London Mathematical Society
Lecture Note Series 275

Computational and Geometric Aspects of Modern Algebra

Edited by Michael Atkinson, Nick Gilbert, James Howie, Steve Linton & Edmund Robertson

London Mathematical Society

CAMBRIDGE
UNIVERSITY PRESS

London Mathematical Society Lecture Note Series. 275

Computational and Geometric Aspects of Modern Algebra

Edited by

Michael Atkinson
University of St Andrews

Nick Gilbert
Heriot-Watt University

James Howie
Heriot-Watt University

Steve Linton
University of St Andrews

Edmund Robertson
University of St Andrews

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011–4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 2000

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

ISBN 0 521 78889 7 paperback

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor N.J. Hitchin, Mathematical Institute,
University of Oxford, 24–29 St Giles, Oxford OX1 3LB, United Kingdom

The titles below are available from booksellers, or, in case of difficulty, from Cambridge University Press.

46 *p*-adic Analysis: a short course on recent work, N. KOBLITZ
59 Applicable differential geometry, M. CRAMPIN & F.A.E. PIRANI
66 Several complex variables and complex manifolds II, M.J. FIELD
86 Topological topics, I.M. JAMES (ed)
88 FPF ring theory, C. FAITH & S. PAGE
90 Polytopes and symmetry, S.A. ROBERTSON
93 Aspects of topology, I.M. JAMES & E.H. KRONHEIMER (eds)
96 Diophantine equations over function fields, R.C. MASON
97 Varieties of constructive mathematics, D.S. BRIDGES & F. RICHMAN
99 Methods of differential geometry in algebraic topology, M. KAROUBI & C. LERUSTE
100 Stopping time techniques for analysts and probabilists, L. EGGHE
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
107 Compactification of Siegel moduli schemes, C.-L. CHAI
109 Diophantine analysis, J. LOXTON & A. VAN DER POORTEN (eds)
113 Lectures on the asymptotic theory of ideals, D. REES
114 Lectures on Bochner-Riesz means, K.M. DAVIS & Y.-C. CHANG
116 Representations of algebras, P.J. WEBB (ed)
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M.-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR & A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
144 Introduction to uniform spaces, I.M. JAMES
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
148 Helices and vector bundles, A.N. RUDAKOV *et al*
149 Solitons, nonlinear evolution equations and inverse scattering, M. ABLOWITZ & P. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
155 Classification theories of polarized varieties, TAKAO FUJITA
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
171 Squares, A.R. RAJWADE
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M. FOURMAN, P. JOHNSTONE & A. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD *et al*
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
188 Local analysis for the odd order theorem, H. BENDER & G. GLAUBERMAN
189 Locally presentable and accessible categories, J. ADAMEK & J. ROSICKY
190 Polynomial invariants of finite groups, D.J. BENSON
191 Finite geometry and combinatorics, F. DE CLERCK *et al*
192 Symplectic geometry, D. SALAMON (ed)
194 Independent random variables and rearrangement invariant spaces, M. BRAVERMAN
195 Arithmetic of blowup algebras, WOLMER VASCONCELOS

196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI *et al*
198 The algebraic characterization of geometric 4-manifolds, J.A. HILLMAN
199 Invariant potential theory in the unit ball of $\mathbf{C}^n$, MANFRED STOLL
200 The Grothendieck theory of dessins d'enfant, L. SCHNEPS (ed)
201 Singularities, JEAN-PAUL BRASSELET (ed)
202 The technique of pseudodifferential operators, H.O. CORDES
203 Hochschild cohomology of von Neumann algebras, A. SINCLAIR & R. SMITH
204 Combinatorial and geometric group theory, A.J. DUNCAN, N.D. GILBERT & J. HOWIE (eds)
205 Ergodic theory and its connections with harmonic analysis, K. PETERSEN & I. SALAMA (eds)
207 Groups of Lie type and their geometries, W.M. KANTOR & L. DI MARTINO (eds)
208 Vector bundles in algebraic geometry, N.J. HITCHIN, P. NEWSTEAD & W.M. OXBURY (eds)
209 Arithmetic of diagonal hypersurfaces over finite fields, F.Q. GOUVÊA & N. YUI
210 Hilbert C*-modules, E.C. LANCE
211 Groups 93 Galway / St Andrews I, C.M. CAMPBELL *et al* (eds)
212 Groups 93 Galway / St Andrews II, C.M. CAMPBELL *et al* (eds)
214 Generalised Euler-Jacobi inversion formula and asymptotics beyond all orders, V. KOWALENKO *et al*
215 Number theory 1992–93, S. DAVID (ed)
216 Stochastic partial differential equations, A. ETHERIDGE (ed)
217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
218 Surveys in combinatorics, 1995, PETER ROWLINSON (ed)
220 Algebraic set theory, A. JOYAL & I. MOERDIJK
221 Harmonic approximation, S.J. GARDINER
222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
224 Computability, enumerability, unsolvability, S.B. COOPER, T.A. SLAMAN & S.S. WAINER (eds)
225 A mathematical introduction to string theory, S. ALBEVERIO, J. JOST, S. PAYCHA, S. SCARLATTI
226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
228 Ergodic theory of $\mathbf{Z}^d$ actions, M. POLLICOTT & K. SCHMIDT (eds)
229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J.W.S. CASSELS & E.V. FLYNN
231 Semigroup theory and its applications, K.H. HOFMANN & M.W. MISLOVE (eds)
232 The descriptive set theory of Polish group actions, H. BECKER & A.S. KECHRIS
233 Finite fields and applications, S. COHEN & H. NIEDERREITER (eds)
234 Introduction to subfactors, V. JONES & V.S. SUNDER
235 Number theory 1993–94, S. DAVID (ed)
236 The James forest, H. FETTER & B. GAMBOA DE BUEN
237 Sieve methods, exponential sums, and their applications in number theory, G.R.H. GREAVES *et al*
238 Representation theory and algebraic geometry, A. MARTSINKOVSKY & G. TODOROV (eds)
239 Clifford algebras and spinors, P. LOUNESTO
240 Stable groups, FRANK O. WAGNER
241 Surveys in combinatorics, 1997, R.A. BAILEY (ed)
242 Geometric Galois actions I, L. SCHNEPS & P. LOCHAK (eds)
243 Geometric Galois actions II, L. SCHNEPS & P. LOCHAK (eds)
244 Model theory of groups and automorphism groups, D. EVANS (ed)
245 Geometry, combinatorial designs and related structures, J.W.P. HIRSCHFELD *et al*
246 *p*-Automorphisms of finite *p*-groups, E.I. KHUKHRO
247 Analytic number theory, Y. MOTOHASHI (ed)
248 Tame topology and o-minimal structures, LOU VAN DEN DRIES
249 The atlas of finite groups: ten years on, ROBERT CURTIS & ROBERT WILSON (eds)
250 Characters and blocks of finite groups, G. NAVARRO
251 Gröbner bases and applications, B. BUCHBERGER & F. WINKLER (eds)
252 Geometry and cohomology in group theory, P. KROPHOLLER, G. NIBLO, R. STÖHR (eds)
253 The *q*-Schur algebra, S. DONKIN
254 Galois representations in arithmetic algebraic geometry, A.J. SCHOLL & R.L. TAYLOR (eds)
255 Symmetries and integrability of difference equations, P.A. CLARKSON & F.W. NIJHOFF (eds)
256 Aspects of Galois theory, HELMUT VÖLKLEIN *et al*
257 An introduction to noncommutative differential geometry and its physical applications 2ed, J. MADORE
258 Sets and proofs, S.B. COOPER & J. TRUSS (eds)
259 Models and computability, S.B. COOPER & J. TRUSS (eds)
260 Groups St Andrews 1997 in Bath, I, C.M. CAMPBELL *et al*
261 Groups St Andrews 1997 in Bath, II, C.M. CAMPBELL *et al*
263 Singularity theory, BILL BRUCE & DAVID MOND (eds)
264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds)
265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
267 Surveys in combinatorics, 1999, J.D. LAMB & D.A. PREECE (eds)
268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
269 Ergodic theory and topological dynamics of group actions on homogeneous spaces, B. BEKKA & M. MAYER
270 Analysis on Lie Groups, N. T. VAROPOULOS & S. MUSTAPHA
271 Singular perturbations of differential operators, S. ALBERVERIO & P. KURASOV
272 Character theory for the odd order theorem, T. PETERFALVI
273 Spectral theory and geometry, E. B. DAVIES & Y. SAFAROV (eds)
274 The Mandelbrot set, theme and variations, TAN LEI (ed)

FOREWORD

We are pleased to present this selection of articles contributed by participants at a workshop on Computational and Geometric Aspects of Modern Algebra, which took place at Heriot-Watt University, 23-31 July 1998, under the auspices of the International Centre for Mathematical Sciences (ICMS).

The workshop was generously supported by the UK Engineering and Physical Sciences Research Council, with additional financial support from the London Mathematical Society, the Edinburgh Mathematical Society, and Heriot-Watt University. Its organisation was made infinitely smoother by the various assistance of Tracey Dart and Julie Brown of ICMS, and Isobel Johnston and Fiona Paterson of Heriot-Watt.

In the preparation of this volume, we have received invaluable help and advice from Roger Astley and Tamsin van Essen of CUP, numerous anonymous referees, and of course the contributors of the articles. To all of the above we wish to record our gratitude.

M D Atkinson
N D Gilbert
J Howie
S A Linton
E F Robertson
October 1999

CGAMA 98 Participants

I Araujo (St. Andrews)
M D Atkinson (St. Andrews)
R E Arthur (St. Andrews)
L Bartholdi (Geneva)
B Baumeister (Imperial College)
N Billington (Victoria Univ)
S Billington (Warwick)
A Bis (Lodz)
B H Bowditch (Southampton)
N Brady (Cornell)
T Brady (DCU)
M R Bridson (Oxford)
C J B Brookes (Cambridge)
C M Campbell (St. Andrews)
I M Chiswell (QMW)
A Clow (Warwick)
D E Cohen (QMW)
G Conner (BYU)
J S Crisp (Southampton)
D Cruickshank (Glasgow)
R L Curtis (Geneva)
A Cutting (St. Andrews)
M Czarnecki (Lodz)
A Drapal (Prague)
A J Duncan (Newcastle)
M J Dunwoody (Southampton)
M Edjvet (Nottingham)
M Elder (Melbourne)
D B A Epstein (Warwick)
B Everitt (York)
V Felsch (Aachen)
B Fine (Fairfield)
A Fish (Warwick)
R Foord (Warwick)
N D Gilbert (Heriot-Watt)
L Grasselli (Bologna)
R I Grigorchuk (Steklov Inst.)
P Hammond (Nottingham)
J Harlander (Frankfurt)
G Havas (Queensland)
S Hermiller (Las Cruces)
A Heyworth (Bangor)
M P Hitchman (Lewis & Clark)
C Hog-Angeloni (Frankfurt)
D F Holt (Warwick)
S Holub (Prague)
J Howie (Heriot-Watt)
G Huck (Flagstaff)
A Hulpke (St. Andrews)
S P Humphries (BYU)
M R Jerrum (Edinburgh)
D L Johnson (UWI, Mona)
O Kharlampovich (McGill)
J Kortelainen (Oulu)
G Levitt (Toulouse)
S A Linton (St. Andrews)
M Lustig (Bochum)
C Maclachlan (Aberdeen)
K Madlener (Kaiserslautern)
B Makubate (Glasgow)
J Marshall (Warwick)
G Martin (Auckland)
U H M Martin (St. Andrews)
J McCammond (Texas A & M)
V Metaftsis (Univ Aegean)
W Metzler (Frankfurt)
C F Miller (Melbourne)
W D Neumann (Melbourne)
W Nickel (St. Andrews)
J-P Preaux (Marseille)
S J Pride (Glasgow)
A R Prince (Heriot-Watt)
M A Ramirez (Barcelona)
A A Ranicki (Edinburgh)
S E Rees (Newcastle)
A W Reid (Austin)
V N Remeslennikov (Omsk)
K Reynolds (Newcastle)
E F Robertson (St. Andrews)
C Roever (Oxford)

S Rosebrock (Karlsruhe)
G Rosenberger (Dortmund)
A Rosenmann (Tel Aviv)
N Ruskuc (St. Andrews)
M E Sageev (Rutgers)
M V Sapir (Vanderbilt)
Z Sela (Jerusalem)
V Yu Shavrukov (Leicester)
H Short (Marseille)
R Shwartz (Technion)
G C Smith (Bath)
A Solomon (St. Andrews)
E Souche (Marseille)
R Strebel (Fribourg)
V Sushchansky (Gliwice/Kiev)
E Swenson (BYU)
O Talelli (Athens)
R M Thomas (Leicester)
R Thomson (St. Andrews)
E Ventura (Manresa)
K Vogtmann (Cornell)
J Wang (Glasgow)
R Weidmann (Bochum)
A G Williams (Heriot-Watt)

CONTENTS

LIE METHODS IN GROWTH OF GROUPS AND GROUPS OF FINITE WIDTH

LAURENT BARTHOLDI AND ROSTISLAV I. GRIGORCHUK

ABSTRACT. In the first, mostly expository, part of this paper, a graded Lie algebra is associated to every group G given with an N-series of subgroups. The asymptotics of the Poincaré series of this algebra give estimates on the growth of the group G. This establishes the existence of a gap between polynomial growth and growth of type $e^{\sqrt{n}}$ in the class of residually-p groups, and gives examples of finitely generated p-groups of uniformly exponential growth.

In the second part, we produce two examples of groups of finite width and describe their Lie algebras, introducing a notion of *Cayley graph* for graded Lie algebras. We compute explicitly their lower central and dimensional series, and outline a general method applicable to some other groups from the class of branch groups.

These examples produce counterexamples to a conjecture on the structure of just-infinite groups of finite width.

1. INTRODUCTION

The main goal of this paper is to present new examples of groups of finite width and to give a method of proving that some groups from the class of branch groups have finite width. This provides examples of groups of finite width with a completely new origin and answers a question asked by several mathematicians. We also give new examples of Lie algebras of finite width associated to the groups mentioned above.

The first group we study, $\mathfrak{G}$, was constructed in [Gri80] where it was shown to be an infinite torsion group; later in [Gri84] it was shown to be of intermediate growth. The second group, $\tilde{\mathfrak{G}}$, was already considered by the second author in 1979, but was rejected at that time for not being periodic. We now know that it also has intermediate growth [BG98] and finite width.

Our interest in the finite width property comes from the theory of growth of groups. Another important area connected to this property is the theory of finite p-groups and the theory of pro-p-groups; see [Sha95b], [Sha95a, §8]

1991 *Mathematics Subject Classification.* 20F50,20F14,17B50,16P90.

The authors express their thanks to the Swiss National Science Foundation; the second author thanks the Russian Fund for Fundamental Research, research grant 01-00974 for its support.

and [KLP97] with its bibliography. More precisely, the following was discussed by many mathematicians and stated by Zel'manov in Castelvecchio in 1996 [Zel96]:

Conjecture 1.1. *Let G be a just-infinite pro-p-group of finite width. Then G is either solvable, p-adic analytic, or commensurable to a positive part of a loop group or to the Nottingham group.*

Our computations disprove this conjecture by providing a counter-example, the profinite completion of $\mathfrak{G}$ (it is a pro-p-group with $p = 2$). Note that it exhibits a behaviour specific to positive characteristic: indeed it was proved by Martinez and Zel'manov in [MZ99] that unipotence and finite width imply local nilpotence.

Before we give the definition of a group of finite width, let us recall a classical construction of Magnus [Mag40], described for instance in [HB82, Chapter VIII]. Given a group G and $\{G_n\}_{n=1}^{\infty}$ an N-series (i.e. a series of normal subgroups with $G_1 = G$, $G_{n+1} \leq G_n$ and $[G_m, G_n] \leq G_{m+n}$ for all $m, n \geq 1$), there is a canonical way of associating to G a graded Lie ring

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} L_n, \tag{1}$$

where $L_n = G_n/G_{n+1}$ and the bracket operation is induced by commutation in G. Possible examples of N-series are the lower central series $\{\gamma_n(G)\}_{n=1}^{\infty}$; for an integer p, the *lower p-central series* given by $P_1(G) = G$ and $P_{n+1}(G) = P_n(G)^p[P_n(G), G]$; and, for a field $\mathbb{k}$, the series of $\mathbb{k}$*-dimension subgroups* $\{G_n\}_{n=1}^{\infty}$ defined by

$$G_n = \{g \in G | \, g - 1 \in \Delta^n\}, \qquad n = 1, 2, \dots$$

where Δ is the augmentation (or fundamental) ideal of the group algebra $\mathbb{k}[G]$.

Tensoring the $\mathbb{Z}$-modules L_n with a suitable field $\mathbb{k}$, we obtain in (1) a graded Lie algebra $\mathcal{L}_{\mathbb{k}}(G)$. In case the N-series chosen satisfies the additional condition $G_n^p \leq G_{pn}$, and $\mathbb{k}$ is a field of characteristic p, the algebra $\mathcal{L}_{\mathbb{k}}(G)$ will then be a p-algebra or *restricted algebra*; see [Jac41] or [Jac62, Chapter V], the Frobenius operation on $\mathcal{L}_{\mathbb{k}}(G)$ being induced by raising to the power p in G. In this case the quotients G_n/G_{n+1} are elementary p-groups.

Many properties of a group are reflected in properties of its corresponding Lie algebra. For instance, one of the most important results obtained using the Lie method is the theorem of Zel'manov [Zel95a] asserting that if the Lie algebra $\mathcal{L}_{\mathbb{F}_p}(G)$ associated to the dimension subgroups of a finitely generated periodic residually-p group G satisfies a polynomial identity then the group G is finite ($\mathbb{F}_p$ is the prime field of characteristic p). This result gives in fact a positive solution to the Restricted Burnside Problem [VZ93, Zel95b, VZ96, Zel97]. Another example is the criterion of analyticity of pro-p-groups discovered by Lazard [Laz65].

The Lie method also applies to the theory of growth of groups, as was first observed in [Gri89]. There the second author proved that in the class of residually-p groups there is a gap between polynomial growth and growth of type $e^{\sqrt{n}}$. This result was then generalized in [LM91, Theorem D] to the class of residually-nilpotent groups, and in [CG97] the Lie method was also used to prove that certain one-relator groups with exponential-growth Lie algebra $\mathcal{L}_{\Bbbk}(G)$ have uniformly exponential growth. If a group G is finitely generated, then its Lie algebra $\mathcal{L}_{\Bbbk}(G) = \bigoplus L_n \otimes \Bbbk$ is also finitely generated, and the growth of $\mathcal{L}_{\Bbbk}(G)$ is by definition the growth of the sequence $\{b_n = \dim(L_n \otimes \Bbbk)\}_{n=1}^{\infty}$.

The investigation of the growth of graded algebras related to groups has its own interest and is related to other topics. One of the first results in this direction is the Golod-Shafarevich inequality [GS64] which plays an important role in group, number and field theories. The idea of Golod and Shafarevich was used by Lazard in the proof of the aforementioned criterion of analyticity (he even used the notation 'gosha' for the growth of the algebras). Vershik and Kaimanovich observed the relation between the growth of gosha, amenability, and asymptotic behaviour of random walks (see Section 4 below).

For our purposes it will be sufficient to consider only the fields $\mathbb{Q}$ and $\mathbb{F}_p$. Let G_n be the corresponding series of dimension subgroups, which is also an N-series, and let $\mathcal{L}_{\Bbbk}(G)$ be the associated Lie algebra. If $\mathcal{L}_{\Bbbk}(G)$ is of polynomial growth of degree $d \geq 0$, then the growth of G is at least $e^{n^{1-1/(d+2)}}$, and if $\mathcal{L}_{\Bbbk}(G)$ is of exponential growth, then G is of uniformly exponential growth.

If $\Bbbk = \mathbb{Q}$ and G is residually-nilpotent and $b_n = 0$ for some n, then G is nilpotent; indeed G_n must be finite for that n, whence $\gamma_n(G)$ is finite too, and since $\bigcap_{k\geq 1} \gamma_k(G) = 1$ this implies that $\gamma_N(G) = 1$ for some N. It follows that G has polynomial growth [Mil68]. In fact polynomial growth is equivalent to virtual nilpotence [Gro81a].

If $\Bbbk = \mathbb{F}_p$ and G is a residually-p group and $b_n = 0$ for some n, then G is a linear group over a field, by Lazard's theorem [Laz65] and therefore has either polynomial or exponential growth, by the Tits alternative [Tit72].

Finally, if $b_n \geq 1$ for all n then, independent of $\Bbbk$, the growth of G is at least $e^{\sqrt{n}}$. Keeping in mind that polynomial growth $b_n \sim n^d$ of $\mathcal{L}_{\Bbbk}(G)$ implies a lower bound $e^{n^{1-1/(d+2)}}$ for the growth of G, we conclude that examples of groups with growth exactly $e^{\sqrt{n}}$ are to be found amongst the class of groups for which the sequence $\{b_n\}_{n=1}^{\infty}$ is uniformly bounded, or at least bounded in average. This key observation leads to the notion of *groups of finite width.* We present two versions of the definition:

Definition 1.2. Let G be a group and $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$ a field. If $\Bbbk = \mathbb{Q}$, assume G is residually-nilpotent; if $\Bbbk = \mathbb{F}_p$, assume G is residually-p.

1. G has *finite C-width* if there is a constant K with $[\gamma_n(G) : \gamma_{n+1}(G)] \leq K$ for all n.

2. G has *finite D-width with respect to* $\Bbbk$ if there is a constant K with $b_n \leq K$ for all n, where $\{b_n\}_{n=1}^{\infty}$ is the growth of $\mathcal{L}_{\Bbbk}(G)$ constructed from the dimension subgroups.

A third notion can be defined, that of *finite averaged width*; see [Gri89] or [KLP97, Definition I.1.ii]. From our point of view D-width is more natural; but the first notion is more commonly used in the theory of finite p-groups and pro-p-groups, see for instance [KLP97, Definition I.1.i]. The examples we will produce are of finite width according to both definitions. That one of our groups has finite width was conjectured in [Gri89]; it was proven that the numbers b_n are bounded in average. Rozhkov then confirmed this conjecture in [Roz96a] by computing explicitly the b_n; but the proof had gaps, one of which was filled in [Roz96b]. We fix another gap in the "Technical Lemma 4.3.2" of [Roz96b] while simplifying and clarifying Rozhkov's proof, and also outline a general method, connected to ideas of Kaloujnine [Kal46].

We recall in the next section known notions on algebras associated to groups, and construct in Section 3 a torsion group of uniformly exponential growth. Section 5 describes a class of groups acting on rooted trees, and the next two sections detail for two specific examples the indices of the lower central and dimensional series. More specifically, we compute in Theorem 6.6 and 7.6 the indices of these series for the group $\mathfrak{G}$ and an overgroup $\widetilde{\mathfrak{G}}$. We also obtain in the process the structure of the Lie algebras $L(\mathfrak{G})$ (associated to the lower central series) and $\mathcal{L}_{\mathbb{F}_2}(\mathfrak{G})$ (associated to the dimension series) in Theorem 6.7, and that of $L(\widetilde{\mathfrak{G}})$ and $\mathcal{L}_{\mathbb{F}_2}(\widetilde{\mathfrak{G}})$ in Theorem 7.7. They are described using Cayley graphs of Lie algebras, see Subsection 6.1.

Throughout this paper groups shall act on the left. We use the notational conventions $[x,y] = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$.

Both authors wish to thank Aner Shalev and Efim Zelmanov for their interest and generous contribution through discussions.

2. Growth of Groups and Associated Graded Algebras

Let G be a group, $\{\gamma_n(G)\}_{n=1}^{\infty}$ the lower central series of G, $\Bbbk \in \{\mathbb{Q}, \mathbb{F}_p\}$ a prime field, $\Delta = \ker(\varepsilon) < \Bbbk[G]$ the augmentation ideal, where $\varepsilon(\sum k_i g_i) = \sum k_i$ is the augmentation map $\Bbbk[G] \to \Bbbk$, and $\{G_n\}_{n=1}^{\infty}$ the series of dimension subgroups of G [Zas40, Jen41]. Recall that

$$G_n = \{g \in G \,|\, g - 1 \in \Delta^n\}.$$

The restrictions we impose on $\Bbbk$ are not important, as G_n depends only on the characteristic of $\Bbbk$. We suppose throughout that G is residually-nilpotent if $\Bbbk = \mathbb{Q}$ and is residually-p if $\Bbbk = \mathbb{F}_p$.

If $\Bbbk = \mathbb{Q}$, then G_n is the isolator of $\gamma_n(G)$, as was proved in [Jen55] (see also [Pman77, Theorem 11.1.10] or [Pas79, Theorem IV.1.5]); i.e.

$$G_n = \sqrt{\gamma_n(G)} = \{g \in G \,|\, g^{\ell} \in \gamma_n(G) \text{ for an } \ell \in \mathbb{N}\}.$$

Note that in [Pman77] these results are stated for finite p-groups. They nevertheless hold in the more general setting of residually-nilpotent or residually-p groups.

If $\Bbbk = \mathbb{F}_p$, then $\gamma_n(G) \leq G_n \leq \sqrt{\gamma_n(G)}$, and the G_n can be defined in several different ways, for instance by the relation

$$G_n = \prod_{i \cdot p^j \geq n} \gamma_i^{p^j}(G)$$

due to Lazard [Laz53], or recursively as

$$G_n = [G, G_{n-1}] G^p_{\lceil n/p \rceil}, \tag{2}$$

where $\lceil n/p \rceil$ is the least integer greater than or equal to n/p. In characteristic p, the series $\{G_n\}_{n=1}^{\infty}$ is called the lower p-central, Brauer, Jennings, Lazard or Zassenhaus series of G. The quotients G_n/G_{n+1} are elementary abelian p-groups and define the fastest-decreasing central series with this property [Jen55].

Let

$$\mathcal{A}(G) = \mathcal{A}_{\Bbbk}(G) = \bigoplus_{n=0}^{\infty} \Delta^{n+1}/\Delta^n$$

be the associative graded algebra with product induced linearly from the group product (see [Pman77, Pas79] for more details).

If $\Bbbk = \mathbb{Q}$, consider the following graded Lie algebras over $\Bbbk$:

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} (G_n/G_{n+1} \otimes_{\mathbb{Z}} \mathbb{Q}), \qquad L(G) = \bigoplus_{n=1}^{\infty} (\gamma_n(G)/\gamma_{n+1}(G) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

If $\Bbbk = \mathbb{F}_p$, consider the restricted Lie $\mathbb{F}_p$-algebra

$$\mathcal{L}_p(G) = \bigoplus_{n=1}^{\infty} (G_n/G_{n+1}).$$

Then Quillen's Theorem [Qui68] asserts that $\mathcal{A}(G)$ is the universal enveloping algebra of $\mathcal{L}(G)$ in characteristic 0 and is the universal p-enveloping algebra of $\mathcal{L}_p(G)$ in positive characteristic.

Let us introduce the following numbers:

$$a_n(G) = \dim_{\Bbbk}(\Delta^n/\Delta^{n+1}), \qquad b_n(G) = \operatorname{rank}(G_n/G_{n+1}).$$

Here by the rank of the G-module M we mean the torsion-free rank $\dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$ in characteristic 0 and the p-group rank $\dim_{\mathbb{F}_p}(M \otimes \mathbb{F}_p)$, equal to the minimal number of generators, in positive characteristic. Note that in zero-characteristic $b_n = \operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))$, because the natural map

$$\gamma_n(G)/\gamma_{n+1}(G) \to G_n/G_{n+1}$$

has finite kernel and cokernel.

The following result is due to Jennings. The case $\Bbbk = \mathbb{F}_p$ appears in [Jen41] and the case $\Bbbk = \mathbb{Q}$ appears in [Jen55]; but see also [Pman77, Theorem 3.3.6 and 3.4.10].

$$\sum_{n=0}^{\infty} a_n(G)t^n = \begin{cases} \prod_{n=1}^{\infty} \left(\frac{1-t^{pn}}{1-t^n}\right)^{b_n(G)} & \text{if } \Bbbk = \mathbb{F}_p, \\ \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n(G)} & \text{if } \Bbbk = \mathbb{Q}. \end{cases} \tag{3}$$

The series $\sum_{n=0}^{\infty} a_n(G)t^n$ is the Hilbert-Poincaré series of the graded algebra $\mathcal{A}(G)$. The equation (3) expresses this series in terms of the numbers $b_n(G)$; the relation between the sequences $\{a_n(G)\}_{n=0}^{\infty}$ and $\{b_n(G)\}_{n=1}^{\infty}$ is quite complicated. We shall be interested in asymptotic growth of series, in the following sense:

Definition 2.1. Let f and g be two functions $\mathbb{R}_+ \to \mathbb{R}_+$. We write $f \precsim g$ if there is a constant $C > 0$ such that $f(x) \leq C + Cg(Cx + C)$ for all $x \in \mathbb{R}_+$, and write $f \sim g$ if $f \precsim g$ and $g \precsim f$.

A series $\{a_n\}_{n=0}^{\infty}$ defines a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = a_{\lfloor x \rfloor}$, and for two series $a = \{a_n\}$ and $b = \{b_n\}$ we write $a \precsim b$ and $a \sim b$ when the same relations hold for their associated functions.

The main facts are presented in the following statement:

Proposition 2.2. *Let $\{a_n\}$ and $\{b_n\}$ be connected by the one of the relations (3). Then*

1. *$\{b_n\}$ grows exponentially if and only if $\{a_n\}$ does, and we have*

$$\limsup_{n\to\infty} \frac{\ln a_n}{n} = \limsup_{n\to\infty} \frac{\ln b_n}{n}.$$

2. *If $b_n \sim n^d$ then $a_n \sim e^{n^{(d+1)/(d+2)}}$.*

Proof. We first suppose $\Bbbk = \mathbb{Q}$, and prove Part 1 following [Ber83]. Let $A = \limsup(\ln a_n)/n$ and $B = \limsup(\ln b_n)/n$. Clearly $A \geq B$ as $a_n \geq b_n$ for all n; we now prove that $A \leq B$. Define

$$f(z) = \prod_{n=1}^{\infty} (1 - e^{-nz})^{-b_n},$$

viewed as a complex analytic function in the half-plane $\Re(z) > B$. We have $|1 - e^{-nz}|^{-1} \leq (1 - e^{-n\Re z})^{-1}$, from which $|f(z)| \leq f(\Re z)$. Now applying the Cauchy residue formula,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u+iv)e^{n(u+iv)}dv \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(u+iv)|e^{nu}dv \leq e^{nu}f(u)$$

for all $u > B$, so

$$A = \limsup_{n\to\infty} \frac{\ln a_n}{n} \leq \limsup_{u>B, n\to\infty} \left(u + \frac{\ln f(u)}{n}\right) = B.$$

For $\Bbbk = \mathbb{F}_p$, Part 1 holds *a fortiori.*

Part 2 for $\Bbbk = \mathbb{Q}$ is a consequence of a result by Meinardus ([Mei54]; see also [And76, Theorem 6.2]). More precisely, when $b_n = n^d$, his result implies that

$$a_n \approx \frac{e^{\zeta'(-d)}}{\sqrt{2\pi(d+2)n}} \left(\frac{(d+1)!\zeta(d+2)}{n}\right)^{\frac{1-2\zeta(-d)}{2+4d}} e^{n\frac{d+2}{d+1}\left(\frac{(d+1)!\zeta(d+2)}{n}\right)^{\frac{1}{1+2d}}},$$

where '$\approx$' means that the quotient tends to 1 as $n \to \infty$, and ζ is the Riemann zeta function.

We sketch the proof for $\Bbbk = \mathbb{Q}$ below: we suppose that $b_n \sim n^d$, so $A = B = 0$ by Part 1, and compute

$$\begin{aligned}\frac{d}{du}\ln f(u) = \sum_{n=1}^{\infty} -b_n \frac{-ne^{-nu}}{1-e^{-nu}} &\sim \frac{1}{u^{d+2}} \sum_{n=1}^{\infty} \frac{(nu)^{d+1}}{e^{nu}-1} u \\ &\sim \frac{1}{u^{d+2}} \int_0^{\infty} \frac{w^{d+1}}{e^w - 1} dw = \frac{C}{u^{d+2}}.\end{aligned}$$

Thus $\ln f(u) \sim C/u^{d+1}$, and the inequality

$$\log a_n \leq nu + \log f(u) \sim nu + C/u^{d+1}$$

is tight by the saddle-point principle when the right-hand side is minimized. This is done by choosing $u = n^{-1/(d+2)}$, whence as claimed $\log a_n \sim n^{1-1/(d+2)}$.

Finally, we show that (3) yields the same asymptotics when $\Bbbk = \mathbb{F}_p$ as when $\Bbbk = \mathbb{Q}$. Clearly

$$\prod_{n=1}^{\infty}(1+t^n)^{b_n} \leq \prod_{n=1}^{\infty}(1+t^n+\cdots+t^{(p-1)n})^{b_n} \leq \prod_{n=1}^{\infty}(1+t^n+\dots)^{b_n}$$

for all $p \geq 2$, where for two power series $\sum e_t^n$ and $\sum f_n t^n$ the inequality $\sum e_t^n \leq \sum f_n t^n$ means that $e_n \leq f_n$ for all n. It thus suffices to consider the case $p = 2$. For this purpose define

$$g(z) = \prod_{n=1}^{\infty}(1+e^{-nz})^{b_n},$$

and compare the series developments of $\log(f)$ and $\log(g)$ in e^{-z}. From $-\log(1-z) = \sum_{n\geq 1} \frac{z^n}{n}$ it follows that

$$\log f(z) = \sum_{n\geq 1} f_n e^{-nz}, \quad f_n = \sum_{d|n} \frac{1}{d},$$

$$\log g(z) = \sum_{n\geq 1} g_n e^{-nz}, \quad g_n = \sum_{d|n} \frac{(-1)^{d+1}}{d},$$

so both series have the same odd-degree coefficients, and thus $\log f \sim \log g$. Their exponentials then have the same asymptotics; more precisely, $f_n \leq g_{2n-1}$ for all n, so $e^z \log f(2z) \leq \log g(z)$ termwise, and $f(2z) \leq g(z)$. □

2.1. **Growth of Groups.** Let G be a finitely generated group with a fixed semigroup system S of generators (i.e. such that every element $g \in G$ can be expressed a product $g = s_1 \dots s_n$ for some $s_i \in S$). Let $\gamma_G^S(n)$ be the growth function of (G, S); recall that it is

$$\gamma_G^S(n) = \#\{g \in G \mid |g| \leq n\},$$

where $|g|$ is the minimal number of generators required to express g as a product.

The following observations are well-known:

Lemma 2.3. *Let G be a group and consider two finite generating sets S and T. Then $\gamma_G^S \sim \gamma_G^T$, with $\sim$ given in Definition 2.1.*

It is then meaningful to consider the *growth* γ_G of G, which is the $\sim$-equivalence class containing its growth functions γ_G^S.

Lemma 2.4. *Let G be a finitely generated group, $H < G$ a finitely generated subgroup and K a quotient of G. Then $\gamma_H \precsim \gamma_G$ and $\gamma_K \precsim \gamma_G$.*

Proof. Let S be a finite generating set for H; choose a generating set $T \supset S$ for G. Apply Definition 2.1 with $C = 1$ to obtain $\gamma_H^S \precsim \gamma_G^T$. Clearly $\gamma_K^T(n) \leq \gamma_G^T(n)$ for all n. □

Lemma 2.5 ([Gri89]). *For any field $\Bbbk$ and any group G with generating set S the inequalities $a_n(G) \leq \gamma_G^S(n)$ hold for all $n \geq 0$.*

Proof. Fix a generating set S. The identities

$$xy-1 = (x-1)+(y-1)+(x-1)(y-1), \qquad x^{-1}-1 = -(x-1)-(x-1)(x^{-1}-1)$$

show that

$$xy - 1 \equiv (x-1) + (y-1), \qquad x^{-1} - 1 \equiv -(x-1) \mod \Delta^2,$$

so Δ^n is generated over $\Bbbk$ by Δ^{n+1} and elements of the form

$$x_0(s_1 - 1)x_1(s_2 - 1) \dots (s_n - 1)x_n,$$

for all $s_i \in S$ and $x_i \in \Bbbk[G]$. Now $x_i \equiv \varepsilon(x_i) \in \Bbbk$ modulo Δ, so Δ^n/Δ^{n+1} is spanned by the

$$(s_1 - 1)(s_2 - 1) \dots (s_n - 1), \qquad s_i \in S.$$

All these elements are in the vector subspace S_n of $\Bbbk[G]$ spanned by products of at most n generators, and by definition S_n is of dimension $\gamma_G^S(n)$. □

Corollary 2.6. $\{a_n(G)\}_{n=0}^{\infty} \precsim \gamma_G$.

Combining Proposition 2.2 and Lemma 2.5, we obtain as

Corollary 2.7. *If there exist $C > 0$ and $d \geq 0$ such that $b_n \geq Cn^d$ for all n, then $\gamma_G(n) \succsim e^{1-1/(d+2)}$. In particular, if $b_n \neq 0$ for all n, then $\gamma_G(n) \succsim e^{\sqrt{n}}$.*

We shall say a group G is of *subradical growth* if $\gamma_G \not\succsim e^{\sqrt{n}}$.