CONTEMPORARY MATHEMATICS

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Rings, Modules and Representations

International Conference on Rings and Things
in Honor of Carl Faith and Barbara Osofsky
June 15–17, 2007
Ohio University-Zanesville
Zanesville, OH

Nguyen Viet Dung
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2000 *Mathematics Subject Classification.* Primary 13–XX, 16–XX, 18–XX.

Library of Congress Cataloging-in-Publication Data

International Conference on Rings and Things in Honor of Carl Faith and Barbara Osofsky (2007 : Ohio University-Zanesville)

Rings, modules, and representations : International Conference on Rings and Things in Honor of Carl Faith and Barbara Osofsky, June 15–17, 2007, Ohio University-Zanesville / Nguyen Viet Dung... [et al.], editors.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 480)

Includes bibliographical references.

ISBN 978-0-8218-4370-3 (alk. paper)

1. Associative rings. 2. Modules (Algebra) 3. Representations of algebras. I. Faith, Carl Clifton, 1927– II. Osofsky, Barbara. III. Nguyen, Viet Dung.

QA251.5.I63 2009
512′.46—dc22 2008039856

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Rings, Modules and Representations

Dedicated to
Carl Faith and Barbara Osofsky

Preface

Continuing the tradition of algebra conferences hosted by Ohio University, the summer of 2007 saw the addition of yet another chapter - an international conference on rings and modules. The conference, attended by over seventy mathematicians of international repute from more than twenty countries, provided an excellent opportunity for the experts in Theory of Rings and Modules and other related topics to exchange ideas and discuss new developments in these rapidly growing areas of research. The conference also stimulated new exciting collaborations among the participants.

The conference, named *International Conference on Rings and Things*, is a reference to Carl Faith's well-known book *Rings and Things.* The conference was hosted by the Zanesville campus of Ohio University and was held in honor of Carl Faith's 80th birthday and Barbara Osofsky's 70th birthday. It was a pleasure to host the conference and then present these Proceedings dedicated to such outstanding algebraists. Their work through the years speaks for itself and is certainly of the highest quality.

This volume represents some of the recent work of the invited speakers and other participants of the conference. It is our hope that the articles presented in this volume will be an important source of inspiration for the researchers interested in Theory of Rings and Modules, Representation Theory and applications.

We would like to thank all of the mathematicians who participated by either attending or presenting their work and generating new ideas for further research. We would especially like to thank those who graciously accepted our invitations and those who submitted their work for publication in these proceedings.

We would like to thank the National Security Agency, the Deans of the Chillicothe, Lancaster and Zanesville Campuses of Ohio University for their financial support. Thanks are also due to the Vice President for Regional Higher Education and the Office of Research at Ohio University for their continued support of scholarly activities at Ohio University and its regional campuses.

Of great assistance and inspiration for this conference and subsequent proceedings is the Center for Ring Theory and its Applications (CRA) at Ohio University. CRA has been a driving force that keeps the State of Ohio on the forefront of research in Algebra. The Director of CRA, S.K. Jain, not only offered financial support but also gave many helpful suggestions in the organization of the conference.

On a personal and professional level we would like to thank our colleagues Dinh V. Huynh and Sergio R. Lopez-Permouth at Ohio University for their assistance. We would also like to thank the staff of the American Mathematical Society for their hard work and we are especially indebted to Ms. Christine Thivierge for her effort and professionalism in getting this project through.

Finally, we thank all of our colleagues who served as anonymous referees for the papers presented here. Their meticulous review and thoughtful suggestions were extremely helpful in the editing of this volume.

Editors

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Contemporary Mathematics
Volume **480**, 2009

Subgroups of Direct Products of Groups, Ideals and Subrings of Direct Products of Rings, and Goursat's Lemma

D. D. Anderson and V. Camillo

Abstract. We give an exposition of Goursat's Lemma which describes the subgroups of a direct product of two groups. A ring version giving the subrings and ideals of a direct product of two rings is also given.

In group theory there are three important constructions of new groups from old groups: (1) the subgroup H of a group G, (2) the quotient or factor group G/H where H is a normal subgroup of G (denoted $H \triangleleft G$), and (3) the direct product $G_1 \times G_2$ of two groups G_1 and G_2. For each of these three constructions, we can ask what are the subgroups? The answer in the first two cases is easy. A subgroup L of H is just a subgroup L of G contained in H (a subgroup of a subgroup is a subgroup!) and by the Correspondence Theorem a subgroup of G/H has the form J/H where J is a subgroup of G with $H \subseteq J \subseteq G$ (moreover, $J/H \triangleleft G/H$ if and only if $J \triangleleft G$). The third case is more difficult and is the focus of this article: Given groups G_1 and G_2, find all the (normal) subgroups of $G_1 \times G_2$.

Given two groups G_1 and G_2, the direct product $G_1 \times G_2$ of G_1 and G_2 is the set of ordered pairs $\{(g_1, g_2) | g_i \in G_i\}$ with coordinate-wise product $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$. Here $(1,1)$ is the identity element and $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$. If H_i is a subgroup of G_i, then $H_1 \times H_2$ is easily checked to be a subgroup of $G_1 \times G_2$. Moreover, $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$ if and only if each $H_i \triangleleft G_i$. Let us call a subgroup of $G_1 \times G_2$ of the form $H_1 \times H_2$ a *subproduct* of $G_1 \times G_2$. A beginning abstract algebra student may be tempted to conjecture that every (normal) subgroup of a direct product of two groups is a subproduct. The standard counterexample is $\mathbb{Z}_2 \times \mathbb{Z}_2$ with normal subgroup $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ where $(\mathbb{Z}_2, +)$ is the integers mod 2 under addition.

There is a way to describe the subgroups of $G_1 \times G_2$ going back to É. Goursat [4] in 1889 which involves isomorphisms between factor groups of subgroups of G_1 and G_2. Briefly, given subgroups $H_{i1} \triangleleft H_{i2} \subseteq G_i$ and an isomorphism $f: H_{12}/H_{11} \to H_{22}/H_{21}$, $H = \{(a, b) \in H_{12} \times H_{22} | f(aH_{11}) = bH_{21}\}$ is a subgroup of $G_1 \times G_2$ and each subgroup of $G_1 \times G_2$ is of this form; moreover, a criterion for H to be normal is given. This is presented in Theorem 4.

Math Subject Classification (2000): primary 20-02, secondary 20D99, 16-02
Keywords: Direct product, Goursat's Lemma

In contrast, if R_1 and R_2 are rings with identity, then every ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ where I_i is an ideal of R_i. Of course, if R_1 and R_2 do not have an identity, this result is no longer true. Indeed, if we endow $(\mathbb{Z}_2, +)$ with the zero product $\bar{0} \cdot \bar{0} = \bar{0} \cdot \bar{1} = \bar{1} \cdot \bar{0} = \bar{1} \cdot \bar{1} = \bar{0}$, then $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ is an ideal of $\mathbb{Z}_2 \times \mathbb{Z}_2$. In Theorem 11 we give a ring version of Goursat's Lemma that describes the ideals and subrings of the direct product $R_1 \times R_2$ of two rings and in Theorem 14 we give a module version.

We also consider the following two questions:

(1) What pairs of groups G_1, G_2 have the property that every (normal) subgroup of $G_1 \times G_2$ is a subproduct of $G_1 \times G_2$ (recall that a subgroup $H_1 \times H_2$ is called a subproduct of $G_1 \times G_2$)?

(2) What pairs of rings R_1, R_2 have the property that every subring (ideal) of $R_1 \times R_2$ is a subproduct of $R_1 \times R_2$?

At the end of the paper we give a brief biographical sketch of É. Goursat and a history of his lemma.

It is our contention that Goursat's Lemma is a useful result that deserves to be more widely known. Indeed, we show that the Zassenhaus Lemma (Theorem 8) is a corollary of Goursat's Lemma. This article could be used in a first abstract course as it makes good use of the three important constructions, subgroups, factor groups, and direct products, and of isomorphisms. The only prerequisites from group theory are a good understanding of subgroups, normal subgroups, factor groups, direct products, and isomorphism. In one place (the proof of Theorem 4(1)) the First Isomorphism Theorem for groups is used, but this can easily be avoided. For the ring portion one only needs to be familiar with rings, subrings, ideals, ring isomorphisms, and direct products of rings.

Goursat's Lemma

As a warm up, we first determine the pairs of groups G_1, G_2 for which every subgroup of $G_1 \times G_2$ is a subproduct. A group G is *nontrivial* if $|G| > 1$. The proof is based on the well known result given below that a direct product $G_1 \times G_2$ of two nontrivial groups is cyclic if and only if G_1 and G_2 are finite cyclic groups with $|G_1|$ and $|G_2|$ relatively prime.

LEMMA 1. *Let G_1 and G_2 be nontrivial groups. Then $G_1 \times G_2$ is cyclic if and only if G_1 and G_2 are finite cyclic groups with* $\gcd(|G_1|, |G_2|) = 1$.

PROOF. ($\Rightarrow$) Suppose that $G_1 \times G_2$ is cyclic, say $G_1 \times G_2 = \langle (g_1, g_2) \rangle$. Let $g \in G_1$, so $(g, 1) = (g_1, g_2)^n$ for some integer n. So $g = g_1^n$ and $g_2^n = 1$. This gives that $G_1 = \langle g_1 \rangle$ and g_2 has finite order. Likewise, g_1 has finite order and $G_2 = \langle g_2 \rangle$. Let $o(g_i) = n_i$ where $o(g_i)$ denotes the order of g_i. Then $o((g_1, g_2)) = |G_1 \times G_2| = n_1 n_2$. However, if $\ell = \operatorname{lcm}(n_1, n_2)$, then $(g_1, g_2)^\ell = (1, 1)$. So $\ell = n_1 n_2$, that is, $\gcd(n_1, n_2) = 1$. ($\Leftarrow$) Suppose that $G_i = \langle g_i \rangle$ where $|G_i| = n_i$ with $\gcd(n_1, n_2) = 1$. Certainly $(g_1, g_2)^{n_1 n_2} = ((g_1^{n_1})^{n_2}, (g_2^{n_2})^{n_1}) = (1, 1)$. But $(1, 1) = (g_1, g_2)^\ell$ implies $g_i^\ell = 1$; so $n_i | \ell$. Since $\gcd(n_1, n_2) = 1$, $n_1 n_2 | \ell$. So (g_1, g_2) has order $n_1 n_2$ and hence $\langle (g_1, g_2) \rangle = G_1 \times G_2$. □

THEOREM 2. *Let G_1 and G_2 be nontrivial groups. Then every subgroup of $G_1 \times G_2$ is a subproduct if and only if for $g_i \in G_i$, g_i has finite order $o(g_i)$ and* $\gcd(o(g_1), o(g_2)) = 1$.

PROOF. ($\Rightarrow$) Let $g_i \in G_i - \{1\}$. Then $\langle(g_1, g_2)\rangle$ is a subproduct of $G_1 \times G_2$, so $\langle(g_1, g_2)\rangle = \langle g_1\rangle \times \langle g_2\rangle$. By Lemma 1, g_i has finite order and $\gcd(o(g_1), o(g_2)) = 1$. ($\Leftarrow$) Let H be a subgroup of $G_1 \times G_2$ and let $(g_1, g_2) \in H$. Since $\gcd(o(g_1), o(g_2)) = 1$, the proof of Lemma 1 gives that $\langle(g_1, g_2)\rangle = \langle g_1\rangle \times \langle g_2\rangle$. So $(g_1, 1)$, $(1, g_2) \in H$. Thus $H = H_1 \times H_2$ where $H_1 = \{g \in G_1 | (g, 1) \in H\}$ and $H_2 = \{g \in G_2 | (1, g) \in H\}$. □

We shift gears for a moment and look at subrings of $R_1 \times R_2$ where R_1 and R_2 are rings with identity. Since $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ is a subring of $\mathbb{Z}_2 \times \mathbb{Z}_2$, a subring of $R_1 \times R_2$ having the same identity as $R_1 \times R_2$ need not be a subproduct. With Theorem 2 (or Lemma 1) in mind, it is easy to characterize the pairs of rings R_1, R_2 with identity such that every subring of $R_1 \times R_2$ containing the identity of $R_1 \times R_2$ is a subproduct. Recall that the characteristic of a ring R, denoted by $\operatorname{char} R$, is the least positive integer n with $na = 0$ for all $a \in R$ (or just with $n1 = 0$ if R has an identity) or 0 if no such n exists.

THEOREM 3. *Let R_1 and R_2 be rings with identity. Then every subring of $R_1 \times R_2$ with identity $(1, 1)$ is a subproduct of $R_1 \times R_2$ if and only if each R_i has nonzero characteristic* $\operatorname{char} R_i$ *and* $\gcd(\operatorname{char} R_1, \operatorname{char} R_2) = 1$.

PROOF. ($\Rightarrow$) The prime subring $\mathbb{Z}(1, 1) = \{n(1, 1) | n \in \mathbb{Z}\}$ of $R_1 \times R_2$ is a subring of $R_1 \times R_2$ containing $(1, 1)$. So $\mathbb{Z}(1, 1) = S_1 \times S_2$ where S_i is necessarily the prime subring of R_i. Hence the direct product of the two additive cyclic groups S_1 and S_2 is cyclic. By Lemma 1, S_1 and S_2 are finite with $\gcd(|S_1|, |S_2|) = 1$; that is, $\operatorname{char} R_i \neq 0$ and $\gcd(\operatorname{char} R_1, \operatorname{char} R_2) = 1$.

($\Leftarrow$) Conversely, by Lemma 1 the conditions on $\operatorname{char} R_i$ give that $\mathbb{Z}(1, 1) = \mathbb{Z}1_{R_1} \times \mathbb{Z}1_{R_2}$. Let S be a subring of $R_1 \times R_2$ with $(1, 1) \in S$. Then $(1, 0), (0, 1) \in \mathbb{Z}(1, 1) \subseteq S$. So if $(s_1, s_2) \in S$, then $(s_1, 0)$, $(0, s_2) \in S$. Thus $S = S_1 \times S_2$ where $S_1 = \{s \in R_1 | (s, t) \in S$ for some $t \in R_2\}$ (resp., $S_2 = \{t \in R_2 | (s, t) \in S$ for some $s \in S_1\}$) is a subring of R_1 (resp., R_2) containing the identity of R_1 (resp., R_2). □

We next give a version of Goursat's Lemma for groups.

THEOREM 4. (*Goursat's Lemma for Groups*) Let G_1 *and* G_2 *be groups.*

(1) *Let H be a subgroup of $G_1 \times G_2$. Let $H_{11} = \{a \in G_1 | (a, 1) \in H\}$, $H_{21} = \{a \in G_2 | (1, a) \in H\}$, $H_{12} = \{a \in G_1 | (a, b) \in H$ for some $b \in G_2\}$, and $H_{22} = \{b \in G_2 | (a, b) \in H$ for some $a \in G_1\}$. Then $H_{i1} \subseteq H_{i2}$ are subgroups of G_i with $H_{i1} \lhd H_{i2}$ and the map $f_H: H_{12}/H_{11} \to H_{22}/H_{21}$ given by $f_H(aH_{11}) = bH_{21}$ where $(a, b) \in H$ is an isomorphism. Moreover, if $H \lhd G_1 \times G_2$, then $H_{i1}, H_{i2} \lhd G_i$ and $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$, the center of G_i/H_{i1}.*

(2) *Let $H_{i1} \lhd H_{i2}$ be subgroups of G_i and let $f: H_{12}/H_{11} \to H_{22}/H_{21}$ be an isomorphism. Then $H = \{(a, b) \in H_{12} \times H_{22} | f(aH_{11}) = bH_{21}\}$ is a subgroup of $G_1 \times G_2$. Further suppose that $H_{i1}, H_{i2} \lhd G_i$ and $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$. Then $H \lhd G_1 \times G_2$.*

(3) *The constructions given in* (1) *and* (2) *are inverses to each other.*

PROOF. (1) It is easily checked that $H_{i1} \subseteq H_{i2}$ are subgroups of G_i; or observe that we may identify H_{i1} with $H \cap G_i$ and $H_{i2} = \pi_i(H)$ where $\pi_i: G_1 \times G_2 \to G_i$ is the projection map. Also, $H_{i1} \lhd H_{i2}$. We do the case $i = 1$. Now $a \in H_{11} \Rightarrow (a, 1) \in H$.

Let $c \in H_{12}$; so there exists b with $(c,b) \in H$. Now $(c^{-1}ac, 1) = (c,b)^{-1}(a,1)(c,b) \in H$ so $c^{-1}ac \in H_{11}$. Define $\hat{f}: H_{12} \to H_{22}/H_{21}$ by $\hat{f}(a) = bH_{21}$ where $(a,b) \in H$. If $(a,b), (a,c) \in H$, then $(1, b^{-1}c) = (a,b)^{-1}(a,c) \in H$ implies $b^{-1}c \in H_{21}$ and hence $bH_{21} = cH_{21}$. So $\hat{f}$ is well-defined. It is easily checked that $\hat{f}$ is a surjective homomorphism with $\ker \hat{f} = H_{11}$. Indeed, if $a \in \ker \hat{f}$, then for $(a,b) \in H$, $b \in H_{21}$. But then $(1,b) \in H$, so $(a,1) = (a,b)(1,b)^{-1} \in H$ which gives $a \in H_{11}$. So by the First Isomorphism Theorem, $f_H: H_{12}/H_{11} \to H_{22}/H_{21}$ is an isomorphism. It is easily checked that $H \lhd G_1 \times G_2$ implies $H_{ij} \lhd G_i$; or use the previous identification with the intersection with G_i and projection onto G_i. Also, for $g \in G_1$ and $a \in H_{12}$ with $(a,b) \in H$, $(g^{-1}ag, b)$, $(a^{-1}, b^{-1}) \in H$ give $(a^{-1}g^{-1}ag, 1) \in H$ and hence $a^{-1}g^{-1}ag \in H_{11}$. So $H_{12}/H_{11} \subseteq Z(G_1/H_{11})$. Likewise, $H_{22}/H_{21} \subseteq Z(G_2/H_{21})$.

(2) It is easily checked that H is a subgroup of $H_{12} \times H_{22}$ and hence of $G_1 \times G_2$. Indeed, if $(a,b), (c,d) \in H$, then $f(aH_{11}) = bH_{21}$ and $f(cH_{11}) = dH_{21}$. So $f(acH_{11}) = f(aH_{11}cH_{11}) = f(aH_{11})f(cH_{11}) = bH_{21}dH_{21} = bdH_{21}$ and $f(a^{-1}H_{11}) = f((aH_{11})^{-1}) = (f(aH_{11}))^{-1} = (bH_{21})^{-1} = b^{-1}H_{21}$. So (ac, bd), $(a^{-1}, b^{-1}) \in H$. Further, suppose that $H_{ij} \lhd G_i$ and $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$. We show that $H \lhd G_1 \times G_2$. Let $(a_1, a_2) \in H$ and $(g_1, g_2) \in G_1 \times G_2$. So $H_{i2} \lhd G_i$ gives $(g_1^{-1}a_1g_1, g_2^{-1}a_2g_2) \in H_{12} \times H_{22}$. And $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$ gives $g_i^{-1}a_ig_iH_{i1} = a_iH_{i1}$, so $f(g_1^{-1}a_1g_1H_{11}) = g_2^{-1}a_2g_2H_{21}$. Thus $(g_1^{-1}a_1g_1,\ g_2^{-1}a_2g_2) \in H$.

(3) Clear. □

The reader may have noticed that the proof in Theorem 4(2) that H is a subgroup did not use the fact that f is a bijection. Indeed, for any homomorphism $f: H_{12}/H_{11} \to H_{22}/H_{21}$, H is a subgroup. However, if we set $\operatorname{im} f = H'_{22}/H_{21}$ where H'_{22} is a subgroup of H_{22} with $H_{21} \lhd H'_{22}$ and $\ker f = H'_{11}/H_{11}$ where $H_{11} \subseteq H'_{11} \lhd H_{12}$, then f induces the isomorphism $f': H_{12}/H'_{11} \to H'_{22}/H_{21}$ and the same subgroup H.

Note that the subproduct $H = H_1 \times H_2$ corresponds to $H_{11} = H_{12} = H_1$ and $H_{21} = H_{22} = H_2$; so $f_H: H_1/H_1 \to H_2/H_2$ is the trivial map. We illustrate Goursat's Lemma by finding all (normal) subgroups of $S_3 \times S_3$.

EXAMPLE 5. ***The subgroups of*** $\mathbf{S_3 \times S_3}$. *First, the subgroups of* S_3 *are* $\langle(1)\rangle$, $\langle(12)\rangle$, $\langle(13)\rangle$, $\langle(23)\rangle$, $\langle(123)\rangle = A_3$, *and* S_3. *Of these,* $\langle(1)\rangle$, A_3, *and* S_3 *are normal. We have the following subnormal quotient groups* H/K *where* $K \lhd H \subseteq S_3$ *grouped by order*: *(a)* $|H/K| = 1$; $\langle(1)\rangle/\langle(1)\rangle$, $\langle(12)\rangle/\langle(12)\rangle$, $\langle(13)\rangle/\langle(13)\rangle$, $\langle(23)\rangle/\langle(23)\rangle$, A_3/A_3, S_3/S_3; *(b)* $|H/K| = 2$: $\langle(12)\rangle/\langle(1)\rangle$, $\langle(13)\rangle/\langle(1)\rangle$, $\langle(23)\rangle/\langle(1)\rangle$, S_3/A_3; *(c)* $|H/K| = 3$: $A_3/\langle(1)\rangle$; *(d)* $|H/K| = 6$: $S_3/\langle(1)\rangle$. *Note that within each of the four classes, the quotient groups are all isomorphic. Class (a) has only the identity maps between the 6 different quotients; so there are 36 different isomorphisms* $f: H_{12}/H_{11} \to H_{22}/H_{21}$ *yielding the 36 different subproducts* $\langle(1)\rangle \times \langle(1)\rangle, \cdots, S_3 \times S_3$. *Of these 9 are normal. Class (b) has 4 groups of order 2. Since there is a unique isomorphism between two groups of order 2, there are 16 different isomorphisms* $f: H_{12}/H_{11} \to H_{22}/H_{21}$ *yielding 16 distinct subgroups. For example, the isomorphism* $\langle(12)\rangle/\langle(1)\rangle \to \langle(13)\rangle/\langle(1)\rangle$ *gives the subgroup* $\{((1),(1)),\ ((12),(13))\}$. *There are 9 such subgroups each of order 2. The isomorphism* $\langle(12)\rangle/\langle(1)\rangle \to S_3/A_3$ *gives the subgroup* $\{((1),(1)),\ ((1),(123)),\ ((1),(132)),\ ((12),(12)),\ ((12),(13)),\ ((12),(23))\}$ *and there are 6 such subgroups* *each isomorphic to* S_3. *The isomorphism* $S_3/A_3 \to S_3/A_3$ *gives rise to the subgroup* $E = \{(a,b) \in S_3 \times S_3 | (a,b) \in A_3 \times A_3$ *or* $o(a) = o(b) = 2\}$. *Note that* E *has 9 elements of order* 2 *and hence is* not *isomorphic to a subproduct of* $S_3 \times S_3$. *So this gives a total of 16 subgroups in this class and as only the last one satisfies* $H_{i2}/H_{i1} \subseteq Z(G/H_{i1})$, *only* E *is a normal subgroup. Class* (*c*) *has only one group of order* 3 *but two isomorphisms* $\langle(123)\rangle/\langle(1)\rangle \to \langle(123)\rangle/\langle(1)\rangle$. *The identity map gives the subgroup* $\{(x,x)|x \in A_3\}$ *which is isomorphic to* A_3 *and the isomorphism given by* $(123) \to (132)$ *gives the subgroup* $\{((1),(1)), ((123),(132)), ((132),(123))\}$ *which is isomorphic to* A_3. *Neither subgroup is normal. Finally, for class* (*d*) *there are 6 isomorphisms* $S_3 \to S_3$, *each given by conjugation.* (*It is well known that for* $n \neq 2, 6$ *each automorphism of* S_n *is inner, that is, a conjugation on* S_n, *and* $\mathrm{Aut}(S_n) \approx S_n$; *see for example,* [**9**, Theorem 7.4].) *Thus this class has 6 subgroups:* $G_\sigma = \{(a, \sigma^{-1}a\sigma)|a \in S_3\}$ *for* $\sigma \in S_3$. *Note that each is isomorphic to* S_3, *but none are normal.*

In summary $S_3 \times S_3$ *has 60 distinct subgroups, 10 of which are normal. Moreover, each of these subgroups except* E *is isomorphic to a subproduct.*

This example raises the following question.

QUESTION 6. *What pairs of groups* G_1 *and* G_2 *have the property that every* (*normal*) *subgroup of* $G_1 \times G_2$ *is isomorphic to a subproduct of* $G_1 \times G_2$?

It is not hard to prove that a pair of finitely generated abelian groups G_1 and G_2 satisfy the condition of Question 6. But we have shown that the pair S_3, S_3 does not. Also, if G is a rank two indecomposable abelian group, then G is isomorphic to a subgroup of $\mathbb{Q} \times \mathbb{Q}$, but not to a subproduct of $\mathbb{Q} \times \mathbb{Q}$.

We next show how you can use Goursat's Lemma to prove Theorem 2. By Goursat's Lemma every subgroup of $G_1 \times G_2$ is a subproduct if and only if the only pairs of normal subgroups $H_{i1} \lhd H_{i2} \subseteq G_i$ with H_{12}/H_{11} and H_{22}/H_{21} isomorphic are the trivial ones $H_{i1} = H_{i2}$. So suppose that every subgroup of $G_1 \times G_2$ is a subproduct. Then for $g_i \in G_i$ and $m_i > 1$ with $\langle g_i^{m_i}\rangle \neq \langle g_i\rangle$, $\langle g_1\rangle/\langle g_1^{m_1}\rangle$ and $\langle g_2\rangle/\langle g_2^{m_2}\rangle$ can not be isomorphic. From this it follows that each order $o(g_i)$ is finite and $\gcd(o(g_1),\ o(g_2)) = 1$. Conversely, suppose that each $g_i \in G_i$ has finite order and $\gcd(o(g_1), (o(g_2)) = 1$. Then H_{i2}/H_{i1} are groups in which every element has finite order and the order of each element of H_{12}/H_{11} is relatively prime to the order of each element of H_{22}/H_{12}. Thus we can have H_{12}/H_{11} and H_{22}/H_{12} isomorphic only in the trivial case that $H_{i1} = H_{i2}$. But then every subgroup of $G_1 \times G_2$ is a subproduct.

We next use Goursat's Lemma to determine the pairs of nontrivial groups G_1 and G_2 with every normal subgroup of $G_1 \times G_2$ a subproduct. Certainly if every subgroup of $G_1 \times G_2$ is a subproduct, every normal subgroup is a subproduct. However, note that $G_1 \times G_2$ may have every normal subgroup a subproduct without having every subgroup a subproduct. For let G be any nonabelian simple group. Then certainly $G \times G$ has subgroups that are not subproducts. However, the only proper, nontrivial normal subgroups of $G \times G$ are $G \times \{1\}$ and $\{1\} \times G$. This easily follows from Theorem 4.

THEOREM 7. *Let* G_1 *and* G_2 *be nontrivial groups. Then the following are equivalent.*

(1) *Every normal subgroup of* $G_1 \times G_2$ *is a subproduct.*

(2) *There do not exist normal subgroups* $H_{i1} \subsetneq H_{i2} \subseteq G_i$ *with* $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$ *and* H_{12}/H_{11} *isomorphic to* H_{22}/H_{21}.

(3) *One of the following two conditions holds.*

(a) G_1 *or* G_2 *is residually centerless.* (*A group* G *is residually centerless if for each homomorphic image* $\bar{G}$, $Z(\bar{G}) = \{1\}$.)

(b) *For each normal subgroup* $H_i \lhd G_i$, $Z(G_i/H_i)$ *is torsion and* $\gcd(o(g_1), o(g_2)) = 1$ *for* $g_i \in G_i/H_i$.

PROOF. (1)⇔(2) This follows from Theorem 4. (2)⇒(3) Suppose that (a) does not hold. Suppose that there is a normal subgroup $H_1 \lhd G_1$ with $Z(G_1/H_1)$ having an element $\bar{g}$ ($g \in G$) of infinite order. Let $H_2 \lhd G_2$ with $Z(G_2/H_2) \neq \{\bar{1}\}$. Let $\bar{1} \neq \bar{h} \in Z(G_2/H_2)(h \in G_2)$. Since $\bar{g} \in Z(G_1/H_1)$ (resp., $\bar{h} \in Z(G_2/H_2)$), $H_1 \subsetneq \langle g\rangle H_1 \lhd G_1$ with $\langle \bar{g}\rangle = \langle g\rangle H_1/H_1 \subseteq Z(G_1/H_1)$ and $H_2 \subsetneq \langle h\rangle H_2 \lhd G_2$ with $\langle \bar{h}\rangle = \langle h\rangle H_2/H_2 \subseteq Z(G_2/H_2)$. If $\bar{h}$ has infinite order, then $\langle g\rangle H_1/H_1$ and $\langle h\rangle H_2/H_2$ are isomorphic, a contradiction. So $o(\bar{h}) = n > 1$. But then $\langle g^n\rangle H_1$ and $\langle g\rangle H_1$ are normal subgroups of G_1 with $\langle g\rangle H_1/\langle g^n\rangle H_1 \subseteq Z(G_1/\langle g^n\rangle H_1)$ and $\langle g\rangle H_1/\langle g^n\rangle H_1$ is cyclic of order n. Thus $\langle g\rangle H_1/\langle g^n\rangle H_1$ and $\langle h\rangle H_2/H_2$ are isomorphic; a contradiction. Thus for each normal subgroup $H_i \lhd G_i$, $Z(G_i/H_i)$ must be torsion. Suppose that there are normal subgroups $H_i \lhd G_i$ with elements $g_i \in G_i$ such that $g_iH_i \in Z(G_i/H_i)$ and $\gcd(o(g_1H_1), o(g_2H_2)) \neq 1$. Then $Z(G_1/H_1)$ and $Z(G_2/H_2)$ have elements $g_i'H_i$ ($g_i' \in G_i$) with $o(g_1'H_1) = o(g_2'H_2) > 1$. But then $\langle g_1'\rangle H_1/H_1$ and $\langle g_2'\rangle H_2/H_2$ are isomorphic; a contradiction. (3)⇒(2) Suppose there exist normal subgroups $H_{i1} \subsetneq H_{i2} \subseteq G_i$ with $H_{i2}/H_{i1} \subseteq Z(G_i/H_{i1})$ and H_{12}/H_{11} isomorphic to H_{22}/H_{21}. Then there are elements $\bar{h}_i \in H_{i2}/H_{i1}$ with $1 < o(\bar{h}_1) = o(\bar{h}_2) < \infty$, a contradiction. □

We next show how Goursat's Lemma can be used to prove the Zassenhaus Lemma [**9**, Lemma 5.8]: if $A \lhd A^*, B \lhd B^*$ are subgroups of a group G, then the groups $A(A^* \cap B^*)/A(A^* \cap B)$ and $B(B^* \cap A^*)/B(B^* \cap A)$ are isomorphic. The Zassenhaus Lemma plays a key role in the proof of the Schreier Refinement Theorem [**9**, Theorem 5.9] which states that two subnormal series for a group have equivalent refinements which in turn is used to prove the Jordan-Hölder Theorem [**9**, Theorem 5.10]: any two composition series of a group G are equivalent.

THEOREM 8. (*Zassenhaus Lemma*) *Let* G *be a group and* $A \lhd A^*$ *and* $B \lhd B^*$ *subgroups of* G. *Then* $A(A^* \cap B) \lhd A(A^* \cap B^*)$, $B(B^* \cap A) \lhd B(B^* \cap A^*)$ *and the quotient groups* $A(A^* \cap B^*)/A(A^* \cap B)$ *and* $B(B^* \cap A^*)/B(B^* \cap A)$ *are isomorphic.*

PROOF. Let $H = \{(ac, bc) \in G \times G | a \in A,\ b \in B,\ c \in A^* \cap B^*\}$. We first show that H is a subgroup of $G \times G$. Let (ac, bc), $(a'c', b'c') \in H$ where $a, a' \in A$, $b, b' \in B$, and $c, c' \in A^* \cap B^*$. Now $A \lhd A^*$ gives $ca' = \bar{a}c$ and $c^{-1}a^{-1} = a^*c^{-1}$ for some $\bar{a}, a^* \in A$ and $B \lhd B^*$ gives $cb' = \bar{b}c$ and $c^{-1}b^{-1} = b^*c^{-1}$ for some $\bar{b}, b^* \in B$. So $(ac, bc)(a'c', b'c') = (aca'c', bcb'c') = (a\bar{a}cc', b\bar{b}cc') \in H$ and $(ac, bc)^{-1} = (c^{-1}a^{-1}, c^{-1}b^{-1}) = (a^*c^{-1}, b^*c^{-1}) \in H$. Now using the notation of Goursat's Lemma (Theorem 4), we determine the H_{ij}'s. Certainly $H_{12} = A(A^* \cap B^*)$ and $H_{22} = B(B^* \cap A^*)$ (which shows that they are subgroups of G). Now $H_{11} = \{ac | a \in A,\ c \in A^* \cap B^*,\ (ac, 1) \in H\} = \{ac | a \in A,\ c \in A^* \cap B^*,\ c = b^{-1}$ for some $b \in B\} = \{ac | a \in A,\ c \in A^* \cap B\} = A(A^* \cap B)$. Likewise $H_{21} = B(B^* \cap A)$. Thus from $H_{i1} \lhd H_{i2}$ we get that $A(A^* \cap B)$ is a normal subgroup of $A(A^* \cap B^*)$,

$B(B^* \cap A)$ is a normal subgroup of $B(B^* \cap A^*)$ and since H_{12}/H_{11} and H_{22}/H_{21} are isomorphic, the proof is complete. □

We next turn to direct products of rings and their ideals and subrings. If R_1 and R_2 are rings, and I_1 and I_2 are ideals of R_1 and R_2, respectively, then $I_1 \times I_2$ is an ideal of $R_1 \times R_2$. Similar statements hold for right and left ideals. If R_1 and R_2 have an identity, then it is well known that every ideal (right, left, or two-sided) has this form. This is our next proposition.

PROPOSITION 9. *Let R_1 and R_2 be rings with identity. Then every (right, left, two-sided) ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ where I_i is a (right, left, two-sided) ideal of R_i.*

PROOF. Let I be a left ideal of $R_1 \times R_2$. Let $I_1 = \{a \in R_1 | (a,0) \in I\}$ and $I_2 = \{a \in R_2 | (0,a) \in I\}$. It is easily checked that I_i is a left ideal of R_i. Let $(a,b) \in I_1 \times I_2$, then $(a,0),(0,b) \in I$, so $(a,b) = (a,0) + (0,b) \in I$. Hence $I_1 \times I_2 \subseteq I$. Conversely, suppose that $(a,b) \in I$. Then $(a,0) = (1,0)(a,b) \in I$; so $a \in I_1$. Likewise $(0,b) = (0,1)(a,b) \in I$; so $b \in I_2$. Hence $(a,b) \in I_1 \times I_2$ and $I \subseteq I_1 \times I_2$. □

Of course Proposition 9 may fail if R_1 and R_2 do not have an identity. For example, if we take $R_1 = R_2 = (\mathbb{Z}_2, +)$ where $\mathbb{Z}_2$ has the zero product, then $\{(\bar{0},\bar{0}),(\bar{1},\bar{1})\}$ is an ideal of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that is not a subproduct. We next give a partial converse to Proposition 9. Let us call a ring R a *left e-ring* if for each $r \in R$ there exists an element $e_r \in R$, depending on r, with $e_r r = r$. Note that R is a left e-ring if and only if $RI = I$ for each left ideal I of R. The next result comes from [**1**].

THEOREM 10. *For a ring R the following statements are equivalent.*

(1) *R is a left e-ring.*
(2) *For each ring S, every left ideal of $R \times S$ is a subproduct of left ideals.*
(3) *Every left ideal of $R \times R$ is a subproduct of left ideals.*

PROOF. (1)⇒(2) Let I be a left ideal of R. For $(a,b) \in I$ choose $e_a \in R$ with $e_a a = a$. Then $(a,0) = (e_a,0)(a,b) \in I$. Hence $(0,b) = (a,b) - (a,0) \in I$. Then as in the proof of Proposition 9 we have $I = I_1 \times I_2$ where $I_1 = \{a \in R | (a,0) \in I\}$ and $I_2 = \{b \in S | (0,b) \in I\}$. (2)⇒(3) Clear. (3)⇒(1) Let $a \in R$. Then the principal left ideal generated by (a,a), $((a,a))_\ell = \{(r,s)(a,a) + n(a,a) | r,s \in R, n \in \mathbb{Z}\}$, is a left ideal of $R \times R$. So $((a,a))_\ell = I_1 \times I_2$ where I_1 and I_2 are left ideals of R. Certainly $a \in I_i$, so $(a)_\ell \subseteq I_i$. And as $((a,a))_\ell \subseteq (a)_\ell \times (a)_\ell$, we have $((a,a))_\ell = (a)_\ell \times (a)_\ell$. Thus $(a,0) \in ((a,a))_\ell$; so $(a,0) = (e_1,e_2)(a,a) + n(a,a)$ for some $e_i \in R$ and $n \in \mathbb{Z}$. Hence $a = e_1 a + na$ and $0 = e_2 a + na$; so $a = e_1 a + na = e_1 a - e_2 a = (e_1 - e_2)a$. □

We leave it to the reader to define a right e-ring and to state versions of Theorem 10 for right ideals and two-sided ideals. Note that if $\{R_\alpha\}$ is any nonempty family of left e-rings, then their direct sum $\oplus R_\alpha$ with coordinate-wise operations is again a left e-ring. In particular, an infinite direct sum of rings each having an identity is both a left and right e-ring, but does not have an identity.

We next give versions of Goursat's Lemma for ideals and subrings of a direct product of rings.

THEOREM 11. (*Goursat's Lemma for Ideals and Subrings*) *Let S_1 and S_2 be rings.*

(1) Let T be an additive subgroup of $S_1 \times S_2$. Let $T_{11} = \{s \in S_1 | (s,0) \in T\}$, $T_{12} = \{s \in S_1 | (s,t) \in T$ for some $t \in S_2\}$, $T_{21} = \{t \in S_2 | (0,t) \in T\}$, and $T_{22} = \{t \in S_2 | (s,t) \in T$ for some $s \in S_1\}$. Then $T_{i1} \subseteq T_{i2}$ are subgroups of $(S_i, +)$ and the map $f_T: T_{12}/T_{11} \to T_{22}/T_{21}$ given by $f_T(s+T_{11}) = t+T_{21}$ for $(s,t) \in T$ is an abelian group isomorphism.

(a) Suppose that T is a left (resp., right, two-sided) ideal of $S_1 \times S_2$. Then $T_{i1} \subseteq T_{i2}$ are left (resp., right, two-sided) ideals of S_i with $S_i T_{i2} \subseteq T_{i1}$ (resp., $T_{i2} S_i \subseteq T_{i1}$, $S_i T_{i2} \subseteq T_{i1}$ and $T_{i2} S_i \subseteq T_{i1}$).

(b) Suppose that T is a subring of $S_1 \times S_2$. Then T_{i2} is a subring of S_i and T_{i1} is an ideal of T_{i2}. Moreover, $f_T: T_{12}/T_{11} \to T_{22}/T_{21}$ is a ring isomorphism. Further, suppose that each S_i has an identity 1_{S_i}. Then T contains $(1_{S_1}, 1_{S_2})$ if and only if $1_{S_i} \in T_{i2}$ and $f_T(1_{S_1} + T_{11}) = 1_{S_2} + T_{21}$.

(2) Conversely, suppose that $T_{i1} \subseteq T_{i2} \subseteq S_i$ are additive subgroups with abelian group isomorphism $f: T_{12}/T_{11} \to T_{22}/T_{21}$.

(a) If $S_i T_{i2} \subseteq T_{i1}$ (resp., $T_{i1} S_i \subseteq T_{i1}$, $S_i T_{i2} \subseteq T_{i1}$ and $T_{i2} S_i \subseteq T_{i1}$), then $T = \{(a,b) \in T_{12} \times T_{22} | f(a + T_{11}) = b + T_{21}\}$ is a left (resp., right, two-sided) ideal of $S_1 \times S_2$.

(b) Suppose that T_{i2} is a subring of S_i, that T_{i1} is an ideal of T_{i2}, and $f: T_{12}/T_{11} \to T_{22}/T_{21}$ is a ring isomorphism. Then $T = \{(a,b) \in T_{12} \times T_{22} | f(a + T_{11}) = b + T_{21}\}$ is a subring of $S_1 \times S_2$. Moreover, if each S_i has an identity 1_{S_i}, then $(1_{S_1}, 1_{S_2}) \in T$ if and only if $1_{S_i} \in T_{i2}$ and $f(1 + T_{11}) = 1 + T_{21}$.

(3) The constructions given in (1) and (2) are inverses to each other.

PROOF. (1) The facts that $T_{i1} \subseteq T_{i2}$ are subgroups of $(S_i, +)$ and that f_T is an abelian group isomorphism follow from Theorem 4.

(a) Suppose that T is a left ideal of $S_1 \times S_2$. Let $s \in S_1$ and $t \in T_{12}$ where $(t, t') \in T$ for some $t' \in S_2$. Then $(st, 0) = (s, 0)(t, t') \in T$ implies $st \in T_{11}$. Hence $S_1 T_{12} \subseteq T_{11}$. This also shows that $T_{11} \subseteq T_{12}$ are left ideals of S_1. The proofs of the other cases are similar.

(b) Suppose that T is a subring of $S_1 \times S_2$. Let $t_1, t_2 \in T_{12}$ where $(t_1, t_1'), (t_2, t_2') \in T$ for some $t_1', t_2' \in S_2$. Then $(t_1 t_2, t_1' t_2') = (t_1, t_1')(t_2, t_2') \in T$ gives $t_1 t_2 \in T_{12}$. Hence T_{12} is a subring of S_1. If actually $t_2 \in T_{11}$, then we can take $t_2' = 0$, so $(t_1 t_2, 0) \in T$ gives $t_1 t_2 \in T_{11}$ and hence T_{11} is a left ideal. Similarly T_{11} is a right ideal and hence is a two-sided ideal. Likewise, T_{22} is a subring of S_2 and T_{21} is an ideal of T_{22}. Now $f_T(t_i + T_{11}) = t_i' + T_{21}$; so $f_T((t_1 + T_{11})(t_2 + T_{11})) = f_T(t_1 t_2 + T_{11}) = t_1' t_2' + T_{21} = (t_1' + T_{21})(t_2' + T_{21}) = f_T(t_1 + T_{11}) f_T(t_2 + T_{11})$; hence f_T is a ring isomorphism. Suppose that each S_i has an identity 1_{S_i}. Certainly $(1_{S_1}, 1_{S_2}) \in T$ gives $1_{S_i} \in T_{i2}$ and that f_T preserves identities. Conversely, if $1_{S_i} \in T_{i2}$, then $f_T(1_{S_1} + T_{11}) = 1_{S_2} + T_{21}$ gives $(1_{S_1}, 1_{S_2}) \in T$.

(2) Suppose that $T_{i1} \subseteq T_{i2}$ are additive subgroups of $(S_i, +)$ and that $f: T_{12}/T_{11} \to T_{22}/T_{21}$ is an isomorphism. By Theorem 4, $T = \{(a,b) \in T_{12} \times T_{22} | f(a + T_{11}) = b + T_{21}\}$ is an additive subgroup of $S_1 \times S_2$.