# calculus with analytic geometry

# Calculus with Analytic Geometry

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## Preface

As noted in the title, this book includes analytic geometry as well as calculus. The analytic geometry is continued in chapters that are separate from the calculus and is presented from time to time as needed in the development of the calculus. This is done so that the book may be used either as a text for a combined course in calculus and analytic geometry, or as a text for calculus alone. We give somewhat more analytic geometry than do most of the recent books on the combined subjects, but less than is given in books on analytic geometry. Differential and integral calculus of polynomials is presented along with the needed analytic geometry before any work on transcendental functions is given.

Considerable effort has been expended to make the book teachable. The work tends toward the traditional, but we have not lost sight of recent trends and have used modern terminology and concepts when they seemed appropriate. We hope that any lack of sophistication is offset by the readability of the book from the student's point of view. Besides believing that the student can understand the book, we anticipate that he can apply the principles that are presented.

We have seriously attempted to include sufficient worked-out examples to illustrate the text material they follow and the problems they precede. We hope that the discussions in connection with the examples are such that the student will understand and become interested in them.

Exercises have been placed a lesson apart, for ease of assignment each day. Many of the concepts and techniques are of such a nature that more than one day is required for their mastery. We think that enough problems have been included for this purpose. There are about 3,700 problems in 116 exercises, so that more than one day can be spent on a considerable number of them. The problems are in groups of four of about the same order of difficulty and requiring essentially the same concepts and techniques; the order of difficulty increases from group to group. With this arrangement a good assignment could consist of each fourth problem in an exercise.

We wish to express our appreciation to CUPM and to many of our colleagues whose recommendations we have considered in deciding what topics to include and in deciding how to treat them. But in the final analysis, the selection of topics and method of treatment are ours and have been determined in the light of our years of teaching collegiate mathematics.

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*Fred W. Sparks*

## Contents

# 1 Topics from algebra

## 1.1 Sets

One of the basic and useful concepts of mathematics is denoted by
*set* the word "set." A *set* is a collection of well-defined objects or symbols called *elements* or members of the set. By "well defined" we mean
*elements*
that there is a criterion that enables us to decide whether an object or symbol is or is not a member of the set. For example, suppose that S is the set of all bicycles that are green. We can conclude: first, a green bicycle is an element of S; second, a tricycle is not an element of S; third, a bicycle that is not green is not a member of S.

As implied above, capital letters are frequently employed to designate sets. Lowercase letters and numbers are often used to designate the elements of a set. A set is also denoted by enclosing the elements in braces $\{ \}$. For example, if $A = \{a, b, c, d\}$, then $A$ is a set whose elements are $a$, $b$, $c$, and $d$. Furthermore, the notation $B = \{1, 2, 3, \ldots, 99\}$ means that $B$ is the set of natural numbers, or the numbers used in counting, that are less than 100. Note that the three dots between 3 and 99 indicate that the natural numbers between 3 and 99 are included in the set $B$. Another notation for a set that is frequently employed is illustrated by the following:

$$A = \{x \mid x \text{ is an even natural number less than } 5\}$$

The vertical bar is read "such that." It follows at once that $A = \{2, 4\}$. This is often referred to as set-building notation.

The notation $a \in A$ means that $a$ is an element of the set $A$ and is sometimes read "$a$ belongs to $A$."

*subset*

*proper subset*

If each element of the set $A$ is a member of the set $B$, then $A$ is a *subset* of $B$, and we write $A \subseteq B$. If the set $B$ contains at least one element that is not a member of $A$, then $A$ is a *proper subset* of $B$, and this situation is indicated by $A \subset B$. For example, if $A = \{a, b, c, d\}$ and $B = \{a, b, c, d, e, f\}$ then $A \subset B$. If, however, $B = \{a, b, c, d\}$, then $A \subseteq B$.

*identical sets*

Two sets $A$ and $B$ are *identical* if and only if each is a subset of the other. For example, the sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 2, 3, 1\}$ are identical. This situation is indicated by writing $A = B$.

*null set*

A set that contains no elements is called the *empty* or *null* set and is indicated by the symbol $\varnothing$.

**Example 1** $\{x \mid x \text{ is a woman who has been president of the United States}\} = \varnothing$

**Example 2** $\{x \mid x \text{ is a two-digit natural number less than } 10\} = \varnothing$

It frequently happens that the same set of elements belongs to each of two sets $A$ and $B$. This set is called the intersection of $A$ and $B$ and is designated by $A \cap B$. More precisely we define the intersection below.

*intersection of two sets*

The *intersection* of two sets, $A \cap B$, is the set $\{x \mid x \in A \text{ and } x \in B\}$.

For example, if $A = \{x \mid x \text{ is a natural number less than } 10\}$ and $B = \{x \mid x \text{ is a natural number divisible by } 2\}$, then $A \cap B = \{2, 4, 6, 8\}$.

Obviously if two sets have no elements in common, their intersection is the null set $\varnothing$. For example, since no former governor of Texas has been a governor of California, then $\{x \mid x \text{ is a former governor of Texas}\} \cap \{x \mid x \text{ is a former governor of California}\} = \varnothing$.

*disjoint sets*

Two sets $A$ and $B$ are *disjoint* if $A \cap B = \varnothing$.

The concept of the intersection of sets can be extended to three or more sets. For example

$$A \cap B \cap C = \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}$$

*union*

The *union* of two sets is denoted by $A \cup B$ and is defined to be the set of elements that belong to $A$ or to $B$ or to both $A$ and $B$. In the symbolism of sets,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Similarly,

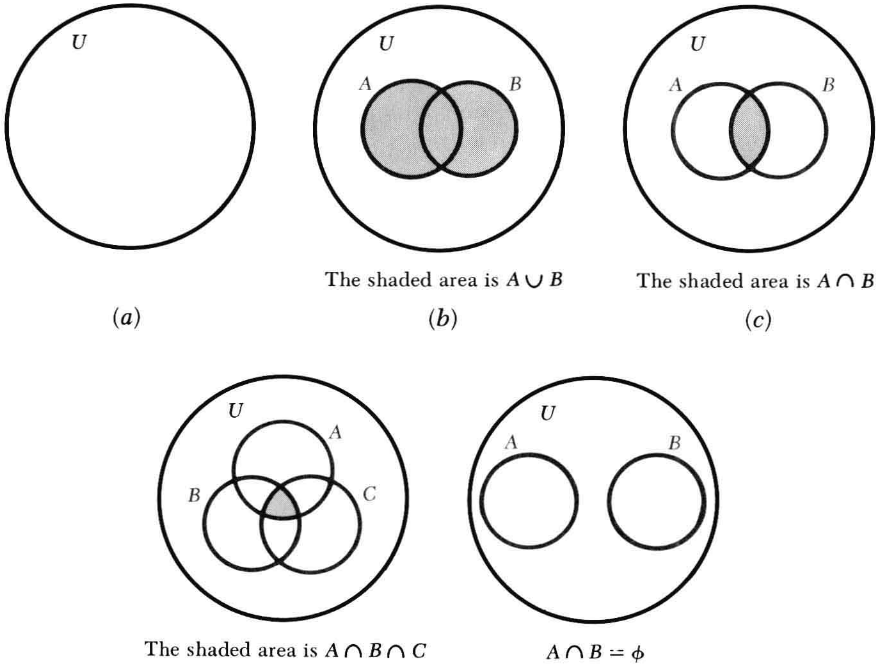
$$A \cup B \cup C = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in C\}$$

For example, if $A = \{1, 3, 5\}$, $B = \{2, 3, 6\}$, and $C = \{4, 5, 7, 9\}$, then $A \cup B = \{1, 2, 3, 5, 6\}$ and $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 9\}$.

*universal set*

The totality of elements that are involved in any specific situation or discussion is called the *universal set* and is designated by $U$. For example, each of the various clubs, athletic teams, academic classes, and any other group whose members are students at a given college are subsets of the universal set composed of the entire student body of the college.

A method for picturing sets and certain relations between them was devised by an Englishman, John Venn (1834–1923). The fundamental idea is to represent a set by a simple plane figure. In order to illustrate the method, we shall use circles. We shall represent the universal set $U$ by a circle $C$ and shall define $U$ to be the set of all points within and on the circumference of $C$. We shall represent the various subsets of $U$ by circles wholly within the circle $C$. Figure 1.1 illustrates the device.




(*a*)

The shaded area is $A \cup B$

(*b*)

The shaded area is $A \cap B$

(*c*)

The shaded area is $A \cap B \cap C$

(*d*)

$A \cap B = \phi$

(*e*)

**Figure 1.1 Venn diagram**

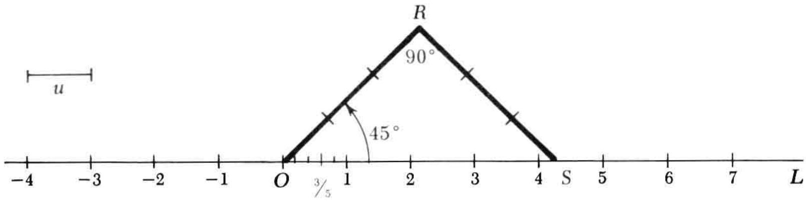



**Figure 1.2 Number line**

## 1.2 Real numbers

In elementary calculus we use the real number system almost exclusively. This system is defined in college algebra, and the numbers are interpreted by use of the number line such as line $L$ shown in Fig. 1.2. In this figure the unit length $u$ is laid off successively to the right and to the left of the point $O$ on $L$. The positive integers 1, 2, 3, 4, 5, 6, . . . are associated with the successive right extremities of the intervals to the right of $O$ and the negative integers $-1$, $-2$, $-3$, $-4$, $-5$, $-6$, . . . are associated with the left extremities of the intervals to the left of $O$. In order to obtain the point associated with $3/5$, we divide the interval from $O$ to 1 into 5 equal parts and then associate $3/5$ with the right extremity of the third of these subintervals.

*rational number* We say that $3/5$ is the quotient of 3 and 5 and call it a *rational number.*

If we construct the right triangle $ORS$ as indicated in the figure, the length of the line segment $OS$ is $\sqrt{3^2 + 3^2} = \sqrt{18}$. Hence the point $S$ is associated with $\sqrt{18}$. Furthermore, $\sqrt{18}$ cannot be expressed as the quotient of two integers.* We can, however, obtain a decimal representation of $\sqrt{18}$ by a repeated application of the square-root process of arithmetic. The process never terminates and the decimal fraction never becomes periodic. Thus we say that $\sqrt{18}$ is a nonterminating, nonperiodic decimal fraction. We call such numbers

*irrational number* *irrational.*

Each point on the line $L$ is associated with one and only one number that is either an integer, a rational number, or an irrational number. Furthermore, each number is associated with one and only one point on the line. Since the number $a$ is associated with only one point on $L$, we shall frequently refer to the number $a$ as the point $a$.

As indicated in Fig. 1.3, we define the sum $a + b$ as the number associated with the point on $L$ that is a distance of $b$ to the right of the point $a$, and the sum $a + (-b)$ as the number associated with the point that is a distance of $b$ to the left of $a$. It can be verified that the point $a + b$ is the same as $b + a$. We shall assume that this is true

* For proof of this statement see Rees and Sparks, College Algebra, 5th ed., McGraw-Hill Book Company, New York, 1967.

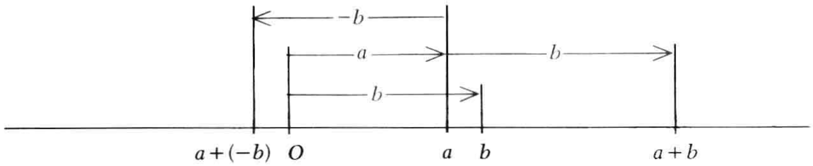



**Figure 1.3 Sum of two numbers**

for any two numbers; thus we have the axiom

$$a + b = b + a \tag{1}$$

The positive numbers are the numbers associated with the points to the right of $O$ on $L$, and the negative numbers are those associated with the points to the left of $O$.

We say that $a > b$ if the point $a$ is to the right of the point $b$ on $L$ and that $c < d$ if the point $c$ is to the left of the point $d$.

By the above interpretation the point associated with the sum $a + (-a)$ is the point $O$. Furthermore the point $O + a$ is the point $a$. Heretofore, we have not called $O$ a number, but have used the symbol to designate the reference point on $L$. We now define the number *zero* to be the number such that

$$\text{Zero} + a = a \tag{2}$$

and shall designate it with the symbol 0. It is consistent with the above reasoning to assign the number zero to the point $O$.

*negative of a number*

We shall now define the *negative* of the number $a$ to be the number $-a$ such that

$$a + (-a) = 0 \tag{3}$$

As implied by the above discussion, the positive integers are the natural numbers, or the numbers used in counting. The negative integers are the negatives of the natural numbers, and 0 is the number such that $0 + a = a$. We are now in a position to define *the set of integers I* as follows:

*the set of integers*

$$I = \{p \mid p \text{ is a natural number}\} \cup \{0\} \cup \{n \mid n \text{ is the negative of a natural number}\}$$

*rational number*

A *rational number* is a number that can be expressed as the quotient of two integers. Since the integer $n = n/1$, any integer is also a rational number. Hence, if $J$ is the set of rational numbers, then $I \subset J$.

*irrational number*

An *irrational* number is a number whose decimal representation is a nonterminating, nonperiodic decimal. Such numbers cannot be expressed as the quotient of two integers. We shall represent the set of irrational numbers by $K$.