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Lectures on Polytopes

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Günter M. Ziegler

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Preface

The aim of this book is to introduce the reader to the fascinating world of convex polytopes. The book developed from a course that I taught at the Technische Universität Berlin, as a part of the Graduierten-Kolleg "Algorithmische Diskrete Mathematik." I have tried to preserve some of the flavor of lecture notes, and I have made absolutely no effort to hide my enthusiasm for the mathematics presented, hoping that this will be enough of an excuse for being "informal" at times.

There is no P2C2E in this book.*

Each of the ten lectures (or chapters, if you wish) ends with extra notes and historical comments, and with exercises of varying difficulty, among them a number of open problems (marked with an asterisk*), which I hope many people will find challenging. In addition, there are lots of pointers to interesting recent work, research problems, and related material that may sidetrack the reader or lecturer, and are intended to do so.

Although these are notes from a two-hour, one-semester course, they have been expanded so much that they will easily support a four-hour course. The lectures (after the basics in Lectures 0 to 3) are essentially independent from each other. Thus, there is material for quite different two-hour courses in this book, such as a course on "duality, oriented matroids, and zonotopes" (Lectures 6 and 7), or one on "polytopes and polyhedral complexes" (Lectures 4, 5 and 9), etc.

Still, I have to make a disclaimer. Current research on polytopes is very

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*P2C2E = "Process too complicated to explain" [386] much alive, treating a great variety of different questions and topics. Therefore, I have made no attempt to be encyclopedic in any sense, although the notes and references might appear to be closer to this than the text. The main pointers to current research in the field of polytopes are the book by Grünbaum (in its new edition [212]) and the handbook chapters by Klee & Kleinschmidt [269] and by Bayer & Lee [50].

To illustrate that behind all of this mathematics (some of it spectacularly beautiful) there are REAL PEOPLE, I have attempted to compile a bibliography with REAL NAMES (i.e., including first names). In the few cases where I couldn't find more than initials, just assume that's all they have (just like T. S. Garp).

In fact, the masters of polytope theory are really nice and supportive people, and I want to thank them for all their help and encouragement with this project. In particular, thanks to Anders Björner, Therese Biedl, Lou Billera, Jürgen Eckhoff, Eli Goodman, Martin Henk, Richard Hotzel, Peter Kleinschmidt, Horst Martini, Peter McMullen, Ricky Pollack, Jörg Rambau, Jürgen Richter-Gebert, Hans Scheuermann, Tom Shermer, Andreas Schulz, Oded Schramm, Mechthild Stoer, Bernd Sturmfels, and many others for their encouragement, comments, hints, corrections, and references. Thanks especially to Gil Kalai, for the possibility of presenting some of his wonderful mathematics. In particular, in Section 3.4 we reproduce his paper [242],

- GIL KALAI:
  A simple way to tell a simple polytope from its graph,
  J. Combinatorial Theory Ser. A **49** (1988), 381–383;
  ©1988 by Academic Press Inc.,

with kind permission of Academic Press.

My typesetting relies on LaTeX; the drawings were done with `xfig`. They may not be perfect, but I hope they are clear. My goal was to have a drawing on (nearly) every page, as I would have them on a blackboard, in order to illustrate that this really is geometry.

Thanks to everybody at ZIB and to Martin Grötschel for their continuing support.

Berlin, July 2, 1994
Günter M. Ziegler

Contents

Lecture 0
Introduction and Examples

Convex polytopes are fundamental geometric objects: to a large extent the geometry of polytopes is just that of $\mathbb{R}^d$ itself. (In the following, the letter d usually denotes dimension.)

The "classic text" on convex polytopes by Branko Grünbaum [212] has recently celebrated its twenty-fifth anniversary — and is still inspiring reading. Some more recent books, concentrating on f-vector questions, are McMullen & Shephard [335], Brøndsted [114], and Yemelichev, Kovalev & Kravtsov [482]. See also Stanley [427] and Hibi [228]. For very recent developments, some excellent surveys are available, notably the handbook articles by Klee & Kleinschmidt [269] and by Bayer & Lee [50]. See also Ewald [168] for a lot of interesting material, and Croft, Falconer & Guy [135] for more research problems.

Our aim is the following: rather than being encyclopedic, we try to present an introduction to some basic methods and modern tools of polytope theory, together with some highlights (mostly with proofs) of the theory. The fact that we can start from scratch and soon reach some exciting points is due to recent progress on several aspects of the theory that is unique in its simplicity. For example, there are several striking papers by Gil Kalai (see Lecture 3!) that are short, novel, and probably instant classics. (They are also slightly embarrassing, pointing us to "obvious" (?) ideas that have long been overlooked.)

For these lectures we concentrate on combinatorial aspects of polytope theory. Of course, much of our geometric intuition is derived from life in $\mathbb{R}^3$ (which some of us might mistake for the "real world," with disastrous results, as everybody should know). However, here is a serious warning:

part of the work (and fun) consists in seeing how intuition from life in three dimensions can lead one (i.e., everyone, but not us) astray: there are many theorems about 3-dimensional polytopes whose analogues in higher dimensions fail badly. Thus, one of the main tasks for polytope theory is to develop tools to analyze and, if possible, "visualize" the geometry of higher-dimensional polytopes. Schlegel diagrams, Gale diagrams, and the Lawrence construction are prominent tools in this direction — tools for a more solid analysis of what polytopes in d-space "really look like."

Notation 0.0. *We stick to some special notational conventions. They are designed in such a way that all the expressions we write down are "clearly" invariant under change of coordinates.*

In the following $\mathbb{R}^d$ *represents the vector space of all column vectors of length* d *with real entries. Similarly,* $(\mathbb{R}^d)^*$ *denotes the dual vector space, that is, the real vector space of all linear functions* $\mathbb{R}^d \longrightarrow \mathbb{R}$. *These are given by the real row vectors of length* d.

The symbols $\boldsymbol{x}, \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{y}, \boldsymbol{z}$ *always denote column vectors in* $\mathbb{R}^d$ *(or in* $\mathbb{R}^{d\pm 1}$*) and represent (affine) points. Matrices* $X, Y, Z, \ldots$ *represent sets of column vectors; thus they are usually* $(d \times m)$*- or* $(d \times n)$*-matrices. The order of the columns is not important for such a set of column vectors.*

Also, we need the unit vectors $\boldsymbol{e}_i$ *in* $\mathbb{R}^d$*, which are column vectors, and the column vectors* $\mathbf{0}$ *and* $\mathbf{1} = \sum_i \boldsymbol{e}_i$ *of all zeroes, respectively all ones.*

The symbols $\boldsymbol{a}, \boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{b}, \boldsymbol{c}, \ldots$ *always denote row vectors in* $(\mathbb{R}^d)^*$*, and represent linear forms. In fact, the row vector* $\boldsymbol{a} \in (\mathbb{R}^d)^*$ *represents the linear form* $\ell = \ell_{\boldsymbol{a}} : \mathbb{R}^d \longrightarrow \mathbb{R}$, $\boldsymbol{x} \longmapsto \boldsymbol{a}\boldsymbol{x}$. *Here* $\boldsymbol{a}\boldsymbol{x}$ *is the scalar obtained as the matrix product of a row vector (i.e., a* $(1 \times d)$*-matrix) with a column vector (a* $(d \times 1)$*-matrix). Matrices like* $A, A', B, \ldots$ *represent a set of row vectors; thus they are usually* $(n \times d)$*- or* $(m \times d)$*-matrices. Furthermore, the order of the rows is not important.*

We use $\mathbb{1} = (1, \ldots, 1)$ *to denote the all-ones row vector in* $(\mathbb{R}^d)^*$*, or in* $(\mathbb{R}^{d\pm 1})^*$*. Thus,* $\mathbb{1}\boldsymbol{x}$ *is the sum of the coordinates of the column vector* $\boldsymbol{x}$*. Similarly,* $\mathbb{O} = (0, \ldots, 0)$ *denotes the all-zeroes row vector.*

Boldface type is reserved for vectors; scalars appear as italic symbols, such as $a, b, c, d, x, y \ldots$*. Thus the coordinates of a column vector* $\boldsymbol{x}$ *will be* $x_1, \ldots, x_d \in \mathbb{R}$*, and the coordinates of a row vector* $\boldsymbol{a}$ *will be* $a_1, \ldots, a_d$.

Basic objects for any discussion of geometry are points, lines, planes and so forth, which are *affine subspaces*, also called *flats*. Among them, the vector subspaces of $\mathbb{R}^d$ (which contain the origin $\mathbf{0} \in \mathbb{R}^d$) are referred to as *linear subspaces*. Thus the nonempty affine subspaces are the translates of linear subspaces.

The *dimension* of an affine subspace is the dimension of the corresponding linear vector space. Affine subspaces of dimensions 0, 1, 2, and $d-1$ in $\mathbb{R}^d$ are called *points*, *lines*, *planes*, and *hyperplanes*, respectively.

For these lectures we need no special mathematical requirements: we just assume that the listener/reader feels (at least a little bit) at home in the real affine space $\mathbb{R}^d$, with the construction of coordinates, and with affine maps $\boldsymbol{x} \longmapsto A\boldsymbol{x} + \boldsymbol{x}_0$, which represent an affine change of coordinates if A is a nonsingular square matrix, or an arbitrary affine map in the general case.

Most of what we do will, in fact, be invariant under any affine change of coordinates. In particular, the precise dimension of the ambient space is usually not really important. If we usually consider "a d-polytope in $\mathbb{R}^d$," then the reason is that this feels more concrete than any description starting with "Let V be a finite-dimensional affine space over an ordered field, and ...."

We take for granted the fact that affine subspaces can be described by affine equations, as the affine image of some real vector space $\mathbb{R}^k$, or as the set of all affine combinations of a finite set of points,

$$F = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} = \lambda_0\boldsymbol{x}_0 + \ldots + \lambda_n\boldsymbol{x}_n \text{ for } \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1\}.$$

That is, every affine subspace can be described both as an intersection of affine hyperplanes, and as the *affine hull* of a finite point set (i.e., as the intersection of all affine flats that contain the set). A set of $n \geq 0$ points is *affinely independent* if its affine hull has dimension $n-1$, that is, if every proper subset has a smaller affine hull.

A point set $K \subseteq \mathbb{R}^d$ is *convex* if with any two points $\boldsymbol{x}, \boldsymbol{y} \in K$ it also contains the straight line segment $[\boldsymbol{x}, \boldsymbol{y}] = \{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} : 0 \leq \lambda \leq 1\}$ between them. For example, in the drawings below the shaded set on the right is convex, the set on the left is not. (This is one of very few nonconvex sets in this book.)

Clearly, every intersection of convex sets is convex, and $\mathbb{R}^d$ itself is convex. Thus for any $K \subseteq \mathbb{R}^d$, the "smallest" convex set containing K, called the *convex hull* of K, can be constructed as the intersection of all convex sets that contain K:

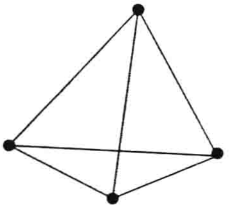
$$\mathrm{conv}(K) := \bigcap \{K' \subseteq \mathbb{R}^d : K \subseteq K',\ K' \text{ convex}\}.$$

Our sketch shows a subset K of the plane (in black), and its convex hull $\mathrm{conv}(K)$, a convex 7-gon (including the shaded part).

For any finite set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq K$ and parameters $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \ldots + \lambda_k = 1$, the convex hull $\operatorname{conv}(K)$ must contain the point $\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_k\boldsymbol{x}_k$: this can be seen by induction on k, using

$$\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_k\boldsymbol{x}_k = (1-\lambda_k)\Big(\frac{\lambda_1}{1-\lambda_k}\boldsymbol{x}_1 + \ldots + \frac{\lambda_{k-1}}{1-\lambda_k}\boldsymbol{x}_{k-1}\Big) + \lambda_k\boldsymbol{x}_k$$

for $\lambda_k < 1$. For example, the following sketch shows the lines spanned by four points in the plane, and the convex hull (shaded).



Geometrically, this says that with any finite subset $K_0 \subseteq K$ the convex hull $\operatorname{conv}(K)$ must also contain the projected simplex spanned by K_0. This proves the inclusion "$\supseteq$" of

$$\operatorname{conv}(K) = \Big\{\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_k\boldsymbol{x}_k : \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq K,\ \lambda_i \geq 0,\ \sum_{i=1}^{k}\lambda_i = 1\Big\}.$$

But the right-hand side of this equation is easily seen to be convex, which proves the equality.

Now if $K = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^d$ is itself finite, then we see that its convex hull is

$$\operatorname{conv}(K) = \Big\{\lambda_1\boldsymbol{x}_1 + \ldots + \lambda_n\boldsymbol{x}_n : n \geq 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{n}\lambda_i = 1\Big\}.$$

The following gives two different versions of the definition of a polytope. (We follow Grünbaum and speak of *polytopes* without including the word "convex": we do not consider nonconvex polytopes in this book.) The two versions are mathematically — but not algorithmically — equivalent. The proof of equivalence between the two concepts is nontrivial, and will occupy us in Lecture 1.

Definition 0.1. A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some $\mathbb{R}^d$.

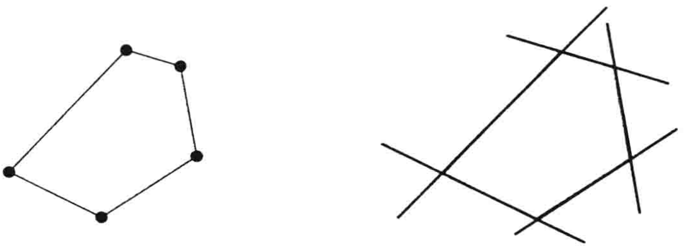
An *$\mathcal{H}$-polyhedron* is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$. An *$\mathcal{H}$-polytope* is an $\mathcal{H}$-polyhedron that is *bounded* in the sense that it does not contain a ray $\{\boldsymbol{x} + t\boldsymbol{y} : t \geq 0\}$ for any $\boldsymbol{y} \neq \boldsymbol{0}$. (This definition of "bounded" has the advantage over others that it does not rely on a metric or scalar product, and that it is obviously invariant under affine change of coordinates.)

A *polytope* is a point set $P \subseteq \mathbb{R}^d$ which can be presented either as a $\mathcal{V}$-polytope or as an $\mathcal{H}$-polytope.

The *dimension* of a polytope is the dimension of its affine hull.

A *d-polytope* is a polytope of dimension d in some $\mathbb{R}^e$ $(e \geq d)$.

Two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are *affinely isomorphic*, denoted by $P \cong Q$, if there is an affine map $f : \mathbb{R}^d \longrightarrow \mathbb{R}^e$ that is a bijection between the points of the two polytopes. (Note that such a map need not be injective or surjective on the "ambient spaces.")



Our sketches try to illustrate the two concepts: the left figure shows a pentagon constructed as a $\mathcal{V}$-polytope as the convex hull of five points; the right figure shows the same pentagon as an $\mathcal{H}$-polytope, constructed by intersecting five lightly shaded halfspaces (bounded by the five fat lines).

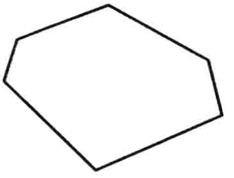
Usually we assume (without loss of generality) that the polytopes we study are full-dimensional, so that d denotes both the dimension of the polytope we are studying, and the dimension of the ambient space $\mathbb{R}^d$.

The emphasis of these lectures is on combinatorial properties of the *faces* of polytopes: the intersections with hyperplanes for which the polytope is entirely contained in one of the two halfspaces determined by the hyperplane. We will give precise definitions and characterizations of faces of polytopes in the next two lectures. For the moment, we rely on intuition from "life in low dimensions": using the fact that we know quite well what a 2- or 3-polytope "looks like." We consider the polytope itself as a trivial face; all other faces are called *proper faces*. Also the empty set is a face for every polytope. Less trivially, one has as faces the *vertices* of the polytope, which are single points, the *edges*, which are 1-dimensional line segments, and the *facets*, i.e., the maximal proper faces, whose dimension is one less than that of the polytope itself.

We define two polytopes P, Q to be *combinatorially equivalent* (and denote this by $P \simeq Q$) if there is a bijection between their faces that preserves the inclusion relation. This is the obvious, nonmetric concept of equivalence that only considers the combinatorial structure of a polytope; see Section 2.2 for a thorough discussion.

Example 0.2. Zero-dimensional polytopes are points, one-dimensional polytopes are line segments. Thus any two 0-polytopes are affinely isomorphic, as are any two 1-polytopes.

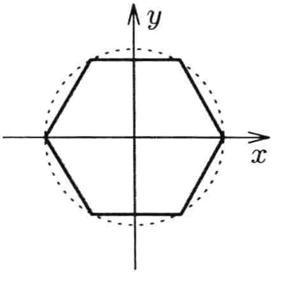
Two-dimensional polytopes are called *polygons*. A polygon with n vertices is called an n-*gon*. Convexity here requires that the interior angles (at the vertices) are all smaller than π. The following drawing shows a convex 6-gon, or *hexagon*.



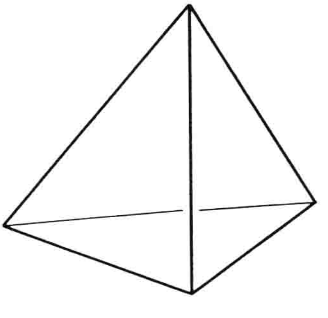
Two 2-polytopes are combinatorially equivalent if and only if they have the same number of vertices. Therefore, we can use the term "the convex n-gon" for the combinatorial equivalence class of a convex 2-polytope with exactly n vertices. There is, in fact, a nice representative for this class: the *regular* n-*gon*,

$$P_2(n) \ := \ \operatorname{conv}\left\{(\cos(\tfrac{2\pi k}{n}), \sin(\tfrac{2\pi k}{n})) : \ 0 \leq k < n\right\} \ \subseteq \ \mathbb{R}^2.$$

The following drawing shows the regular hexagon $P_2(6)$ in $\mathbb{R}^2$. It is combinatorially equivalent, but not affinely isomorphic, to the hexagon drawn above.



Example 0.3. The *tetrahedron* is a familiar geometric object (a 3-dimensional polytope) in $\mathbb{R}^3$:



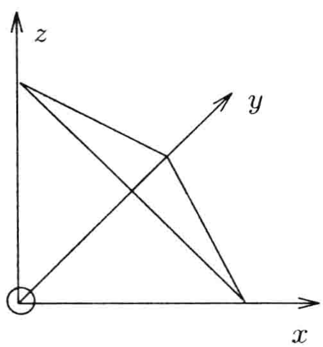
Similarly, its d-dimensional generalization forms the first (and simplest) infinite family of higher-dimensional polytopes we want to consider. We define a *d-simplex* as the convex hull of any $d+1$ affinely independent points in some $\mathbb{R}^n$ $(n \geq d)$.

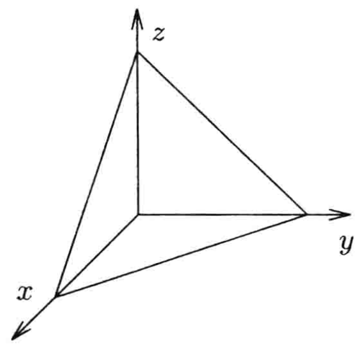
Thus a d-simplex is a polytope of dimension d with $d+1$ vertices. Naturally the various possible notations for the d-simplex lead to confusion, in particular since various authors of books and papers have their own, inconsistent ideas about whether a lower index denotes dimension or number of vertices. In the following, we consistently use lower indices to denote dimension of a polytope (which should account for our awkward $P_2(n)$ for an n-gon...).

It is easy to see that any two d-simplices are affinely isomorphic. However, it is often convenient to specify a canonical model. For the d-simplex, we use the *standard d-simplex* Δ_d with $d+1$ vertices in $\mathbb{R}^{d+1}$,

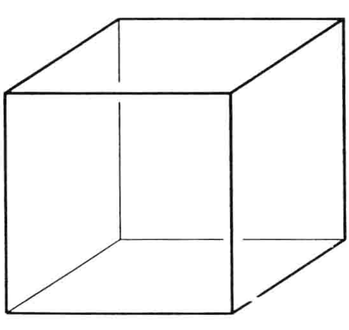
$$\Delta_d \;:=\; \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} : \mathbb{1}\,\boldsymbol{x} = 1,\; x_i \geq 0 \right\} \;=\; \operatorname{conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d+1}\}$$

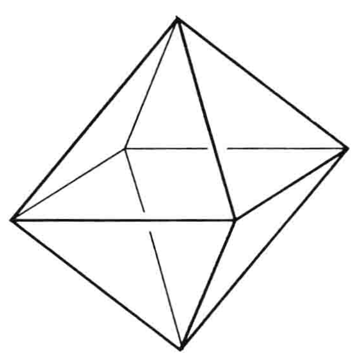
Our figures illustrate the construction of Δ_2 in $\mathbb{R}^3$:





Example 0.4. The three-dimensional *cube* C_3 and the *octahedron* C_3^{Δ} are familiar objects as well:





Their generalization to d dimensions is straightforward. We arrive at the d-dimensional *hypercube* (or the *d-cube*, for short):

$$C_d \;:=\; \{\boldsymbol{x} \in \mathbb{R}^d : -1 \leq x_i \leq 1\} \;=\; \operatorname{conv}\{\{+1, -1\}^d\},$$

and the d-dimensional *crosspolytope*:

$$C_d{}^{\Delta} := \{\boldsymbol{x} \in \mathbb{R}^d : \sum_i |x_i| \leq 1\} = \operatorname{conv}\{\boldsymbol{e}_1, -\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d, -\boldsymbol{e}_d\}.$$

We have chosen our "standard models" in such a way that they are symmetric with respect to the origin. In this version there is a very close connection between the two polytopes C_d and $C_d{}^{\Delta}$: they satisfy

$$\begin{aligned} C_d{}^{\Delta} &\cong \{\boldsymbol{a} \in (\mathbb{R}^d)^* : \boldsymbol{a}\boldsymbol{x} \leq 1 \text{ for all } \boldsymbol{x} \in C_d\ \} \\ C_d &\cong \{\boldsymbol{a} \in (\mathbb{R}^d)^* : \boldsymbol{a}\boldsymbol{x} \leq 1 \text{ for all } \boldsymbol{x} \in C_d{}^{\Delta}\}, \end{aligned}$$

that is, these two polytopes are *polar* to each other (see Section 2.3).

Now it is easy to see that the d-dimensional crosspolytope is a *simplicial* polytope, all of whose proper faces are simplices, that is, every facet has the minimal number of d vertices. Similarly, the d-dimensional hypercube is a *simple* polytope: every vertex is contained in the minimal number of only d facets.

These two classes, simple and simplicial polytopes, are very important. In fact, the convex hull of any set of points that are in general position in $\mathbb{R}^d$ is a simplicial polytope. Similarly, if we consider any set of inequalities in $\mathbb{R}^d$ that are generic (i.e., they define hyperplanes in general position) and whose intersection is bounded, then this defines a simple polytope. Finally the two concepts are linked by polarity: if P and P^{Δ} are polar, then one is simple if and only if the other one is simplicial.

(The terms "general position" and "generic" are best handled with some amount of flexibility — you supply a precise definition only when it becomes clear how much "general position" or "genericity" is really needed. One can even speak of "sufficiently general position"! For our purposes, it is usually sufficient to require the following: a set of $n > d$ points in $\mathbb{R}^d$ is in *general position* if no d of them lie on a common affine hyperplane. Similarly, a set of $n > d$ inequalities is *generic* if no point satisfies more than d of them with equality. More about this in Section 3.1.)

Here is one more aspect that makes the d-cubes and d-crosspolytopes remarkable: they are regular polytopes — polytopes with maximal symmetry. (We will not give a precise definition here.) There is an extensive and very beautiful theory of regular polytopes, which includes a complete classification of all regular and semi-regular polytopes in all dimensions. A lot can be learned from the combinatorics and the geometry of these highly regular configurations ("wayside shrines at which one should worship on the way to higher things," according to Peter McMullen).

At home (so to speak) in 3-space, the classification of regular polytopes yields the well-known five platonic solids: the tetrahedron, cube and octahedron, dodecahedron and icosahedron. We do not include here a drawing of the icosahedron or the dodecahedron, but we refer the reader to

Grünbaum's article [217] for an amusing account of how difficult it is to get a correct drawing (and a "How to" as well).

The classic account of regular polytopes is Coxeter's book [140]; see also Martini [317, 318], Blind & Blind [89], and McMullen & Schulte [336] for recent progress. The topic is interesting not only for "aesthetic" reasons, but also because of its close relationship to other parts of mathematics, such as crystallography (see Senechal [406]), the theory of finite reflection groups ("Coxeter groups," see Grove & Benson [209] or Humphreys [236]), and root systems and buildings (see Brown [116]), among others.

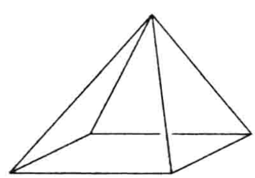
Example 0.5. There are a few simple but very useful *recycling operations* that produce "new polytopes from old ones."

If P is a d-polytope and $\boldsymbol{x}_0$ is a point outside the affine hull of P (for this we embed P into $\mathbb{R}^n$ for some $n > d$), then the convex hull

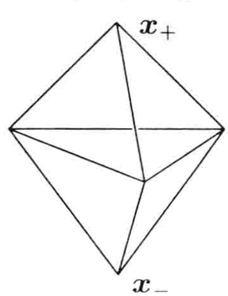
$$\operatorname{pyr}(P) \ := \ \operatorname{conv}(P \cup \{\boldsymbol{x}_0\})$$

is a $(d+1)$-dimensional polytope called the *pyramid* over P. Clearly the affine and combinatorial type of $\operatorname{pyr}(P)$ does not depend on the particular choice of $\boldsymbol{x}_0$ — just change the coordinate system. The faces of $\operatorname{pyr}(P)$ are the faces of P itself, and all the pyramids over faces of P.

Especially familiar examples of pyramids are the simplices (the pyramid over Δ_d is Δ_{d+1}), and the *Egyptian pyramid* $Pyr_3 = \operatorname{pyr}(P_2(4))$: the pyramid over a square.



Similarly we construct the *bipyramid* $\operatorname{bipyr}(P)$ by choosing two points $\boldsymbol{x}_+$ and $\boldsymbol{x}_-$ outside $\operatorname{aff}(P)$ such that an interior point of the segment $[\boldsymbol{x}_+, \boldsymbol{x}_-]$ is an interior point of P. As examples, we get the bipyramid over a triangle



and the crosspolytopes, which are iterated bipyramids over a point,

$$\operatorname{bipyr}({C_d}^{\Delta}) = {C_{d+1}}^{\Delta}.$$

Especially important, it is quite obvious how to define the *product* of two (or more) polytopes: for this we consider polytopes $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$,