 is motivated by wave and Kirchoff elastic plate equations, under the action of point control. It also includes two models of coupled PDE systems, such as they arise in noise reduction problems in structural acoustics. Both systems are subject to point control, which models the action of smart material technology. One couples the wave equation for the pressure in the acoustic chamber with a Kirchoff equation for the elastic displacement of the moving wall. It is an example of hyperbolic/hyperbolic coupling. Instead, in the second system, the elastic wall is modeled by an Euler–Bernoulli equation with structural damping, thus giving rise to a hyperbolic/parabolic coupling. Finally, the abstract model of Chapter 10, which further builds on that of Chapter 9, looks at first artificial and complicated. Actually, it is a natural framework, which simply extracts the correct settings for problems such as second-order hyperbolic equations with Dirichlet-boundary control, numerous other plate equations with a variety of boundary control, as well as the Schrödinger equation with Dirichlet-boundary control. All the relevant regularity theory, some of which is new, of these dynamical PDEs is provided in detail, subject to the exclusions noted below. Indeed, in contrast with parabolic theory, the regularity theory of hyperbolic and Petrowski-type equations (such as plate equations and Schrödinger equations) demand a broader array of purely PDE techniques to obtain sharp/optimal interior- and trace-regularity properties. They include energy methods, or multipliers methods, at the differential level or pseudo-differential/microlocal analysis level, which were discovered much more recently than parabolic techniques. This contrast between the two basic classes of dynamical systems – parabolic-like versus hyperbolic- or Petrowski-type equations – was already emphasized in the preface to the authors' Lectures Notes. Accordingly, Volume II contains in detail most of the needed regularity theory (both interior and trace regularity) of the many hyperbolic-like PDE systems here considered. Exceptions include the more recent regularity theory of first-order hyperbolic systems and of second-order hyperbolic equations with Neumann boundary datum, which require a treatment based on the technical apparatus of pseudo-differential operators and microlocal analysis. For these, appropriate references to the recent literature are given.

As already noted, Volume III (in preparation) will cover optimal control problems for hyperbolic-like dynamics (both continuous and numerical approximation theory) and for coupled PDE systems, over an infinite time horizon.

Further information on this treatise in the context of available books is contained in the introductory section of Chapter 0.

Acknowledgments for the First Two Volumes

Though almost all of the results presented in the first two volumes are taken from the authors' original research work in both optimal control theory and PDE theory, there are a numbers of friends whom we wish to thank very warmly, and to whom we wish to express our gratitude. In chronological order, first A.V. Balakrishnan, who, with his original work of the mid-1970s, provided the initial spur into abstract modeling for parabolic equations with Dirichlet boundary control, and who soon thereafter graciously introduced us to it. Next, we wish to thank some of our coauthors, from whom and with whom we have learnt about this fascinating subject, for joint work reported in these first two volumes: G. Da Prato on the abstract differential Riccati equation in the hyperbolic case, via a direct approach, and J. L. Lions on the regularity of second-order hyperbolic equations with Dirichlet-boundary datum. Our frequent visits over the years to the mathematically rewarding environment of the Scuola Normale Superiore di Pisa, Italy, and the stimulating exchange of correspondence with J. L. Lions on a priori inequalities of wave equations are also greatly appreciated. Special thanks for reading portions of the manuscript and offering their comments are due to P. Acquistapace (Chapter 1), A. Ichikawa (Chapter 6), S. K. Chang (Chapter 10) and, particularly, to L. Pandolfi, for his insight on the entire first draft.

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