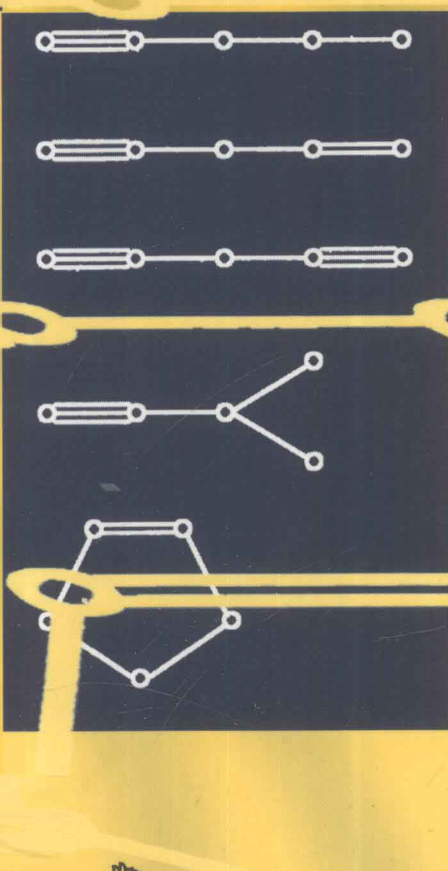
ARNOLD's PROBLEMS



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Arnold's Problems

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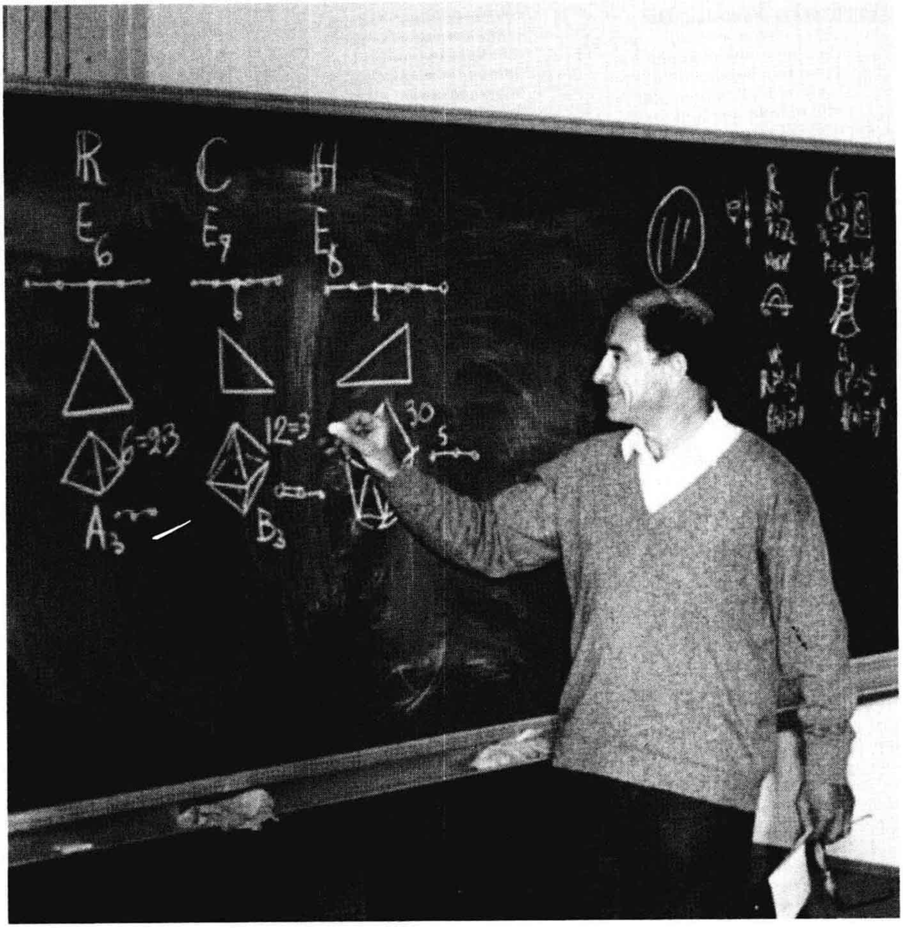

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Author's Preface to the Second Edition

For a working mathematician, it is much more important to know what questions are not answered so far and failed to be solved by the methods already available, than all lists of numbers already multiplied, and than the erudition in the ocean of literature that has been created by previous generations of researchers over twenty thousand years.

To tell the students about the most (in the author's opinion) interesting unsolved problems—this is the purpose of the present book which is composed of problems formulated at seminars in Moscow and Paris starting from 1958. The main body of the book is formed by comments of my former students about the current progress in the problems solution (featuring bibliography inspired by them).

The observed half-life of the problem (of its more or less complete solution) is about seven years on average. Thus, many problems are still open, and even those that are mainly solved keep stimulating new research appearing every year in journals of various countries of the World.

The invariable peculiarity of these problems was that Mathematics was considered there not as a game with deductive reasonings and symbols, but as a part of natural science (especially of Physics), that is, as an experimental science (which is distinguished among other experimental sciences primarily by the low costs of its experiments).

Problems of binary type admitting a "yes–no" answer (like the Fermat problem) are of little value here. One should rather speak of wide-scope programs of explorations of new mathematical (and not only mathematical) continents, where reaching new peaks reveals new perspectives, and where a preconceived formulation of problems would substantially restrict the field of investigations that have been caused by these perspectives. It is not sufficient to know whether there is a river beyond the mountain; it does remain to cross this river! Evolution is more important than achieving records.

In the raw cases where the imperatives of simplicity and beauty contradicted each other, the author usually has chosen the latter, having in mind that it was the beauty rather than the utility of science (including Mathematics) that historically played the role of the main engine leading researchers to the discoveries proved to be most useful nowadays (such as the conic sections for space navigation, or the Maxwell equations for television and radar).

I would wish the reader not to be held back by the fact that such applications are not evident at the beginning: if a result is truly beautiful then it will certainly be of use in due course!

V. I. Arnold

Moscow, 2003

Le monde est soutenu
par les enfants
qui étudient.

Roger Peyrefitte

Les juifs. Paris: Flammarion, 1965, p. 281

Author's Preface to the First Edition

Moscow has a long-standing fame for its mathematical seminars. At the beginning of each academic term I formulate problems, usually a dozen or two. The future analysis shows that the average half-life of a problem (after which it would be more or less solved) is about *seven years*.

Poincaré used to say that precise formulation, as a question admitting a "*yes* or *no*" answer, is possible only for problems of little interest. Questions that are really interesting would not be settled this way: they yield *gradual* forward motion and *permanent* development.

In Poincaré's opinion, the main essence of any problem is to understand what is definitive in its formulation, and what can be varied (like boundary conditions in an elliptic problem).

I. G. Petrovskiĭ, who was one of my teachers in Mathematics, taught me that the most important thing that a student should learn from his supervisor is that some *question is still open*. Further choice of the problem from the set of unsolved ones is made by the student himself. To select a problem for him is the same as to choose a bride for one's son.

Mainly, I did not write my problems down, especially in the sixties; therefore most of them are probably lost. Some problems are included in my papers and books. Sometimes I reconstructed my problems to the seminar from conversations with my colleagues and friends. I hope that below the authors are quoted in most of such situations.

There are two principal ways to formulate mathematical assertions (problems, conjectures, theorems, ...): Russian and French. The *Russian way* is to choose *the most simple and specific* case (so that nobody could simplify the formulation preserving the main point). The *French way* is *to generalize the statement as far as nobody could generalize it further.*

I assume that this division more or less coincides with the division of people into the right-hemisphere resolvers of posed problems, and the left-hemisphere authors of research programs.

Once, when I was a younger student, I asked R. L. Dobrushin (who was a graduate student) a question. "A fool can ask so many questions that a hundred of intellectuals could not answer them," Dobrushin said. As for me, questions should nevertheless be published. By the way, it turned out that the question that I had asked Dobrushin that time—*whether the perimeter of a rectangle can increase as the result of a sequence of foldings and unfoldings*—remains open and is treated as folklore (although, seems to me, I published it, say, 40 years ago).

Ya. B. Zeldovich thought that posing a problem is a much finer art than its solution. "*Once you formulate a precise question*, he said, *there already appears a mathematician able to solve it.* In fact, *mathematicians are like flies*, fit to walk on the ceiling!"

This had led him to a well-known struggle, where Pontryagin and Logunov tried to criticize the mathematical rigor of his theories. It resulted in the following phrase in Pontryagin's book: "Some physicists think that one can make a correct use of the mathematical analysis without full knowledge of its foundation. And I do agree with them."

Zeldovich was offended by this phrase. "Why hasn't he named me?" Yakov Borisovich said to me then.

I am deeply indebted to a large number of my former and present students who have written this book. I tried to quote them appropriately.

Mathematical training in Moscow usually begins before the school age. Here is a couple of excercises (children 4–5 years old would have solved them in half an hour):

1) From a barrel of wine, a spoon was poured into a cup of tea, and then the same spoon of the obtained (nonhomogeneous!) mixture was poured from the cup back into the barrel. Where did the amount of the foreign beverage become greater?

2) On a chess board, two opposite angle squares (a1, h8) are cut off. Can the 62 remaining squares be covered by 31 domino pieces (without overlaps), every piece covering two (neighboring) squares?

Leibnitz thought that a curve intersects its curvature circle at four coincident points and that $d(ab) = (da)(db)$.

Hilbert argued that a really interesting work in mathematics rarely happens to be correct. For example, in his survey of relativity theory, he affirmed that "*simultaneity* exists by itself." His description of geometry of numbers from the article dedicated to Minkowski is beyond any critics at all.

A. Weil wrote that his famous dissertation had been read only by two opponents; but even they understood too little because of their lack of proficiency (the work was erroneous). And this is yet one of the most important works of our century (1926–1928) in number theory.

Errors committed by Poincaré himself are too widely known to be recalled here: he confused homology with homotopy and missed the 3-manifold of dodecahedral lens type which is now named after him. Many questions in the theory of differential equations, dynamical systems, and celestial mechanics "solved" by him still remain open.

Descartes wrote to Huygens: "If I see any vacuum in Nature, it is only in Pascal's head."

Mathematician N refused to correct misprints while re-editing his book, in order not to rob the reader of pleasure in finding errors.

It seems, Napoleon said that a person, who is unable to *think*, cannot also be taught anything.

I hope that the present book will teach at least anybody to think (by the above problems 1 and 2 though).

V. I. Arnold

Garches (France), 1999

Editorial to the Second Edition

You are looking at the second edition of the title "Arnold's Problems," which is now in English. Its size has noticeably grown compared with the first Russian edition of 2000—by more than a one third; for new problems and comments have appeared, and some old comments have been supplemented. The number of authors of comments has doubled, from 29 to 59.

The format of the comments has also been modified. The name of the comment's author is now shown at the beginning of the comment (beside the problem's number), no longer at its end. If there are several comments to a problem, then the problem number in every comment is preceded by a symbol indicating if this comment is the first one (▽), an intermediate one ($\frac{\triangle}{\triangledown}$) or the last one (△). Each comment is opened by a notation indicating its nature: the letter $\mathcal{H}$ means that the comment is *historic*, and $\mathcal{R}$ means that the comment is devoted to the *results* of the research on the problem.

Just as in the first (Russian) edition of this book, twin problems appear here (see the explanation on page XIII).

For the problems appearing in the first edition, the numbers have been preserved. In cases when problems of the preceding years forgotten in the former edition have since been discovered, they are appended at the end of the list of problems of the corresponding year.

We also point out a feature of the bibliography. If an article was published in a journal in Russian that is translated into English on a regular basis (cover-to-cover), then its bibliographical description includes only the translation of the article (since the original is easily found in this case). In the cases when it might be difficult to find respectively the English translation or the Russian original of an article, the references to both of them are provided.

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We acknowledge our pleasant duty to thank Professors M. S. P. Eastham, A. G. Khovanskiĭ, L. P. Kotova, M. B. Sevryuk, and O. V. Sipacheva who have contributed to this edition by improving the English text.

All formulations of the problems and all the comments have been checked by Vladimir Igorevich Arnold. Some comments, in comparison with the first edition, have been reduced by excluding the descriptions of unpublished and unverified results. Unfortunately, not all potential authors of comments accepted our suggestion to write comments to the problems they had studied. Now we keep on inviting all the colleagues to participate in commenting Arnold's Problems. For more information, see the Internet site `http://www.phasis.ru`.

In order to make the author's famous Russian original edition accessible to readers worldwide, PHASIS and Springer-Verlag have collaborated in the publication of this enlarged and updated English edition using the know-how, experience and abilities of both publishers.

V. Philippov
A. Yakivchik

M. Peters

Heidelberg, 2004

Moscow, 2004

To ask the right question
is harder than to answer it.

Georg Cantor

Editorial to the First Edition

The present title represents the problems that have been posed by Vladimir Igorevich Arnold during a period of over 40 years.

This is principally a fairly complete list of problems presented by him at his seminar on the theory of singularities of differentiable mappings, twice a year at the beginning of each academic term. (This famous seminar has been working at the Department of Mechanics and Mathematics of Moscow State University for over 30 years and deserves the title of one of the leading World centers of mathematical science.) In addition, there are problems published by Vladimir Igorevich in his numerous papers and books. It is clear, however, that not all Arnold's problems have been collected so far, and we would be grateful to those readers who will report to us any problems not appearing in the present volume.

The book consists of two parts. The first part comprises the formulations of the problems; brief explanations that are italicized there are due to the author. The second part is a collection of comments including a survey of results on the given problem or, in some cases, a historic reference. Almost all the comments are signed by their authors (which are mostly the former students of Vladimir Igorevich); the brief unsigned comments belong either to the author or to the editor. In a few cases, the authors include a description of their unpublished and unverified results in their comments, sometimes even those on classical problems; such assertions should be regarded as conjectures. The bibliography to all comments has been carefully checked by the editor.

For the sake of historic certainty, we preserve the so-called twin problems, i. e., the problems that date back to different years but are almost identical in their essence. Only one of these problems (and not always the earliest) is commented on in such a case, the other twin problems being supplied with a reference: "See the comment to problem ⟨*number*⟩." Such references are used in some other cases when information from the comment to one problem applies to another problem as well.

All mathematical notations appearing in the book are commonly used. However, the notations for spheres and balls of various dimensions must be clarified. Non-parallelizable n-dimensional spheres (i. e., for $n \notin \{0,1,3,7\}$) are always denoted by S^n. The spheres of dimensions $n = 0,1,3,7$ are generally denoted by $\mathbb{S}^n$, but in some exceptional cases (either pointed out by V. I. Arnold or, for example, in dealing with the bouquet of spheres $S^2 \vee S^1$) also by S^n. The closed ball of dimension $n \geq 3$ is denoted by B^n. For the two-dimensional ball (disk) the notation D^2 is mainly used. Finally, the one-dimensional ball (line segment) $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ is denoted by $[a;b]$.

We hope the reader appreciates the tough work that we had to perform while preparing this title, and we would like to thank all participants in this project, especially the authors of comments. We are not entirely satisfied by the quality of our own efforts, but our main desire was the early appearance of the book. Many problems have been left without comments; with several exceptions this means only that nobody has undertaken the task to write such a comment so far.

At the same time, we believe that the work on this project is still only at its first steps, and we would be indebted to everybody who will contribute to the next edition of this title with remarks, suggestions, corrections, new comments or historic references.

V. B. Philippov
M. B. Sevryuk

Moscow, 1999

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Et à quoi bien exécuter des projets,
puisque le projet est en lui-même
une jouissance suffisante?

Charles Baudelaire

Le spleen de Paris, XXIV (Les projets)

The Problems

1956

1956-1. "The rumpled dollar problem": is it possible to increase the perimeter of a rectangle by a sequence of foldings and unfoldings?

1958

1958-1. Let us consider a partition of the closed interval $[0;1]$ into three intervals Δ_1, Δ_2, Δ_3 and rearrange them in the order Δ_3, Δ_2, Δ_1. Explore the resulting dynamical system $[0;1] \to [0;1]$: is it true that the mixing rate and similar ergodic characteristics are the same for almost all lengths $(\Delta_1, \Delta_2, \Delta_3)$ of the partition intervals?

An analogous question may be asked for n intervals and for arbitrary permutations as well (changing the orientation of some intervals also being allowed).

1958-2. Let all four faces of a tetrahedron have equal areas. Prove that the lengths of opposite edges are equal (and all faces are congruent!). *The idea is quite simple: cut along three edges from a vertex and develop.*

1958-3. Find a multidimensional version of the Hilbert conjecture on the number of limit cycles of a polynomial vector field. *For instance, one is interested in the number of integral curves connecting two algebraic or invariant manifolds and sufficiently "monotone."*

1959

1959-1. Let the biholomorphic mapping $z \mapsto z + a + b\sin z \mod 2\pi$ of the circle $\operatorname{Im} z = 0$ onto itself be not conjugate analytically to a rotation but have an irrational rotation number. Is it true that in any neighborhood of the circle, there is a periodic orbit?

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