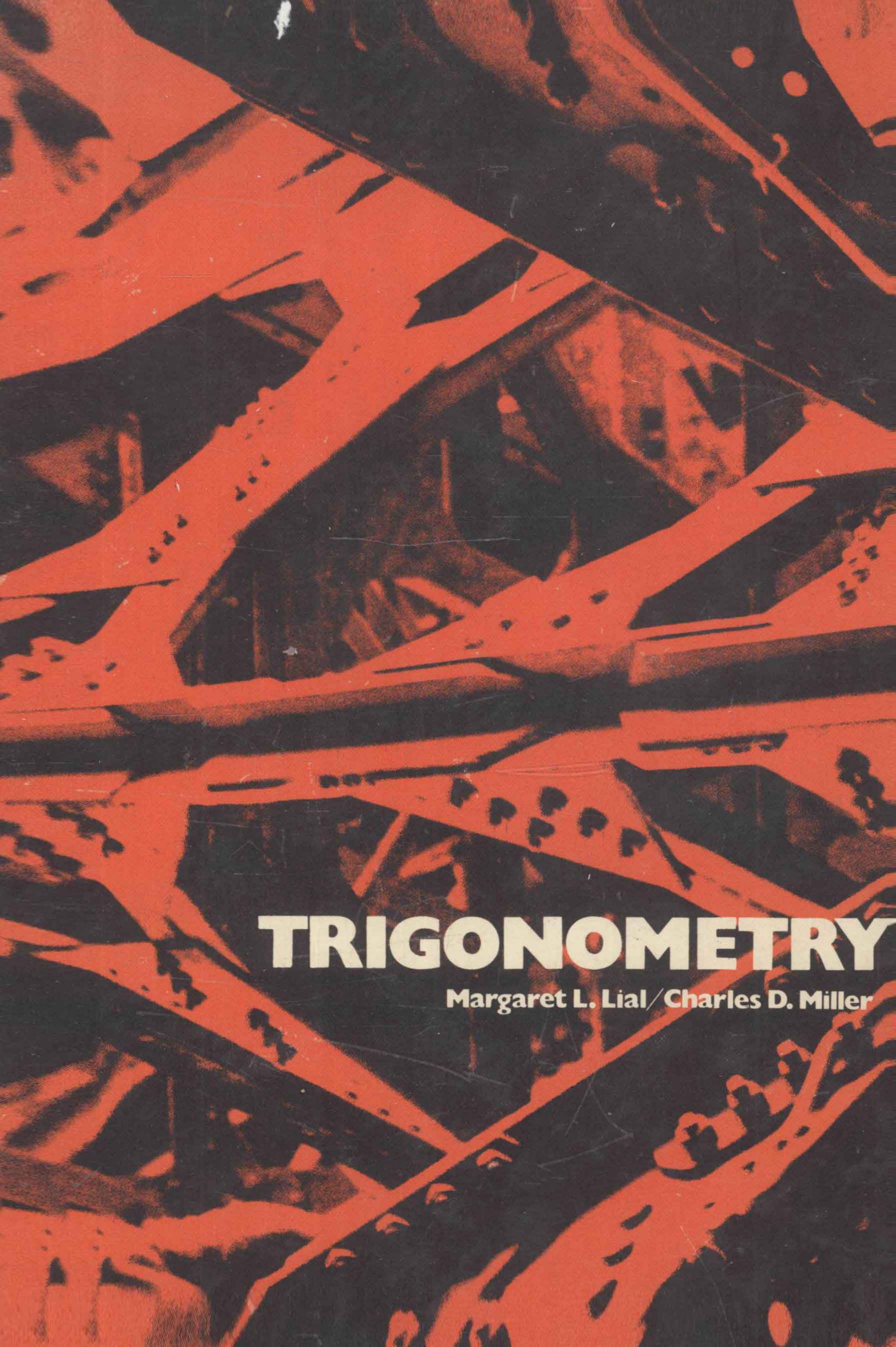
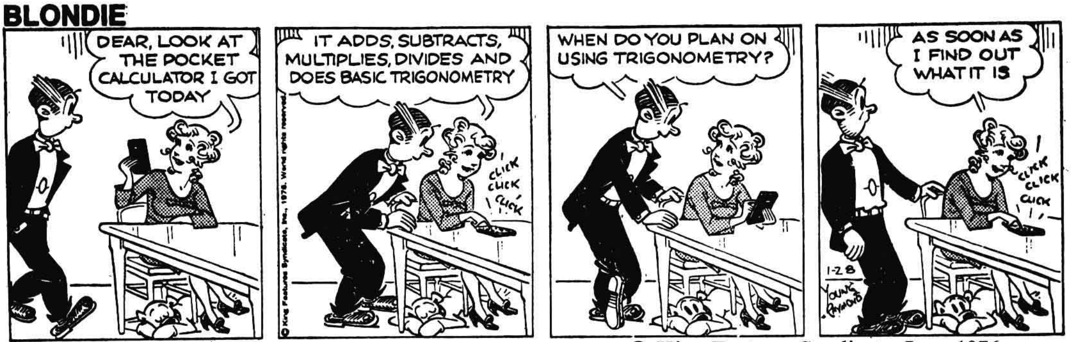


TRIGONOMETRY

Margaret L. Lial / Charles D. Miller



TRIGONOMETRY

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American River College
Sacramento, California

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PREFACE

Trigonometry is designed for the standard one-semester or one- or two-quarter course in the fundamental ideas needed for success in later mathematical work or in technical fields. Trigonometry students have diverse goals. Some plan to become engineers or chemists. Others are technical students. A growing number will go into biological science. Because of the diverse population of students that the course must serve, we have been careful to include many different types of applications throughout the book.

We assume a background of intermediate algebra. A course in geometry is a desirable prerequisite, but many students today take trigonometry with little or no geometry background. For this reason, all necessary geometric ideas are explained as needed.

Triangles are presented early. We show how triangles are used to obtain the trigonometric functions for acute angles, and then show some applications of trigonometry. This quickly gives the student a feel for the usefulness of trigonometry.

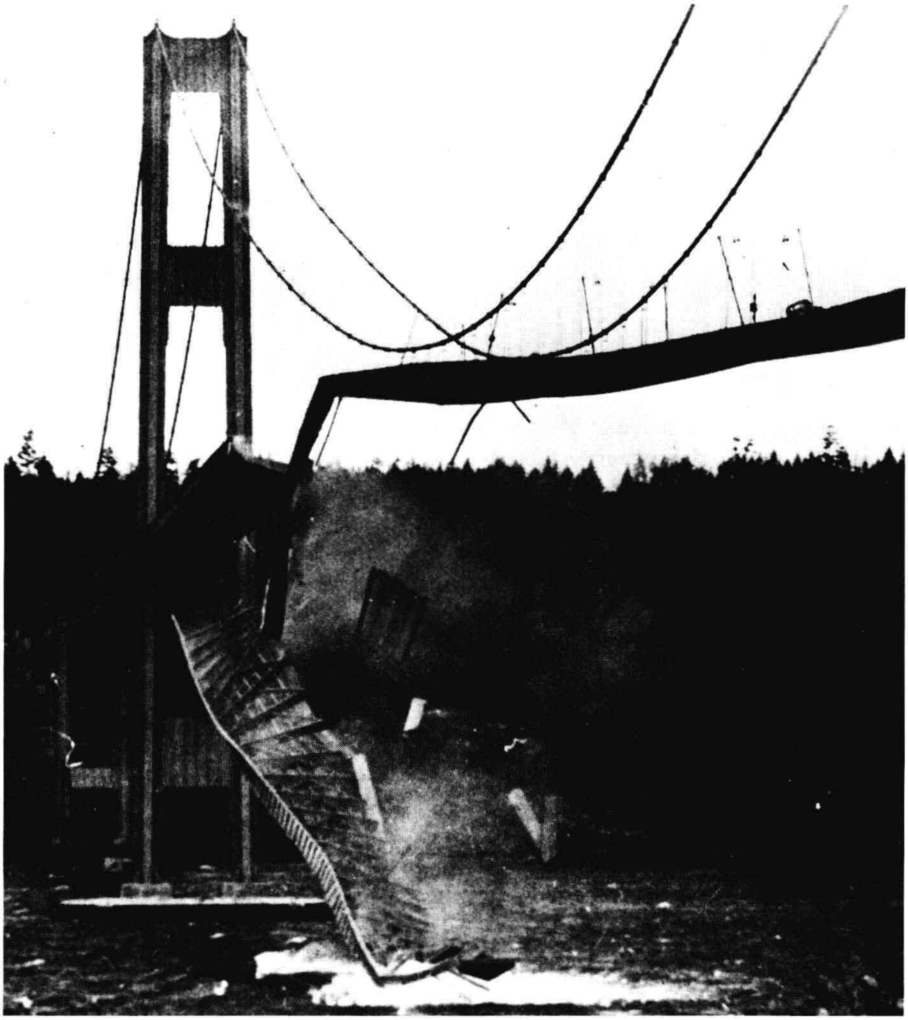
The book contains 200 examples and 2300 exercises. Because of this abundance, it normally suffices to assign only the odd numbered exercises, or perhaps only the multiples of 3 or 4.

Each chapter ends with a summary of the key ideas and formulas of that chapter, and with a *chapter test* which helps students check their understanding of the ideas of the chapter.

Many people have helped us create this book. Vern Heeren, of American River College, cleared up many points. Much helpful advice was also received from Gary R. Penner, Richland College; John C. Lanz, Harrisburg Area Community College; Andrew L. Leake and LeRoy Stoldt, College of DuPage; and Joseph J. Staley, Daytona Beach Community College.

Scott, Foresman's excellent editorial staff—Robert Runck, Pamela Conaghan, and Lynn Lawson—provided the impetus and assistance necessary to produce the book.

Margaret L. Lial
Charles D. Miller



Trigonometry has many practical uses. One use is in the design of suspension bridges. These bridges are subject to periodic vibrations which, if the bridges are not designed correctly, may cause their collapse. One such bridge in Europe collapsed from the vibrations caused by a troop of soldiers marching across it.

The most famous bridge collapse was that of the Tacoma Narrows Bridge, in Washington state. Four months after it was opened in 1940, *this bridge collapsed in a moderately strong wind. Previous vibrations of the bridge had given warning of the collapse, so that no one was hurt. (The occupant of the car which can be seen on the bridge crawled to safety.) After the collapse, government researchers worked out the complete mathematical theory of suspension bridges to prevent future collapses.*

CONTENTS

3 Radian Measure and Circular Functions 63

4 Graphs of Trigonometric Functions 84

5 Trigonometric Identities 106

6 Trigonometric Equations and Inverse Trigonometric Functions 148

7 Logarithms 169

8 Triangles and Vectors 206

test on 6 and 7

1

The Trigonometric Functions

Trigonometry is over 3000 years old. The ancient Egyptians, Babylonians, and Greeks developed trigonometry to find the lengths of the sides of triangles and the measures of their angles. For example, in Egypt trigonometry was used to re-establish land boundaries after the annual flood of the Nile River. In Babylonia it was used in astronomy.

Today trigonometry is used in electronics, surveying, and other engineering areas, and in calculus. This course prepares students for further work in all these areas.

We discuss in the first section some of the basic ideas needed to study trigonometry. In the second section we discuss angles, and in the rest of the chapter we define the basic trigonometric functions.

1.1 Basic Terms

Number lines are basic to all of mathematics. To make a **number line,** draw a straight horizontal line. Choose any point on the line and label it 0. Now choose a point to the right of 0 and label it 1. The distance from 0 to 1 gives a unit measure which can be used to find points representing other numbers, labeled 2, 3, 4, 5, . . . , going to the right, and -1, -2, -3, -4, . . . , to the left. A number line and points representing several different numbers are shown in Figure 1.1.

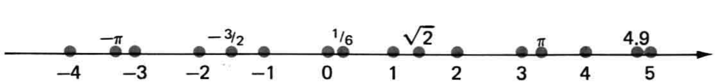


Figure 1.1

The set of all numbers which represent points on a number line is called the set of **real numbers.** The numbers shown in Figure 1.1 are examples of real numbers.

The distance between a point on the number line and 0 is called the **absolute value** of the number represented by the point. The distance from 0 to -6 is 6, so the absolute value of -6 is 6. We write absolute value using vertical bars, so that

$$|-6| = 6.$$

Also,

$$|5| = 5.$$

Example 1 Find each of the following.

(a) $|-\pi| = \pi$

(b) $|-6+8| = |2| = 2$

(c) $-|2-(-9)| - |-4| = -|11| - |-4|$
$= -11 - 4$
$= -15$
(Work first inside each pair of absolute value bars.)

(d) $|0| = 0$ ■

We can give a more formal definition of absolute value as follows.

$$|x| = \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$$

(Recall: $\geqslant$ means "greater than or equal to" and $<$ means "less than.")

A pair of numbers inside parentheses, such as $(-2, 4)$, is called an **ordered pair.** In the ordered pair $(-2, 4)$, -2 is called the *first component*, or *first element*, and 4 is called the *second component*, or *second element*.

A real number, such as -4, can be represented by a point on a number line. An ordered pair of numbers, such as $(-2, 4)$ can be represented in a coordinate system. To make a **coordinate system,** use two number lines, called *axes*, which are placed at right angles. See Figure 1.2. To locate the point $(-2, 4)$, start at the point O, called the **origin**. Because the first component is a negative number, -2, go 2 units to the left along the horizontal number line, or ***x*-axis.** Because the second component is a positive number, $+4$, turn and go 4 units up, parallel to the vertical number line, or ***y*-axis.** The point $(-2, 4)$ and other sample points are shown in Figure 1.2.

The axes divide the coordinate system into four regions called **quadrants.** The quadrants are numbered in a counterclockwise direction, as shown in Figure 1.3. The points on the coordinate axes themselves belong to none of the quadrants.

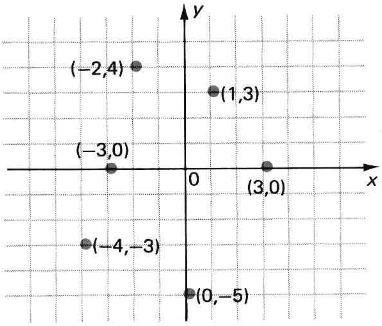


Figure 1.2

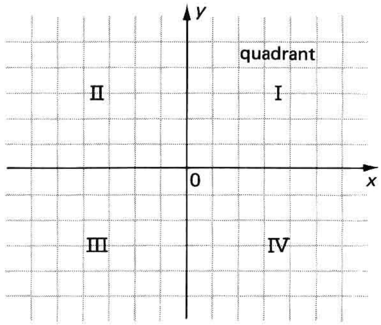


Figure 1.3

Example 2 In which quadrants do the points in Figure 1.4 lie?

(a) $(-3, 2)$ is in quadrant II

(b) $(-4, -2)$ is in quadrant III

(c) $(3, -5)$ is in quadrant IV

(d) $(-6, 0)$ is on the negative x-axis and thus is not in any quadrant

(e) $(0, 5)$ is in no quadrant ■

Given two points in the coordinate system, we often need to find the distance between them. To do this, we first need to know the **Pythagorean theorem** of geometry.

If the two shorter sides of a right triangle have lengths a and b and the length of the longest side, the hypotenuse, is c, then

$$a^2 + b^2 = c^2.$$

Figure 1.5 shows a right triangle with sides a, b, and c (the hypotenuse).

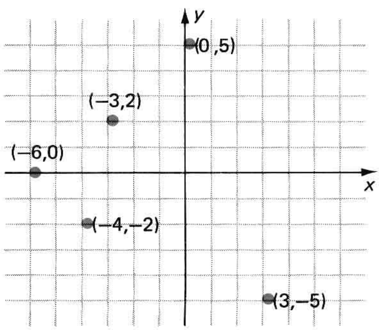


Figure 1.4

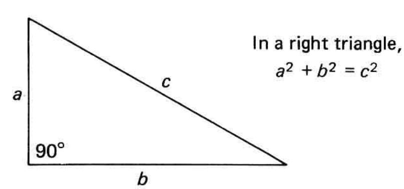


Figure 1.5

The coordinate system in Figure 1.6 shows the points (x_1, y_1) and (x_2, y_2). Construct a right triangle by drawing a line through (x_1, y_1) parallel to the x-axis and through (x_2, y_2) parallel to the y-axis. The ordered pair at the right angle of this triangle is (x_2, y_1).

The horizontal side of the right triangle in Figure 1.6 has length $x_2 - x_1$, while the vertical side has length $y_2 - y_1$. We now use the Pythagorean theorem to find the length of the hypotenuse. (This gives us the distance between the two original points.)

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This result is called the **distance formula.**

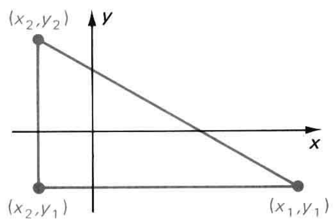


Figure 1.6

Example 3 Use the distance formula to find the distance, d, between each of the following pairs of points.

(a) (2, 6) and (5, 10)

We can use either point as (x_1, y_1). If we choose (2, 6) as (x_1, y_1), then we have $x_1 = 2$, $y_1 = 6$, $x_2 = 5$, and $y_2 = 10$. Thus,

$$\begin{aligned} d &= \sqrt{(5-2)^2 + (10-6)^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ d &= 5. \end{aligned}$$

(b) (−7, 2) and (3, −8)

Here

$$\begin{aligned} d &= \sqrt{[3 - (-7)]^2 + (-8-2)^2} \\ &= \sqrt{10^2 + (-10)^2} \\ &= \sqrt{100 + 100} \\ &= \sqrt{200} \\ d &= 10\sqrt{2}. \end{aligned}$$

(Notice that $\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$.) ■

A **relation** is a set of ordered pairs. Almost all relations have a rule or formula showing the connection between the two components of the ordered pairs. For example, the formula

$$y = -5x + 6$$

shows that we can get a value of y from a given value of x by multiplying x by -5 and then adding 6 to the result. For example, if $x = 2$, then $y = -5 \cdot 2 + 6 = -4$, so that $(2, -4)$ belongs to the relation defined by this formula. In the relation $y = -5x + 6$, the variable y depends on the value of x, so that y is the **dependent variable** and x is the **independent variable.**

Most of the relations in trigonometry are also functions. A **function** is a relation in which each value of x gives exactly one value of y. For example, $y = -5x + 6$ is a function. If we choose one value of x, then $y = -5x + 6$ gives us exactly one value of y. On the other hand, $y^2 = x$ is a relation that is not a function. If we choose $x = 16$, then $y^2 = x$ becomes $y^2 = 16$, from which $y = 4$ or $y = -4$. The one x-value, 16, leads to *two* y-values, 4 and -4. Thus, $y^2 = x$ is not a function.

Functions are often named with letters such as f, g, or h. For example, the function $y = -5x + 6$ can be written as

$$f(x) = -5x + 6,$$

where $f(x)$ is read "f of x." The $f(x)$ notation is used to show that x is the independent variable. For the function $f(x) = -5x + 6$, if $x = 2$ then $f(x) = -5 \cdot 2 + 6 = -10 + 6 = -4$. This can be written as

$$f(2) = -4.$$

Also,

$$f(-6) = -5 \cdot (-6) + 6$$
$$f(-6) = 36.$$

Example 4 Let $f(x) = -x^2 + |x + 5|$. Find each of the following.

(a) $f(0)$

Use $f(x)$ and replace x with 0.

$$f(0) = -0^2 + |0 + 5|$$
$$f(0) = 5$$

(b) $f(-4) = -(-4)^2 + |-4 + 5|$
$= -16 + |1|$
$f(-4) = -15$

(c) $f(a) = -a^2 + |a + 5|$ (Replace x with a.)

(d) Is f a function?

For each value of x, there is exactly one value of $f(x)$. Thus f is a function. ■

For a relation to be a function, for each value of x we must be able to find exactly one value of y. Figure 1.7 shows the graph of a relation. A point x_1 has been chosen on the x-axis. When we draw a vertical line through x_1, the line cuts the graph in more than one point. Thus, for the x-value x_1 we have more than one y-value, so this relation is not a function. In summary, we have the **vertical line test for a function.**

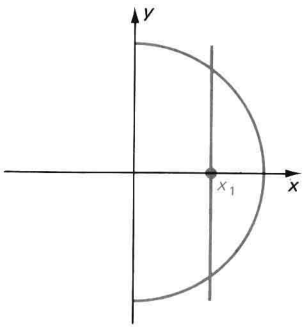


Figure 1.7

If any vertical line cuts the graph of a relation in more than one point, then the relation is not a function.

The set of all possible values that can be used as a replacement for the independent or x variable is called the **domain** of the relation. The set of all possible values for the dependent or y variable is called the **range** of the relation.

Example 5 Find the domain and range for each of the following. Is each of the following a function?

(a) $y = x^2$

Here x can take on any value, so the domain is the set of all real numbers. Since y equals the square of x, and a square is never negative, the range is the set of all nonnegative numbers, $y \geqslant 0$. Here, each value of x leads to exactly one value of y, so $y = x^2$ is a function.

(b) $3x + 2y = 6$

In this function x and y can take on any value at all. Thus, both the domain and the range are the set of all real numbers. For any value of x that we might choose, we could find exactly one value of y. Thus, $3x + 2y = 6$ is a function.

(c) $x = y^2 + 2$

Since $y^2 \geqslant 0$ for all values of y, we have $x = y^2 + 2 \geqslant 0 + 2 = 2$. Thus the domain of the relation is $x \geqslant 2$. Any real number can be squared, so the range is the set of all reals. If we choose the single real number 6 for x, we get

$$6 = y^2 + 2$$
$$4 = y^2$$
$$y = 2 \qquad \text{or} \qquad y = -2.$$

Since one x-value, 6, leads to two y-values, 2 and -2, the relation $x = y^2 + 2$ is not a function. ■

1.1 EXERCISES

Find each of the following. (*See Example* 1.)

1. $|-6| + 2$

2. $-|-2| + 4$

3. $-|8| + |-3|$

4. $-|12| + |-9|$

5. $-|-8| - |7|$

6. $-|-3| - |-2|$

7. $-|-8-9| + 4$

8. $-|16-(-2)| - 3$

9. $|-14-(-3)| - |-2+1|$

10. $|-8-(-9)| - |-4-(-2)|$

Find the quadrant in which each of the following points lie. (*See Example* 2.)

11. $(-4, 2)$
12. $(-3, -5)$
13. $(-9, -11)$
14. $(8, -5)$
15. $(9, 0)$
16. $(-2, 0)$
17. $(0, -2)$
18. $(0, 6)$
19. $(-5, \pi)$*
20. $(\pi, -3)$

Suppose that the point (x, y) *is, in turn, in each of the following quadrants. Decide if* x *is positive or negative and if* y *is positive or negative.*

21. quadrant I **22.** II **23.** III **24.** IV

Suppose r *is a positive number and the point* (x, y) *is in the indicated quadrant. Decide if the given ratio is positive or negative.*

Example Quadrant II, ratio y/x. In quadrant II, x is negative and y is positive. Thus, the quotient y/x is negative. ■

---

* π is the ratio of the circumference to the diameter of a circle. π is approximately 3.14159.

25. II, y/r **28.** III, x/r **31.** IV, x/r **33.** IV, y/x

26. II, x/r **29.** III, y/x **32.** IV, y/r **34.** IV, x/y

27. III, y/r **30.** III, x/y

Find the distance between each of the following pairs of points. (*See Example* 3.)

35. $(-2, 7)$ and $(1, 4)$

36. $(8, -2)$ and $(4, -5)$

37. $(2, 1)$ and $(-3, -4)$

38. $(-5, 2)$ and $(3, -7)$

39. $(-1, 0)$ and $(-4, -5)$

40. $(-2, -3)$ and $(-6, 4)$

41. $(3, -7)$ and $(-2, -5)$

42. $(-5, 8)$ and $(-3, -7)$

43. $(-3, 6)$ and $(-3, 2)$

44. $(5, -2)$ and $(5, -4)$

45. $(7, -2)$ and $(3, -2)$

46. $(-4, -1)$ and $(2, -1)$

Let $f(x) = -2x^2 + 4x + 6$. *Find each of the following.* (*See Example* 4.)

47. $f(0)$ **49.** $f(-1)$ **51.** $f(a)$ **53.** $f(1 + a)$

48. $f(-2)$ **50.** $f(3)$ **52.** $f(-m)$ **54.** $f(2 - p)$

For each of the following, replace x, *in turn, by* -2, -1, 0, 1, 2, *and* 3. *Then plot the resulting points in a coordinate system.*

55. $y = -3x + 5$

56. $y = 2x - 4$

57. $y = -x^2 + 2x$

58. $y = x^2 - 4x + 1$

Find the domain and range of each of the following. Identify any which are functions. (*See Example* 5.)

59. $y = 4x - 3$

60. $2x + 5y = 10$

61. $y = x^2 + 4$

62. $y = 2x^2 - 5$

63. $x = y^2$

64. $-x = y^2$

65. $y = \sqrt{4 + x}$

66. $y = \sqrt{x - 2}$

Find the domain only for each of the following.

67. $y = 1/x$ **68.** $y = -2/(x + 1)$

1.2 Angles

One basic idea in trigonometry is the angle, which we define in this section. Figure 1.8 shows a line through the two points A and B. This line is named *line AB*.