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Editor-in-Chief

Madan G Singh

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CONTENTS

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Singular Control: Higher-Order Conditions

The maximum principle (see Theorem 2 in the article *Maximum Principle*) provides a necessary, but not sufficient, condition that a control u^* has a solution $x(\cdot, u^*)$ of

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1}$$

with $x(t, u^*) \in \partial\mathcal{A}(t)$. Thus it is possible to have a solution pair $x(\cdot, u)$, $\eta(\cdot)$ of Eqn. (1) and

$$\dot{\eta}(t) = -\eta(t)\frac{\partial f}{\partial x}(t, x(t, u^*), u^*(t)) \tag{2}$$

respectively, such that on an interval $[0, t_1]$,

$$H(t, x(t, u), \eta(t), u(t)) \geqslant H(t, x(t, u), \eta(t), v), \qquad \forall v \in U$$

yet $x(t, u) \in \operatorname{int} \mathcal{A}(t)$ for $t \in (0, t_1]$. This inequality could not occur if the approximating cone $K^1_{t_1}$ to $\mathcal{A}(t_1)$ at $x(t_1, u)$ were all of $\mathbb{R}^n$, motivating one definition of singularity found in the literature, i.e., that the first-order cone K^1_T at $x(T, u)$ does not provide a sufficient condition to determine whether or not $x(T, u) \in \partial\mathcal{A}(T)$. In general for such a solution we shall see that

$$H(t, x(t, u), \eta(t), v)$$

is independent of $v \in U$, i.e., the maximization in

$$\mathcal{M}(t, x, \eta) = \max\{H(t, x, \eta, v) : v \in U\} \tag{3}$$

(see Eqn. 8 of *Maximum Principle*) provides no information about *any* component of the control. It is also possible that some but not all components of the control are determined by the maximization. Another definition of singularity utilizes this determination of control components via the maximization of H, as in Eqn. (3), as its basis. In some instances all definitions are equivalent, in others they are not. Our approach will be to take an overview which combines the essence of the various definitions. In order to do this, we decompose the first-order cone K^1_T as follows. Let $u = (u_1, \ldots, u_m)$ be any admissible control for system (1).

DEFINITION 1. $K^{1,i}_{\tau,T}$ is the first-order cone to $\mathcal{A}(T)$ at $x(T, u)$ obtained from variations only in the ith component of the control u during the interval $\tau \leqslant t \leqslant T$.

In other words, $K^{1,i}_{\tau,T}$ is the smallest closed, convex cone containing tangent vectors arising from perturbation data π (linear in ϵ as in the proof of the maximum principle in the article *Maximum Principle*) to the ith component of u. If $\tau = 0$, we shall not write it. It follows that

$$K^1_T = \sum_{i=1}^{m} K^{1,i}_T = \{v^1 + \cdots + v^m : v^i \in K^{1,i}_T \quad i = 1, \ldots, m\}$$

DEFINITION 2. Consider the system (1) and assume that for $T > 0$, $\mathcal{A}(T)$ has nonempty interior relative to $\mathbb{R}^n$. The solution $x(\cdot, u)$ is singular on the interval $\tau \leqslant t \leqslant T$ if $\dim K^{1,i}_{\tau,T} < n$ for some $1 \leqslant i \leqslant m$.

It is totally singular on the interval $[\tau, \mathrm{T}]$ if $\dim K^1_{\tau,T} < n$. (Note that one may certainly have $\dim K^{1,i}_{\tau,T} < n$ for all $i = 1, \ldots, m$, yet $\dim K^1_{\tau,T} = n$.)

For autonomous smooth systems, one is usually interested in singularity on intervals of the form $[0, T]$, $T > 0$ and small. This occurs in the study of local controllability (see *Local Controllability*). For nonautonomous (even linear) systems, however, singular solutions on intervals $[\tau, T]$, $\tau > 0$, lead to interesting geometrical properties of attainable sets.

Singular solutions arise naturally in systems having the control appearing linearly, i.e., n-dimensional systems of the form

$$\dot{x}(t) = X(x(t)) + \sum_{i=1}^{m} u_i(t) Y^i(x(t)), \quad x(0) = x^0 \tag{4}$$

with, say, $|u_i(t)| \leqslant 1$, $i = 1, \ldots, m$ and $X, Y^1, \ldots, Y^m$ real-analytic vector fields on an n-dimensional manifold M^n. For such systems the approximating cone K^1_T at a point $x(T, u) \in \mathcal{A}(T)$ with u such that

$$|u(t)| < 1, \quad 0 \leqslant t \leqslant T$$

is a subplane. In particular, let $(\exp tX)(x^0)$ denote the solution of Eqn. (4) corresponding to $u \equiv 0$. Then if K^1_T is the first-order cone to $\mathcal{A}(T)$ at $(\exp TX)$ (x^0) where $t \rightarrow (\exp tX)$ (x^0) satisfies the maximum principle, K^1_T must be a proper ($\dim < n$) subplane of $\mathbb{R}^n$.

Remark 1. If the system (1), or (4), has a $(k \leqslant n)$-dimensional integral manifold M^k (i.e., solutions beginning on M^k remain on M^k) and k is minimal, one says that the system admits a k-dimensional, minimal realization. If $\mathcal{A}(T)$ has nonempty interior relative to M^k, then $x(\cdot, u)$ is singular (totally singular) on $[\tau, T]$ if $\dim K^{1,i}_{\tau,T} < k$ for some i ($\dim K^1_{\tau,T} < k$).

For the general system (1) it is difficult to calculate the dimension of integral manifolds, $\mathcal{A}(T)$ and the $K^{1,i}_{\tau,T}$. For smooth systems of the form (4) these computations are easier. This seems to be the underlying reason why most literature on singular problems deals with systems of the form (4). Questions concerning $(k \leqslant n)$-dimensional integral manifolds, $\dim K^{1,i}_T$, $\operatorname{int}\mathcal{A}(T) \neq \varnothing$ etc., for system (4) are most naturally answered within the context of Lie algebras of vector fields. Our approach will be to deal mainly with system (4). Higher-order conditions for more general systems of the form (1) have been treated in Krener (1977) and Knobloch (1981); the basic ideas are similar but computations are more involved.

1. Notation

Let M be a real-analytic, n-dimensional manifold and X and Y be real-analytic, tangent vector fields on M. The Lie product $[X, Y]$ can be defined, via local coordinates $(x_1, \ldots, x_n)$ on M, as

$$[X, Y](x) = X_x(x)Y(x) - Y_x(x)X(x)$$

with X_x, Y_x denoting the Jacobian matrices of partial derivatives. For a coordinate-free definition we can consider a vector field as a differential operator acting on smooth functions $f: M \to \mathbb{R}^1$; we denote this Xf. Then

$$[X, Y]f = Y(Xf) - X(Yf)$$

for all smooth f. The real vector space of all real-analytic vector fields on M, together with the Lie product, is a Lie algebra (in general, infinite-dimensional) which we denote by $V(M)$. Let TM_x denote the tangent space to M at x. For $\mathscr{C} \subset V(M)$, $L(\mathscr{C})$ denotes the Lie algebra generated by $\mathscr{C}$, i.e., the smallest subalgebra of $V(M)$ containing $\mathscr{C}$, and

$$\mathscr{C}(x) = \{v(x) \in TM_x : V \in \mathscr{C}\}$$

Let $(\text{ad}\, X, Y) = [X, Y]$ and inductively

$$(\text{ad}^{k+1} X, Y) = [X, (\text{ad}^k X, Y)]$$

If $\mathscr{C} \subset V(M)$ a solution ϕ for the system $\mathscr{C}$ in an absolutely continuous map $t \to \phi(t) \in M$ such that $\phi(t) \in \mathscr{C}(\phi(t))$ p.p. A submanifold N of M is an integral manifold of $\mathscr{C}$ if all solutions which begin on N remain on N.

2. Theorems

The next, fundamental, theorem may be viewed as a sharpening of the Frobenius theorem (i.e., the hypothesis involves a condition on vector fields evaluated at a single point) for real-analytic vector fields.

THEOREM 1 (Hermann 1963, Nagano 1966). *Let M be a real-analytic manifold, $p \in M$ and $\mathscr{C} = \{X^\alpha : \alpha \in A\}$ a collection of real-analytic vector fields on M. If* $\dim L(\mathscr{C})(p) = k$ *then $\mathscr{C}$ (and $L(\mathscr{C})$) has a k-dimensional integral manifold through p.*

For variations and extensions of this type of result to the $\mathbb{C}^\infty$ category see Hermann (1963) or Sussmann (1973).

Thus if we have a control system defined by a collection of real-analytic vector fields $\mathscr{C}$, as above, the correct state space should be the integral manifold of $L(\mathscr{C})$ through the initial point x^0. For example, system (4) may be viewed as the collection

$$\mathscr{C} = \left\{X + \sum_{i=1}^{m} \alpha_i Y^i : -1 \leq \alpha_i \leq 1, \quad i = 1, \ldots, m\right\}$$

The correct state space is therefore the integral manifold of $L(X, Y^1, \ldots, Y^m)$ through x^0 and relative to this manifold

$$\text{int} \bigcup_{0 \leq t \leq T} \mathscr{A}(t, x^0) \neq \varnothing$$

for any $T > 0$ (Krener 1974). For our discussion of singular solutions we need to know whether $\mathscr{A}(T)$ has nonempty interior relative to this integral manifold. It is possible that if $\dim L(\mathscr{C})(x^0) = k$,

$$\text{int} \bigcup_{0 \leq t \leq T} \mathscr{A}(t) \neq \varnothing$$

yet for each t, $\text{int}\, \mathscr{A}(t) = \varnothing$. Let L be a Lie subalgebra of $V(M)$; its derived algebra

$$L' = \{[V, W] : V, W \in L\}$$

i.e., $L' = [L, L]$. For systems defined by analytic vector fields the next theorem gives definitive results for $\text{int}\, \mathscr{A}(T)$.

THEOREM 2 (Sussmann and Jurdjevic 1972). *Let $\mathscr{C} = \{X^\alpha : \alpha \in A\}$ be a family of real-analytic vector fields on a real-analytic manifold M, $p \in M$ and $\mathscr{A}(t, p)$ denote the set of points attainable at time t by solutions of $\mathscr{C}$ initiating from p at $t = 0$. Let*

$$L_0(\mathscr{C}) = \left\{\sum_{i=1}^{l} \lambda_i X^{\alpha_i} + W : l \text{ any integer}, \quad X^{\alpha_i} \in \mathscr{C}, W \in L'(\mathscr{C}), \lambda_i \text{ real}, \sum \lambda_i = 0\right\}$$

Then $L_0(\mathscr{C})$ is an ideal of codimension at most one in $L(\mathscr{C})$. If $\dim L(\mathscr{C})(p) = k$, *a necessary and sufficient condition that* $\text{int}\, \mathscr{A}(t, p) \neq \varnothing$ *(interior relative to the k-dimensional integral manifold of $L(\mathscr{C})$ through p) for all $t > 0$ is that*

$$\dim L_0(\mathscr{C})(p) = \dim L(\mathscr{C})(p) = k$$

For system (4), which has

$$\mathscr{C} = \{X + \sum_{i=1}^{m} \alpha_i Y^i : |\alpha_i| \leq 1, \quad i = 1, \ldots, m\}$$

we define

$$\mathscr{S}^{1,i} = \{(\text{ad}^\nu X, Y^i) : \nu = 0, 1, \ldots\}$$
$$\mathscr{S}^1 = \{(\text{ad}^\nu X, Y^i) : i = 1, \ldots, m; \nu = 0, 1, \ldots\} \quad (5)$$

It follows that $L_0(\mathscr{C}) = L(\mathscr{S}^1)$ (Hermes 1976) which yields the following corollary.

COROLLARY. *Let $X, Y^1, \ldots, Y^m$ be real-analytic vector fields on a real-analytic, n-dimensional manifold M and $\mathscr{A}(t, x^0)$ denote the attainable set at time t for the associated system* (4). *Suppose*

$$\dim L(X, Y^1, \ldots, Y^m)(x^0) = k$$

Then a necessary and sufficient condition that int $\mathcal{A}(t, x^0) \neq \varnothing$ (int *relative to the integral manifold of* $L(x, Y^1, \ldots, Y^m)$ *through* x^0) *for all* $t > 0$ *is that*

$$\dim L(\mathcal{S}^1)(x^0) = k$$

Insight into this result may be obtained as follows. Let $(\exp tX)(x^0)$ denote the solution at t of $\dot{x} = X(x)$, and $x(0) = x^0$ and u be any admissible control for Eqn. (4). If one attempts to write the solution $x(\cdot, u)$ of Eqn. (4) as the composition

$$x(t, u) = (\exp tX) \circ y(t, u)$$

it is (locally) necessary and sufficient (Chen 1962) that y satisfies

$$\dot{y} = \sum_{i=1}^{m} u_i(t) \sum_{\nu=0}^{\infty} (-t)^\nu/\nu!(\mathrm{ad}^\nu X, Y)(y), \quad y(0) = x^0 \tag{6}$$

We refer to this as the auxiliary equation and denote its attainable set, at time t, by $\mathcal{B}(t, x^0)$. Then the attainable set of Eqn. (4) satisfies

$$\mathcal{A}(t, x^0) = (\exp tX)\mathcal{B}(t, x^0)$$

The map $p \to (\exp tX)(p)$ is a homeomorphism hence $\mathcal{A}(t, x^0)$ has nonempty interior if and only if the same is true for $\mathcal{B}(t, x^0)$. Now if

$$\dim L(\mathcal{S}^1)(x^0) = k < \dim L(X, Y^1, \ldots, Y^m)(x^0)$$

by Theorem 2, $L(\mathcal{S}^1)$ has a k-dimensional integral manifold through x^0. Then all solutions of Eqn. (6) lie on this manifold; $\mathcal{B}(t, x^0)$ has empty interior and the same holds for $\mathcal{A}(t, x^0)$.

To summarize for system (4) we may (and will) assume that M^n is the minimal integral manifold of $L(X, Y^1, \ldots, Y^m)$ through x^0, i.e.,

$$\dim L(X, Y^1, \ldots, Y^m)(x^0) = n$$

Then int $\mathcal{A}(t, x^0) \neq \varnothing$ for all $t > 0$ if and only if $\dim L(\mathcal{S}^1)(x^0) = n$, hence a solution $x(\cdot, u)$ will be totally singular on $[0, T]$ if $\dim K_T^1 < n$ where K_T^1 is the first-order cone to $\mathcal{A}(T, x^0)$ at $x(T, u)$ as defined in the article *Maximum Principle*.

We next study $K_T^{1,i}$ and K_T^1. It is convenient to assume these are calculated at $x(T, u) \in \mathcal{A}(T)$ where $u \equiv 0$, i.e., at $(\exp tX)(x^0) \in \mathcal{A}(T)$. For any $0 < t_1 \leq T$ and perturbation data $\pi_1 = (t_1, l_1, v^1)$ it follows from

$$v_{\pi_1}(t_1) = [f(t_1, x(t_1, u^*), v^1) - f(t_1, x(t_1, u^*), u^*(t_1))]l_1 \tag{7}$$

(see Eqn. (10) in the article *Maximum Principle*), that

$$v_{\pi_1}(t) = \sum_{i=1}^{m} v_i^1 Y^i((\exp t_1 X)(x^0))l_1$$

i.e., $v_{\pi_1}(t_1)$ is a linear combination of

$$Y^1((\exp t_1 X)(x^0)), \ldots, Y^m((\exp t_1 X)(x^0))$$

Now let $\mathrm{D}(\exp tX)$ denote the differential of the map $p \to (\exp tX)(p)$. Then $\mathrm{D}(\exp tX)$ is the fundamental solution of the variational equation

$$\dot{y} = f_x(t, x(t, u^*), u^*(t))y \tag{8}$$

which is the identity at $t = 0$, i.e.,

$$\mathrm{D}(\exp(T - t_1)X) = A_{T, t_1}$$

The element of K_T^1 which corresponds to $v_{\pi_1}(t_1)$ is then

$$\begin{aligned} v_{\pi_1} &= \mathrm{D}(\exp(T - t_1)X) \sum_{i=1}^{m} l_1 v_i^1 Y^i((\exp t_1 X)(x^0)) \\ &= \mathrm{D}(\exp(T - t_1)X) \sum_{i=1}^{m} l_1 v_i^1 Y^i \\ &\quad \times ((\exp(t_1 - T)X) \circ (\exp TX)(x^0)) \\ &= \sum_{i=1}^{m} l_1 v_i^1 \sum_{\nu=0}^{\infty} [(T - t_1)^\nu/\nu!](\mathrm{ad}^\nu X, Y^i) \\ &\quad \times ((\exp TX)(x^0)) \end{aligned} \tag{9}$$

A relatively easy consequence of (9) is the following proposition (Hermes 1976).

Proposition 1. *The first-order approximating cone* K_T^1 *at a point* $(\exp TX)(x^0) \in \mathcal{A}(T)$ *for system* (4) *is*

$$\mathrm{span}\, \mathcal{S}^1((\exp TX)(x^0))$$

Similarly,

$$K_T^{1,i} = \mathrm{span}\, \mathcal{S}^{1,i}((\exp TX)(x^0))$$

Remark 2. The relationships between the various notions of singular solutions are as follows. First, we have restricted attention to the case int $\mathcal{A}(T) \neq \varnothing$ relative to a minimal integral manifold M^k of the system. If this is not done and $k < n$, every point of $\mathcal{A}(T)$ is a boundary point relative to M^n; for any control u the first-order cone K_T^1 to $\mathcal{A}(T)$ at $x(T, u)$ will be contained in the tangent space to M^k at $x(T, u)$, denoted $TM^k_{x(T,u)}$. If $\eta(T) \neq 0$ is in $TM^n_{x(T,u)}$ and orthogonal to $TM^k_{x(T,u)}$, while η is extended to $[0, T]$ via the solution of Eqn. (2) we will automatically have

$$H(t, x(t, u), \eta(t), v) \equiv 0$$

and is hence independent of v (see example 1). Thus the maximization Eqn. (3) would not determine any component of a control, but this occurs because of the extraneous dimensions.

Next, turning to system (4), suppose int $\mathcal{A}(T) \neq \varnothing$ relative to M^n and there exists a solution $x(\cdot, u)$ such that the maximum principle does not determine some component u_i of u via the maximization as given in Eqn. (3); or equivalently that there is a nonzero η satisfying Eqn. (2) such that the Hessian of H, i.e., H_{uu}, is not definite along the solution pair $x(\cdot, u), \eta(\cdot)$. Since

$$H = \eta \cdot X + \eta \cdot \left(\sum_{j=1}^{m} u_j Y^j \right)$$

for Eqn. (4) this would mean there exists a nonzero vector η satisfying

$$\dot{\eta} = -\eta\left(X_x(x(t,u)) + \sum_{j=1}^{m} u_j(t) Y_x(x(t,u))\right)$$

such that

$$\eta(t) Y^i(x(t,u)) \equiv 0$$

Differentiating this with respect to t gives

$$\begin{aligned}\dot{\eta} Y^i + \eta \cdot Y^i_x \dot{x} &= -\eta\left(X_x Y^i + \sum_{j=1}^{m} u_j Y^i_x\right) + \eta Y^i_x\left(X + \sum_{j=1}^{m} u_j Y^j\right)\\ &= -\eta(t)\left\{[X, Y^i](x(t,u)) - \sum_{j=1}^{m} u_j(t)[Y^j, Y^i](x(t,u))\right\}\\ &= 0\end{aligned}$$

In particular, if the solution we examine corresponds to $u = 0$, we have

$$\eta(t) Y^i((\exp tX)(x^0)) = 0$$

$$-\eta(t)[X, Y^i]((\exp tX)(x^0)) = 0$$

and repeated differentiation gives

$$-\eta(t)(\mathrm{ad}^k X, Y^i)((\exp tX)(x^0)) = 0$$

This shows that the nonzero vector $\eta(T)$ is orthogonal to $\mathcal{S}^{1,i}((\exp TX)(x^0))$, hence, by Proposition 1,

$$\dim K_T^{1,i} = \dim \operatorname{span} \mathcal{S}^{1,i}((\exp TX)(x^0)) < n$$

showing that the two notions of singularity agree in this case.

Remark 3. The assumption that the reference solution is $(\exp tX)(x^0)$, i.e., corresponds to $u \equiv 0$, makes the calculation in Eqn. (9) easy. Had we chosen any u and calculated K_T^1 at $x(T,u) \in \mathcal{A}(T)$ one finds

$$v_\pi(t_1) = \sum_{i=1}^{m} (v_i^1 - u_i(t_1)) Y^i(x(t_1,u)) l_1$$

with the sign of $(v_i^1 - u_i(t_1))$, $v^1 \in U$, no longer necessarily arbitrary. One can conclude that K_T^1 is contained in

$$\operatorname{span}\{(\mathrm{ad}^\nu X, W)(x(T,u)) \colon W \in L(Y^1, \ldots, Y^m)\}$$

but computations are difficult. See Krener (1974) for some examples.

In summary, for the systems (4), if $X, Y^1, \ldots, Y^m$ are real-analytic vector fields on a manifold M and

$$\dim L(X, Y^1, \ldots, Y^m)(x^0) = n$$

then the system has an n-dimensional integral manifold (the correct state space) through x^0, call this M^n, such that

$$\operatorname{int} \bigcup_{0 \leq t \leq T} \mathcal{A}(t, x^0) \neq \varnothing$$

for all $T > 0$ (with interior relative to M^n). A necessary and sufficient condition that $\operatorname{int}(T, x^0) \neq 0$ for each $T > 0$ is that $\dim L(\mathcal{S}^1)(x^0) = n$; we assume this. The first-order approximating cone to $\mathcal{A}(T, x^0)$ at $(\exp TX)(x^0)$ is

$$K_T^1 = \operatorname{span} \mathcal{S}^1((\exp TX)(x^0))$$

hence the solution $t \to (\exp TX)(x^0)$ is *totally singular* on $[0, T]$ if

$$\dim \operatorname{span} \mathcal{S}^1((\exp tX)(x^0)) < \dim L(\mathcal{S}^1)((\exp tX)(x^0)) = n$$

for sufficiently small $t > 0$. Since $\mathrm{D}(\exp tX)$ carries span $\mathcal{S}^1(x^0)$ into span $\mathcal{S}^1((\exp tX)(x^0))$ it follows that

$$\dim K_t^1 = \dim \operatorname{span} \mathcal{S}^1((\exp tX)(x^0))$$

is constant with respect to t (Hermes 1976) and hence we obtain the following theorem.

THEOREM 3. *Assume*

$$\dim L(X, Y^1, \ldots, Y^m)(x^0) = \dim L(\mathcal{S}^1)(x^0) = n$$

The solution $t \to (\exp tX)(x^0)$ *is a totally singular solution of Eqn.* (4) *on some interval* $[0, T]$ *if* $\dim \operatorname{span} \mathcal{S}^1(x^0) < n$.

EXAMPLE 1. We consider the system, on $\mathbb{R}^2$,

$$\dot{x}_1 = x_1 + u$$

$$\dot{x}_2 = x_2 + u$$

with $x^0 = (x_1^0, x_2^0)$. Then by subtracting the equation it follows that for any $t \geq 0$, $\mathcal{A}(t, x^0)$ is a subset of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 - x_2 = c\,\mathrm{e}^t\}$$

where $c = (x_1^0 - x_2^0)$. If this system is written in the form of Eqn. (4),

$$X = (x_1, x_2), \qquad Y = (1, 1)$$

$$\dim L(X, Y)(x^0) = 1, \qquad \dim L(\mathcal{S}^1)(x^0) = 1$$

$$\dim \operatorname{span} \mathcal{S}^1(x^0) = 1$$

and $(\exp tX)(x^0)$ is *not* a singular solution. On the other hand, Eqn. (2) gives $\dot{\eta}_1 = -\eta_1$, $\dot{\eta}_2 = -\eta_2$ hence

$$\eta_1(t) + \eta_2(t) = c_1\,\mathrm{e}^{-t}$$

with

$$c_1 = (\eta_1(T) + \eta_2(T))$$

If we choose $\eta_1(T) = -\eta_2(T) \neq 0$ (note that $\eta(T)$ is then orthogonal to the line which contains $\mathcal{A}(T, x^0)$) we have $\eta_1(t) - \eta_2(t) \equiv 0$ and for any control $u \in \mathcal{U}$,

$$H(t, x(t,u), \eta(t), v) = \eta_1(t) x_1(t,u) + \eta_2(t) x_2(t,u)$$

is independent of $v \in [-1, 1]$. Thus if one defined a solution $x(\cdot, u)$ to be singular if there exists a nonzero solution $\eta(\cdot)$ of Eqn. (2) such that the pair satisfy the maximum principle

$$H(t, x(t,u^*), \eta(t), u^*(t)) = \mathcal{M}(t, x(t,u^*), \eta(t)) \quad (10)$$

(see Eqn. (9) in the article *Maximum Principle*), every solution, here, would be singular. Indeed, one could then make any problem singular by adding a "non-controllable" additional dimension, i.e., adjoining an equation $\dot{x}_{n+1} = 1$ to Eqn. (1) and then choosing the $(n+1)$-dimensional vector $\eta(T)$ to be perpendicular to the hyperplane $x_{n+1} = T$ within which $\mathscr{A}(T)$ would lie.

EXAMPLE 2. Consider the system, on $\mathbb{R}^2$, with

$$X(x) = (1, x_2), \qquad Y(x) = (x_2, 1)$$

that is

$$\dot{x}_1 = 1 + u(t)x_2, \qquad \dot{x}_2 = x_2 + u(t)$$

with $x^0 = (0,0)$ and $U = [-1, 1]$. Then

$$(\mathrm{ad}^k X, Y)(x) = ((-1)^k x_2, 1), \quad k = 0, \ldots$$

so $\dim \operatorname{span} \mathscr{S}^1(x^0) = 1$, while

$$(\mathrm{ad}^2 Y, X)(x^0) = (-2, 0)$$

hence $\dim L(\mathscr{S}^1)(x^0) = 2$. This shows that for each $t > 0$, $\operatorname{int} \mathscr{A}(t, x^0) \neq \varnothing$, yet the first-order cone K_T^1 to $\mathscr{A}(T, x^0)$ at $(\exp TX)(x^0)$ is

$$\operatorname{span} \mathscr{S}^1((\exp TX)(x^0)) = \{(0, \alpha) : \alpha \in \mathbb{R}^1\}$$

Thus $t \to (\exp tX)(x^0) \equiv (t, 0)$ is a singular solution and $(\exp TX)(x^0)$ may or may not be on $\partial \mathscr{A}(T, x^0)$. If one chooses $\eta(T) = (1, 0)$, then $\eta(t) \equiv (1, 0)$ and

$$H(t, (\exp tX)(x^0), \eta(t), v) \equiv \eta(t) \equiv 1$$

which shows the pair $(\exp tX)(x^0)$, $\eta(t)$ does satisfy the maximum principle.

We calculate a higher-order tangent vector for this example. Let

$$u_\pi(t,\epsilon) = \begin{cases} -1 \text{ if } T - 4\epsilon \leq t < T - 3\epsilon, & T - \epsilon \leq t \leq T \\ +1 \text{ if } T - 3\epsilon \leq t < T - \epsilon \end{cases}$$

Then u is an admissible control with corresponding solution (using the exponential notation for solutions)

$$x_\pi(T,\epsilon) = (\exp \epsilon(X - Y)) \circ (\exp 2\epsilon(X + Y)) \circ (\exp \epsilon(X - Y)) \circ (\exp(-4\epsilon X)) \circ (\exp TX)(x^0) \quad (*)$$

For $0 \leq 4\epsilon < \mathrm{T}$,

$$x_\pi(T, \epsilon) \in \mathscr{A}(T, x^0)$$

while

$$x_\pi(T, 0) = (\exp TX)(x^0)$$

Using the Campbell–Baker–Hausdorff formula (Varadarajan 1974) on the right-hand side of $(*)$, one can show

$$x_\pi(T,\epsilon) = \exp((2\epsilon^3/3)[[\mathrm{X}, \mathrm{Y}], \mathrm{Y}] + o(\epsilon^3)) \circ (\exp TX)(x^0) \quad (**)$$

from which we conclude

$$\lim_{\epsilon^+ \to 0} \mathrm{d}x_\pi(T,\epsilon)/\mathrm{d}\epsilon = 0$$

$$\lim_{\epsilon^+ \to 0} \mathrm{d}^2 x_\pi(T,\epsilon)/\mathrm{d}\epsilon^2 = 0$$

$$\lim_{\epsilon^+ \to 0} \mathrm{d}^3 x_\pi(T,\epsilon)/\mathrm{d}\epsilon^3 = 4[[X, Y], Y]((\exp TX)(x^0))$$

is a tangent vector to $\mathscr{A}(T, x^0)$ at $(\exp TX)(x^0)$. Since

$$[[X, Y], Y]((\exp TX)(x^0)) = (-2, 0)$$

this tangent vector is *not* in K_T^1; indeed (since it involves two factors Y) it is in K_T^2. As mentioned in Remark 3, one could reparametrize, i.e., let $\epsilon^3 = \sigma$ or $\epsilon = \sigma^{1/3}$ and

$$x_\pi(T,\epsilon) = y(T,\sigma) = \exp((2\sigma/3)[[X, Y]Y] + o(\sigma))(\exp TX)(x^0)$$

and take

$$\lim_{\sigma^+ \to 0} \mathrm{d}y(T,\sigma)/\mathrm{d}\sigma$$

to obtain this tangent vector. With η_T as in the proof of the maximum principle (see *Maximum Principle*), i.e., $\eta_T \cdot v(T) \leq 0$ for all $v(T) \in K_T^1$, it is clearly also necessary that if

$$(\exp TX)(x^0) \in \partial \mathscr{A}(T, x^0)$$

then $\eta_T \cdot v(T) \leq 0$ for all $v(T) \in K_T^1 \cup K_T^2$, etc. This is the essence of the higher-order maximum principle, which will be stated shortly.

A kth-order tangent vector at $(\exp TX)(x^0)$, for system (4), is an element $W \in L(\mathscr{S}^1)$ evaluated at $(\exp TX)(x^0)$ which involves k factors Y, i.e., a product of k elements of $\mathscr{S}^1$. It is interesting, and important, to determine which elements in $L(\mathscr{S}^1)$ can yield tangent vectors. For example, as shown above,

$$[[X, Y]Y]((\exp TX)(x^0)) = -[Y, [X, Y]]((\exp TX)(x^0))$$

can be a second-order tangent vector; its negative cannot. Indeed (Krener 1977) for system (4) one has the following theorem.

THEOREM 4. *If h is the smallest integer such that for some $t_1 \in [0, T]$,*

$$[Y, (\mathrm{ad}^h X, Y)]((\exp t_1 X)(x^0)) \notin \operatorname{span} \mathscr{S}^1((\exp tX)(x^0))$$

then h is odd. A necessary condition that

$$(\exp TX)(x^0) \in \partial \mathscr{A}(T, x^0)$$

is that there exists an η satisfying Eqn. (2) *on* $[0, T]$ *such that*

$$\eta(T)(-1)^{(h+1)/2}[Y, (\mathrm{ad}^h X, Y)]((\exp tX)(x^0)) \leq 0$$

Another way to view this result is that

$$(-1)^{(h+1)/2}[Y, (\mathrm{ad}^h X, Y)]((\exp t_1 X)(x^0))$$

is a second-order tangent vector. This condition is related to a second-order condition (Kelley *et al.* 1967) for a control $u^* \in \mathcal{U}$ having values $u^*(t) \in (-1, 1)$ to lead to a solution $x(\cdot, u^*)$ of Eqn. (4) having

$$x(T, u^*) \in \partial\mathcal{A}(T, x^0)$$

Specifically, suppose u^* is such; there then exists a solution η of Eqn. (2) on $[0, T]$ satisfying the maximum principle. Assume there is a smallest integer h such that

$$\left(\frac{\partial}{\partial u}\right)\left(\frac{d^{h+1}}{dt^{h+1}}\right)\left(\frac{\partial}{\partial u}\right)H(x(t,u^*),\eta(t),\eta(t),u^*(t))\neq 0$$

for some $t \in [0, T]$. Then h is odd and on $[0, T]$,

$$(-1)^{(h+1)/2}\left(\frac{\partial}{\partial u}\right)\left(\frac{d^{h+1}}{dt^{h+1}}\right)\left(\frac{\partial}{\partial u}\right)H(x(t,u^*),\eta(t),u^*(t))\leqslant 0$$

is a necessary condition.

For details of how these conditions are related, and more general results of this nature, see Krener (1977). For further characterizations of which elements of $L(\mathcal{S}^1)$ can yield higher-order tangent vectors, a verification that $[Y, [X, Y]]$ is not such, etc., see Hermes (1978).

We next state the high-order maximum principle for system (4); the statement for more general systems can be found in Krener (1977).

THEOREM 5 (High-order maximum principle for system (4)). *A necessary condition to ensure that $u^* \in \mathcal{U}$ is such that the corresponding solution $x(\cdot, u)$ of Eqn. (4) satisfies $x(T, u^*) \in \partial\mathcal{A}(t, x^0)$ is that there exists an absolutely continuous n-vector function η defined on $[0, T]$ such that*

$$\dot{\eta}(t) = -\eta\left[X_x(x(t, u^*)) + \sum_{i=1}^{m} u_i^*(t)Y_x^i(x(t, u^*))\right]$$

$$\eta(t)\cdot\left[\sum_{i=1}^{m} Y^i(x(t,u^*))(u_i^*(t)-v_i)\right]\geqslant 0, \quad \forall v_i \in [-1,1]$$

and for any perturbation data

$$\pi = (\pi_1(t_1, l_1, v^1), \ldots, \pi_j(t_j, l_j, v^j))$$

with corresponding perturbed control u_π and solution at T, denoted $x_\pi(T, \epsilon) \in \mathcal{A}(T, x^0)$,

$$\lim_{\epsilon^+\to 0} \eta(T)\cdot(d^h/d\epsilon^h)x_\pi(T, \epsilon) \leqslant 0$$

if

$$\lim_{\epsilon^+\to 0} (d^l/d\epsilon^l)x_\pi(T, \epsilon) = 0, \quad 1 \leqslant l \leqslant h-1$$

The first two conditions are the analogs of Eqns. (2) and (10) in the first-order maximum principle (Eqns. (6) and (9) in *Maximum Principle*); the last condition merely states that $\eta(T)$ must be an outward normal to a support plane for a local cone containing higher-order tangent vectors.

For systems having one control component appearing linearly, on two-dimensional manifolds, high-order necessary and sufficient conditions may be obtained via the use of Green's (or Stokes') theorem. Specifically, consider the two-dimensional system

$$\dot{x} = X(x) + u(t)Y(x), \quad x(0) = x^0 \tag{11}$$

with X, Y smooth vector fields on a two manifold M^2 and $|u(t)| \leqslant 1$. We assume the vectors $X(x^0)$, $Y(x^0)$ are linearly independent and that ω is a one form such that if $\langle\omega(x), X(x)\rangle$ denotes its action on X,

$$\langle\omega(x), X(x)\rangle = 1, \qquad \langle\omega(x), Y(x)\rangle = 0 \tag{12}$$

If $x(\cdot, u^1)$ is a solution of Eqn. (11) such that

$$x(0, u^1) = x^0, \qquad x(t(u^1), u^1) = x^1$$

and $\Gamma(u^1)$ is the orbit of $x(\cdot, u^1)$, then from (10), the line integral

$$\int_{\Gamma(u^1)} \omega = t(u^1)$$

Let u^1 be another admissible control such that

$$x(0, u^2) = x^0$$
$$x(t(u^2), u^2) = x^1$$
$$\Gamma = \Gamma(u^1) \cup \Gamma(u^2)$$

bounds a region $\mathcal{R}$ of the plane within which the two form $d\omega$ is well behaved. Assume that by following the solution $x(\cdot, u^1)$ from x^0 to x^1 and then the solution of $x(\cdot, u^2)$ backwards in time, from x^1 to x^0, we traverse the boundary of $\mathcal{R}$ in a counterclockwise direction. Then

$$\int_\Gamma \omega = t(u^1) - t(u^2) = \int_{\mathcal{R}} d\omega$$

Thus the sign of $d\omega$ in the region $\mathcal{R}$ can be used to compare the times taken to reach x^1 via the controls u^1 and u^2. (One can show that this sign is determined, locally, by the sign of $\langle\omega, [X, Y]\rangle$.) It follows that the statement "$d\omega$ is zero along the solution

$$t \to (\exp tX)(x^0)$$

with X, Y real-analytic" implies and is implied by

$$\langle\omega(x^0), (\mathrm{ad}^k X, Y)(x^0)\rangle = 0, \quad \forall k = 0, 1, \ldots$$

This implies dim span $\mathcal{S}^1(x^0) = 1$, i.e., $d\omega$ is zero along a singular solution

$$t \to (\exp tX)(x^0)$$

The method is best illustrated by an example; further details and results can be found in Hermes and Haynes (1963).

EXAMPLE 2 (continued). With

$$X(x) = (1, x_2), \; Y(x) = (x_2, 1)$$

in local coordinates

$$\omega(x) = (1/(x_2^2 - 1))(-dx_1 + x_2\, dx_2)$$

which is well defined for $|x_2| < 1$.

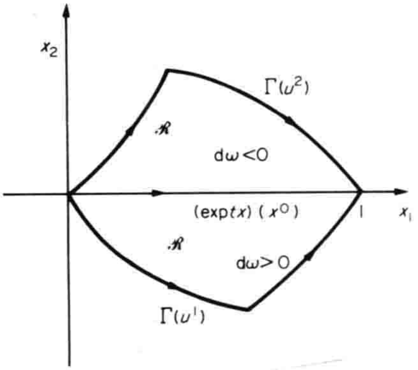

Figure 1
Green's theorem method

Consider the problem of finding the solution which initiates from $x^0 = (0, 0)$ and reaches $x^1 = (1, 0)$ in minimum time. For this system $(\exp tX)(p) = (t, 0)$; let $u^0(t) \equiv 0$ so the time to reach x^1 using control u^0 is $t(u^0) = 1$. Now suppose u^1, u^2 are admissible controls with, respectively, solutions $x(\cdot, u^1), x(\cdot, u^2)$ such that

$$x(t(u^1), u^1) = x(t(u^2), u^2) = x^1$$

and orbits, $\Gamma(u^1)$ in the half plane $x_2 \leqslant 0$ and $\Gamma(u^2)$ in the half plane $x_2 \geqslant 0$ (see Fig. 1). Then

$$t(u^1) - t(u^0) = \int_{\mathcal{R}_1} \mathrm{d}\omega > 0 \text{ or } t(u^0) < t(u^1)$$

$$t(u^0) - t(u^2) = \int_{\mathcal{R}_2} \mathrm{d}\omega < 0 \text{ or } t(u^0) < t(u^2)$$

It is not difficult to conclude that any solution joining x^0 to x^1 and having orbit remaining in the region $|x_2| < 1$ will reach x^1 in a time $t \geqslant 1$, hence

$$(\exp 1X)(x^0) \in \partial\mathcal{A}(1, x^0)$$

i.e., $u^0 \equiv 0$ is the unique time-optimal control for this problem.

Singular problems have not only intrinsic mathematical interest but seem to appear in applications of optimal control. The literature on the subject is extensive, making it virtually impossible to cover all known results here. Some omissions are the work in McDonnell and Powers (1971) on necessary conditions at the junction of an optimal singular and nonsingular solution; the relationship between singular problems, the second variation and the Legendre–Clebsch condition of the classical calculus of variations (Bell and Jacobson 1975). The articles *Bang–Bang Principle* and *Local Controllability* deal with questions related (in the interesting cases) to singular solutions and treat additional high-order conditions and aspects of the singular problem.

See also: Bang–Bang Principle; Local Controllability; Maximum Principle; Optimal Control: Singular Arcs

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Singular Perturbations: Boundary-Layer Problem

A basic problem in control system theory is the mathematical modelling of a physical system. The realistic representation of systems calls for high-order differential equations in which the presence of some "parasitic" parameters, such as masses, inductances, capacitances, resistances and more generally small time constants, is often the cause of the increased order of these systems. If the suppression of small parameters involves the degeneration of dimension, the system is called "singularly perturbed."

Generally the singularly perturbed problem possesses a two-timescale property owing to the simultaneous presence of slow and fast phenomena and can be represented by a model as described by

$$\left.\begin{aligned} dx/dt &= f(x, y, u, \epsilon) \\ \epsilon\, dy/dt &= g(x, y, u, \epsilon) \end{aligned}\right\} \quad (1)$$

where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ denote, respectively, the slow and fast variables, $u \in \mathbb{R}^m$ is the control law, $t \in T = [0, \infty[$ corresponds to current time and $\epsilon \in \mathbb{R}^+$ represents a small parameter.

$$f: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^{n_1}$$

$$g: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^{n_2}$$

and setting $\epsilon = 0$ in Eqn. (1) we obtain the degenerate problem and the solution

$$\left.\begin{aligned} dx_s/dt &= f(x_s, y_s, u, 0), \quad x_s(0) = x_0 \\ 0 &= g(x_s, y_s, u, 0) \end{aligned}\right\} \quad (2)$$

The solution of Eqn. (2) which corresponds to the slow part of the evolution is called the outer solution.

In such a problem there exists a boundary-layer domain where the solution changes rapidly and which corresponds to the inner solution satisfying the initial condition of Eqn. (1).

To study the behavior of $y(\cdot)$ near $t = 0$, the timescale is stretched by introducing the transformation $\tau = t/\epsilon$ in Eqn. (1) to obtain

$$\left.\begin{aligned} dx/d\tau &= \epsilon f(x, y, u, \epsilon), \quad x(\epsilon, 0) = x_0 \\ dy/d\tau &= g(x, y, u, \epsilon), \quad y(\epsilon, 0) = y_0 \end{aligned}\right\} \quad (3)$$

A reasonable approximation of these equations, called the boundary-layer equations, may be obtained, for $t \in T_b = [0, \tau_b]$, by setting $\epsilon = 0$, which gives

$$\left.\begin{aligned} x_f &= x_0 \\ dy_f/d\tau &= g(x_0, y_f, u, 0), \quad y_f(0) = y_0 \end{aligned}\right\} \quad (4)$$

EXAMPLE 1. Consider the singularly perturbed initial problem

$$\left.\begin{aligned} dx/dt &= -x + y + u, \quad x(0) = x_0 \\ \epsilon\, dy/dt &= -x - y + u, \quad y(0) = y_0 \end{aligned}\right\} \quad (5)$$

with $z = x + 2y$ the output of the system and $u(t) \equiv u_0$. The reduced (degenerate) system is obtained by setting $\epsilon = 0$. The solution of Eqn. (2) is then

$$x_0 = e^{-2t} x_0 + (1 - e^{-2t}) u_0$$

$$y_0 = u_0 - e^{-2t} x_0 - (1 - e^{-2t}) u_0$$

Hence there is a discontinuity at $t = 0$ ($y_s(0) \neq y_0$). The evolution of y_f deduced from Eqn. (4) is

$$y_f(\tau) = e^{-\tau} y_0 + (1 - e^{-\tau})(u_0 - x_0)$$

and in the boundary layer the output is approximated by

$$z_b(t) = x_0 + 2[e^{-t/\epsilon} y_0 + (1 - e^{-t/\epsilon})(u_0 - x_0)]$$

1. Boundary-Layer Problem

The singular perturbation method enables the overall model of a process to be broken down into several reduced ones describing in simplified form the successive periods of evolution of the initial process. In considering such an approach, we must note the following points.

(a) Comparison of Eqns. (1) and (2) shows that the reduced solution evaluated at $t = 0$ is an equilibrium point of the zeroth-order boundary-layer equation, hence the problem of stability of this equilibrium point.

(b) Only one initial condition can be met in Eqn. (2) which needs to allow a discontinuity in y at $t = 0$.

A solution to problem (a) has been proposed by Tikhonov (1952), which leads, for example with a constant input u, to the following Tikhonov theorem.

THEOREM 1. *If the following conditions are satisfied:*

(a) *$f(\cdot)$ and $g(\cdot)$ are continuous in some open region Ω of their domains;*

(b) *the degenerate problem, Eqn. (2), and the full problem, Eqn. (1), both have unique solutions in some interval $0 \leq t \leq T$;*

(c) *there exists an isolated root $y_s = \phi(x_s)$ of Eqn. (2) in Ω, that is, $g[x_s, \phi(x_s)] = 0$;*

(d) *the root $y_s = \phi(x_s)$ is an asymptotically stable equilibrium point of Eqn. (3);*

(e) *the initial point (x_0, y_0) belongs to the domain of influence of the root; then $\lim_{\epsilon\to 0^+} x(\epsilon, t) = x_s(t)$ uniformly in $[0, T]$ and $\lim_{\epsilon\to 0^+} y(\epsilon, t) = y_s(t)$ uniformly in any closed subinterval of $[0, T]$.*

This important theorem may be difficult to implement for two reasons:

(a) the stability analysis of nonlinear systems is typically difficult;

(b) when the equation $0 = g[x_s, \phi(x_s)]$ admits several roots, the relevant one has to be identified.

Concerning point (b), three different main approaches have been proposed for solving the boundary-layer problem:

(a) expansion in the small parameter ϵ,

(b) matched asymptotic expansion,

(c) the use of both the singular perturbation technique and reciprocal transformation.