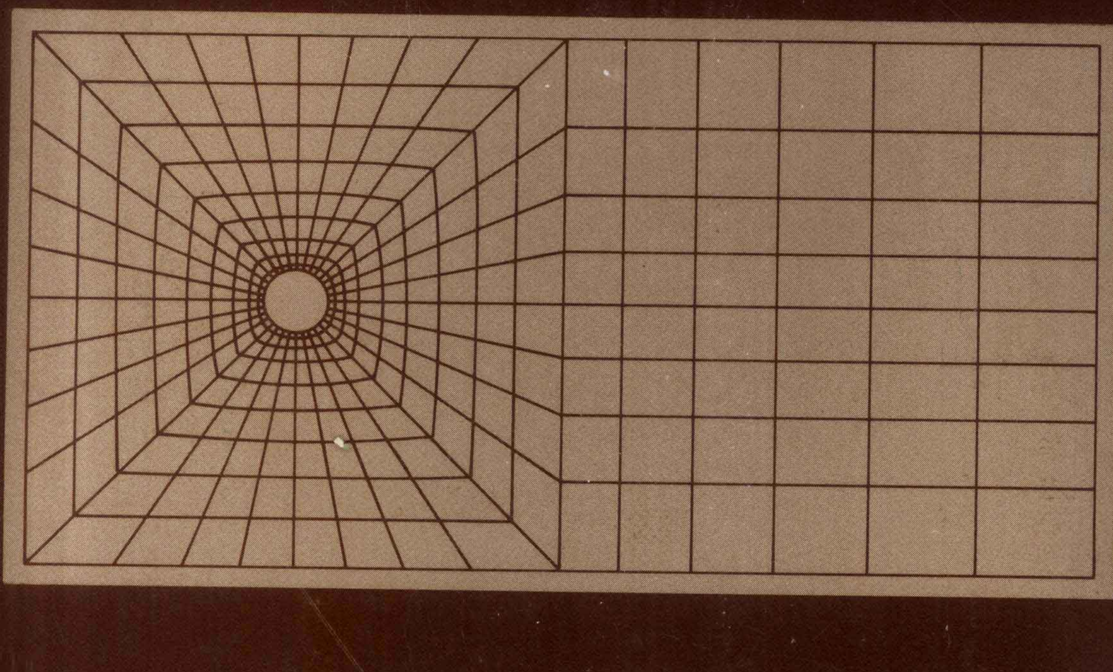
AMD - Vol. 54

Computer Methods for Nonlinear Solids and Structural Mechanics



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Computer Methods for Nonlinear Solids and Structural Mechanics

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PREFACE

It is commonplace in engineering practice today in structural and solid mechanics to utilize linear computer programs for general purpose applications in many areas ranging from automobile problems through to aerospace and machine design, to name a few. While vigorous research activity has persisted for more than a decade on nonlinear phenomena, applications in computational and structural mechanics are occurring in a small but growing number of real situations. Nonlinear computer program developers frequently find themselves at the cutting edge of research as they attempt to refine their complex software products.

We are indeed making progress in new and exciting fields such as automated manufacturing and robotics along with more classical disciplines of buckling, vibration, transient response, etc.; however, new difficulties are emerging with pronounced nonlinear aspects, be they geometrical, material-related, or otherwise. It is important to note as complex nonlinear problems arise with greater frequency, that we are armed with much better tools to resolve them. We might categorize the tools as software (including methods of analysis, solution strategies and algorithms) and hardware.

As we anticipated many years ago, gains in computer hardware have taken place and continue to do so in systems ranging from large mainframes like the Cray series through to desktop micros that were termed mini computers but a few years ago. As a result of the availability of incredibly cheap computing power, many more nonlinear problems are falling within the purview of feasible solution. Rather than hardware, the emphasis in the papers presented in this volume is on improved methods and analytical techniques for nonlinear problems.

Organized through the auspices of the Applied Mechanics Division's Committee on Computing in Applied Mechanics, the Symposium from which this volume emanates is, in a sense, a follow-up to a meeting of a decade ago organized by Dr. Richard F. Hartung (November, 1973, Numerical Solution of Nonlinear Structural Problems).While that particular productive symposium focused on static problems, the present one is more general.

We thank the authors for their cooperation in preparing their manuscripts on schedule and also Mr. Leggitt and his staff at ASME for their help in bringing this volume out on schedule.

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A POST-PROCESSING APPROACH IN THE FINITE ELEMENT METHOD

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ABSTRACT

In this paper we discuss a method for post-processing a finite element solution in order to obtain high accuracy approximations for displacements, stresses, stress intensity factors etc. Rather than take the values of these quantities directly from the finite element solution, as is usually done, we evaluate certain weighted averages of the finite element solution over the entire region or over the boundary. These yield approximations having the same order of accuracy as the error in the strain energy. We briefly discuss some theoretical error estimates. To illustrate the practical effectivity of the technique, we present some numerical examples. In discussing these examples we shall touch upon the important practical issues of adaptive mesh selection and a'posteriori error estimation.

1. INTRODUCTION

The finite element method has become one of the major tools in contemporary computational mechanics. In the context of structural mechanics, the basic theoretical results of finite element analysis relate to the convergence of the strain energy of the error in the finite element solution. Usually, however, one is primarily interested in the values of displacements, stresses or stress intensity factors at some points. The question of the accuracy at a point of the displacement or the stress of the finite element computation is difficult to analyze in general. Nevertheless, both theory and practice show that in many cases the accuracy in the stresses is of the same order as the square root of the strain energy of the error. (The square root of the strain energy is often called the strain energy norm, e.g. a 1% relative accuracy measured in terms of strain energy is the same as a 10% relative accuracy measured in the strain energy norm.) It is well known that the pointwise finite element displacements are considerably more accurate than the corresponding stresses. In linear fracture mechanics, it has been observed that the computation of stress intensity factors by use of the Energy Release principle (see, for example, [1]) leads to good results even in cases where the computed pointwise stresses and displacements are inaccurate. The success is due to the fact that the accuracy of the method is of the order of the strain energy of the error, not the strain energy norm of the error [2]. (The various curve fitting methods used for the computation of the stress intensity factors are much less reliable.)

This experience in fracture mechanics leads, quite naturally, to the question: Is it possible to obtain approximations for displacements, stresses and the individual stress intensity factors K_I and K_{II}, etc. all with an accuracy of the same order as the strain energy of the error? In this paper we shall show that this is indeed possible by a suitable post-processing of the finite element solution. This entails a weighted averaging of the finite element solution, and the loading data of the problem, over the region or its boundary. The stiffness derivative [1] and the J-integral [3] methods of linear fracture mechanics can be viewed as particular forms of this post-processing procedure. In addition we shall see how adaptive finite element techniques and a'posteriori error estimation may be employed in post-processing calculations.

We shall not go into too much detail here, but rather illustrate the fundamental ideas in the setting of a few simple model problems. For further details see [2], [4], [5] and [6].

2. MODEL PROBLEM I (THE MEMBRANE PROBLEM)

Consider the simple model problem

$$\begin{aligned} \nabla^2 w &= f \quad \text{in} \quad \Omega = (-1,1) \times (-1,1) , \\ w &= 0 \quad \text{on the boundary } \partial\Omega \text{ of } \Omega , \end{aligned} \tag{2.1}$$

which describes a square membrane, fixed at its edges and loaded with a body force of surface density f.

2.1 Extraction of displacement $w(0,0)$

Let (r,θ) be polar coordinates centered on $(0,0)$. Define a function

$$\phi = \frac{1}{2\pi} \log r - \phi_0 , \tag{2.2}$$

where ϕ_0 is any function which satisfies $\phi_0 = \frac{1}{2\pi} \log r$ on $\partial\Omega$. In particular then, $\phi = 0$ on $\partial\Omega$, and so an integration by parts gives

$$\int_\Omega \nabla^2 \phi w \, dA = \oint_{\partial\Omega} (\nabla\phi\cdot\hat{n}\, w - \nabla w\cdot\hat{n}\, \phi) \, ds + \int_\Omega \nabla^2 w \phi \, dA = \int_\Omega f\phi \, dA \tag{2.3}$$

by (2.1). Notice, however, that

$$\nabla^2 \left(\frac{1}{2\pi} \log r\right) = \delta_0$$

where δ_0 is the Dirac delta function (concentrated unit load) positioned at the origin. Therefore, we have the alternate expression

$$\int_\Omega \nabla^2 \phi w \, dA = w(0,0) - \int_\Omega \nabla^2 \phi_0 w \, dA .$$

Substituting this into (2.3) and rearranging gives

$$W = w(0,0) = \int_\Omega \nabla^2 \phi_0 w \, dA + \int_\Omega f\phi \, dA \tag{2.4}$$

We shall call this expression an extraction expression for $w(0,0)$. The function $\zeta = \nabla^2 \phi_0$ will be referred to as the extraction function, and the integral $\int_\Omega f\phi \, dA$ (which relates to (2.1) only through the known loading f) will be called the load term.

If $\tilde{w}$ is a finite element approximation to w, then we may replace w in (2.4) by $\tilde{w}$ to obtain

$$\tilde{W} = \int_\Omega (\nabla^2 \phi_0 \tilde{w} + f \phi) \, dA \tag{2.5}$$

as an approximation to $W = w(0,0)$. Our post-processing calculation for W will be based on (2.5).

There are many choices of ϕ_0, and hence ϕ, that may be employed in (2.4) and (2.5). For instance, if we let ϕ be the influence (Green's) function for (2.1) then $\tilde{w}$ would be exact,

$$\tilde{W} = \int_\Omega f\phi \, dA = w(0,0) \quad .$$

Of course, this is unrealistic since we do not have easy access to the influence function. If we set $\phi_0 = \frac{1}{2\pi} \log r$, and hence $\phi = 0$, then (2.5) would give

$$\tilde{W} = \int_\Omega \delta_0 \tilde{w} \, dA = \tilde{w}(0,0) \quad ,$$

and our post-processing would just return the pointwise value of $\tilde{w}$ at (0,0). As indicated in the introduction we hope to achieve higher accuracy than this. The key is to choose ϕ_0 to be smooth. To see why this should be, we shall first obtain an estimate for the difference $|w(0,0)-\tilde{W}|$.

Subtracting (2.5) from (2.4) gives

$$|\tilde{W} - \tilde{W}| = \left| \int_\Omega \zeta(w-\tilde{w}) dA \right| \quad , \tag{2.6}$$

where we have written ζ for the extraction function $\nabla^2\phi_0$. Introduce an auxiliary problem:

$$\begin{aligned} \nabla^2\psi &= -\zeta \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega \quad . \end{aligned} \tag{2.7}$$

Notice that this is of the same form as (2.1), only with a different body loading. An integration by parts shows that

$$\int_\Omega \zeta(w-\tilde{w}) \, dA = \int_\Omega \nabla\psi\cdot\nabla(w-\tilde{w}) \, dA \quad . \tag{2.8}$$

Recall the fundamental orthogonality property of finite element theory,

$$\int_\Omega \nabla v\cdot\nabla(w-\tilde{w}) dA = 0$$

for any admissible finite element function v. In particular, choosing v to be the finite element solution $\tilde{\psi}$ of (2.7), it then follows from (2.6) and (2.8) that

$$|W - \tilde{W}| = \int_\Omega \nabla(\psi-\tilde{\psi}) \cdot \nabla(w-\tilde{w}) dA \quad . \tag{2.9}$$

This integral may be bounded to give the estimate

$$|W - \tilde{W}| \le E(\psi-\tilde{\psi})^{\frac{1}{2}} E(w-\tilde{w})^{\frac{1}{2}} = |E(\psi)-E(\tilde{\psi})|^{\frac{1}{2}} |E(w)-E(\tilde{w})|^{\frac{1}{2}} \quad . \tag{2.10}$$

Here $E(u)$ denotes the strain energy of a function u,

$$E(u) = \int_\Omega \nabla(u)\cdot\nabla(u) \, dA \quad .$$

In other words then, the error in the post-processed value $\tilde{W}$ is bounded by product of the strain energy norm of the errors in $\tilde{w}$ and $\tilde{\psi}$, the simultaneous finite element approximations to the solutions w and ψ of the primary problem (2.1) and the auxiliary problem (2.7) respectively. We emphasize the term simultaneous finite element approximation. It is important to realize that in (2.9) and (2.10) the finite element solutions $\tilde{w}$ and $\tilde{\psi}$ must be defined using the same class of admissible finite element functions.

The estimate (2.10) hints at how we should, at least in principle, select ϕ_0. If ϕ_0, and hence $\zeta = \nabla^2\phi$, could be chosen so that the solution ψ of (2.7) is smooth enough to guarantee that $E(\psi-\tilde{\psi})^{\frac{1}{2}} \sim E(w-\tilde{w})^{\frac{1}{2}}$ or better, then we would have $|W - \tilde{W}| \sim E(w-\tilde{w})$, or better. This is the goal we set in §1 for the accuracy of our post-processing calculations.

Let us examine (2.10) in some further detail for the particular case of $f = -1$. Suppose that we are using a quasi-uniform finite element mesh of mesh size h and C^0 conforming elements of degree p. Our discussion will be broken into two parts. Firstly, we shall work with the h-version of the finite element method, and secondly with the p-version. (For details on the h and p-versions see [7].)

a) The h-version: This is the standard version of the finite element method. In it convergence of the approximate solution is presumed to occur as $h \to 0$ with p fixed.

The solution w has a mild singularity at the vertices of the square Ω. It may be shown that

$$C(p)^{-1}h^{\mu} \leq E(w-\tilde{w})^{\frac{1}{2}} \leq C(p)h^{\mu} \tag{2.11}$$

where $\mu = \min(\frac{3}{2},p)$, and C is a generic constant not depending on h. In general we also have the pointwise estimate

$$|w(0,0) - \tilde{w}(0,0)| \leq C(p)h^{\bar{\mu}-\varepsilon}$$

where $\bar{\mu} = \min(\frac{5}{2},p+1)$ and $\varepsilon > 0$ is any positive number. For a general smooth ϕ_0 we have an estimate similar to (2.11) for $E(\psi-\tilde{\psi})^{\frac{1}{2}}$,

$$E(\psi-\psi)^{\frac{1}{2}} \leq C(p)h^{\mu} \quad . \tag{2.12}$$

Therefore (2.10) becomes

$$|W-\tilde{W}| \leq C(p)h^{2\mu} \quad ,$$

and we see that the error in $\tilde{W}$ is of the same order as the strain energy of the error in $\tilde{w}$. Note that for $p \geq 2$, $|W-\tilde{W}| \sim O(h^3)$ while $|w(0,0)|-\tilde{w}(0,0)| \sim O(h^{5/2-\varepsilon})$.

Let us now be more particular in our choice of ϕ_0. Suppose that $\zeta = \nabla^2\phi_0$ vanishes in a neighborhood of the boundary $\partial\Omega$ and Ω. (This can be achieved if we let ϕ_1 be a smooth function which is unity near $\partial\Omega$ and zero near (0,0), and then set $\phi_0 = (\frac{1}{2\pi}\log r)\phi_1$.) Now instead of (2.12) we have

$$E(\psi-\tilde{\psi})^{\frac{1}{2}} \leq C(p)h^{p} \quad ,$$

and consequently

$$|W-\tilde{W}| \leq C(p)h^{\mu+p} \quad ,$$

and we see that $|W-\tilde{W}|$ has a superior asymtotic (as $h \to 0$) convergence rate than $E(w-\tilde{w})$.

Note however that our analysis above has been asymtotic. We have said nothing about the numerical values of the generic constant C. In any practical post-processing using a non-zero h, these constants may dominate in the above estimates.

b) The p-version: In the p-version of the finite element method the convergence of the approximate solution is presumed to occur as $p \to \infty$ with h fixed.

It may be shown that

$$C(\varepsilon)^{-1}p^{-(3+\varepsilon)} \leq E(w-\tilde{w})^{\frac{1}{2}} \leq C(\varepsilon)p^{-(3-\varepsilon)}$$

for any $\varepsilon > 0$. Likewise for a general smooth ϕ_0

$$E(\psi-\tilde{\psi})^{\frac{1}{2}} \leq C(\varepsilon)p^{-(3-\varepsilon)} \quad . \tag{2.13}$$

Therefore from (2.10)

$$|W-\tilde{W}| \le C(\varepsilon)p^{-(6-\varepsilon)} \quad ,$$

again showing that $|W-\tilde{W}| \sim E(w-\tilde{w})$.

Again making the particular choice for ϕ_0 outlined in (a) above, instead of (2.13) we have

$$E(\psi-\tilde{\psi})^{\frac{1}{2}} \le C(n)p^{-n}$$

with $n > 0$ arbitrary. Therefore, in this case,

$$|W-\tilde{W}| \le C(\varepsilon,n)p^{-(3+n-\varepsilon)}$$

with n arbitrarily large. As in (a), we see that $|W-\tilde{W}|$ now has a superior asymtotic (as $p\to\infty$) convergence rate than $E(w-\tilde{w})$. Once more however, we caution that we have made no attempt to estimate the generic constants C appearing in the above estimates.

2.2 Extraction of the stress component $\frac{\partial w}{\partial x_1}(1,0)$

Let (r,θ) now be polar coordinates centered at (1,0), and define

$$\phi = \frac{1}{\pi}\frac{\cos\theta}{r} - \phi_0$$

where ϕ_0 is a smooth function satisfying $\phi_0 = \frac{1}{\pi}\frac{\cos\theta}{r}$ on $\partial\Omega$. Proceeding much as in §2.1, we obtain an extraction expression for the stress component

$$V = \frac{\partial w}{\partial x_1}(1,0) = -\int_\Omega \nabla^2\phi_0 w\, dA - \int_\Omega f\phi\, dA \quad .$$

The extraction function in this case is $\zeta = -\nabla^2\phi_0$ and the load term is $R = -\int_\Omega f\phi dA$. We may replace w by $\tilde{w}$ in the extraction expression to obtain an approximation

$$\tilde{V} = -\int_\Omega \nabla^2\phi_0\tilde{w}\, dA - \int_\Omega f\phi\, dA \qquad (2.14)$$

to $\frac{\partial w}{\partial x_1}(1,0)$. The analysis of the accuracy of the approximation $\tilde{V}$ is similar to the corresponding analysis for $\tilde{W}$ in §2.1. Just as we saw there, the accuracy of $\tilde{V}$ is closely related to the smoothness of the solution of an auxiliary problem similar to (2.7).

3. MODEL PROBLEM II (SLIT MEMBRANE PROBLEM)

As another example, consider the problem illustrated in Figure 1. This models a circular membrane slit along a radius. The face Γ_1 of the slit is fixed while the other face Γ_2 is load free. The solution w of this problem is known to have an expansion about the origin of the form

$$w = k_1 r^{\frac{1}{4}}\sin\frac{\theta}{4} + k_2 r^{3/4}\sin\frac{3\theta}{4} + k_3 r^{5/4}\sin\frac{5\theta}{4} + O(r^{7/4}) \quad ,$$

where (r,θ) are polar coordinates centered at (0,0). The constants k_1,k_2 and k_3 can be thought of as analogs of the stress intensity factors of linear fracture mechanics.

A number of extraction expressions for the k_m's can be derived. We shall mention two possibilities. Firstly, define

$$\phi = \frac{2}{\pi(2m-1)} r^{-(2m-1)\frac{1}{4}} \sin((2m-1)\frac{\theta}{4}) \qquad (m=1,2,3) \quad .$$

Then it may be shown [5]

$$k_m = \int_{\Gamma_3} (g\varphi - \nabla\phi\cdot\hat{n}\, w)ds \quad . \qquad (3.1)$$

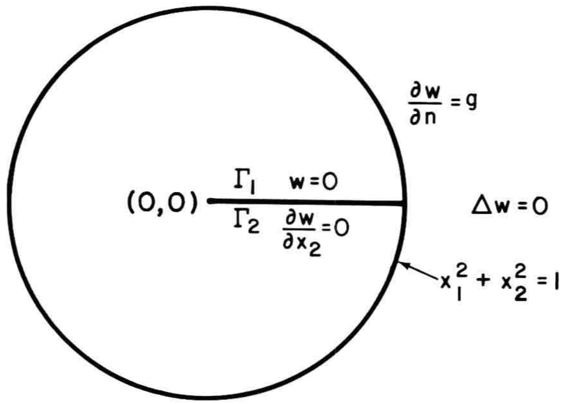

Figure 1. Slit Membrane Problem

Notice that unlike the extraction expressions that we met in §2, (3.1) involves an integration over part of the boundary of Ω .

To describe another extraction expression let us set

$$v = \frac{-8}{\pi(2m-1)(5-2m)} \; r^{(5-2m)^{1/4}} \sin((5-2m)\frac{\theta}{4}) \qquad (m = 1,2,3) \quad .$$

Suppose that ϕ is any sufficiently smooth function that is near (0,0) and vanishes on Γ_3 . In [5] it is shown that

$$k_m = \int_\Omega (\nabla w)^T \begin{bmatrix} -\frac{\partial\phi}{\partial x_1} & -\frac{\partial\phi}{\partial x_2} \\ -\frac{\partial\phi}{\partial x_2} & \frac{\partial\phi}{\partial w_1} \end{bmatrix} (\nabla v) \; dA \qquad (3.2)$$

This extraction expression is related to the stiffness derivative [1] and J-integral [3] methods of fracture mechanics.

As before we may obtain an approximation to k_m by replacing w by $\tilde{w}$ in (3.1) or (3.2). The analysis of the accuracy of these approximations is similar to the corresponding analysis in §2.1.

4. TWO DIMENSIONAL PROBLEMS (SEEPAGE PROBLEM)

4.1 Formulation of the problem

We shall briefly discuss the problem illustrated in Figure 2. This models the seepage under a dam which has been fitted with a grout curtain. The mathematical model is based upon the following assumptions:

(i) a linear relationship (Darcy's law) between flow velocity $v = (v_1, v_2)$ and the gradient of the piezomatric head w , namely $v = -k\nabla w$,where $k > 0$ is the permeability of the (isotropic) ground material.

(ii) the flow is incompressible, that is $\nabla \cdot v = 0$.

We shall restrict our attention to the slit region Ω (the slit models the grout curtain). The region Ω is composed of two subregions Ω_I and Ω_{II} which meet along an oblique interface Γ_0 . We suppose that the permeability is constant in each subregion (zone) taking the values k_I and k_{II} in Ω_I and Ω_{II} respectively. The ground surface downstream from the dam is taken as the zero of piezometric head, and the value of w at the ground surface upstream has been scaled to unity. The region Ω is assumed large enough that a no outflow condition is valid on all the below ground boundaries. Finally, the grout curtain and the dam base are assumed impermeable.

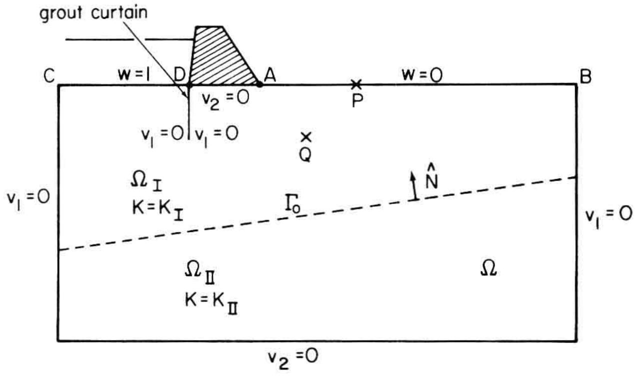

Figure 2. Model for seepage beneath an impermeable dam with a grout curtain. [w = piezometric head; k = permeability; $v = (v_1, v_2) = -k\nabla w$ (Darcy's Law)]

The governing differential equation is

$$\nabla^2 w = 0 \quad \text{in} \quad \Omega_I \text{ and } \Omega_{II} \quad , \tag{4.1}$$

with boundary conditions on $\partial\Omega$ as shown in Figure 2. There is also an interface condition

$$(-k\nabla w \cdot \hat{N})_I = (-k\nabla w \cdot \hat{N})_{II} \quad \text{on } \Gamma_0 \quad , \tag{4.2}$$

where $\hat{N}$ is a unit normal to Γ_0, and the subscripts I, II indicate limiting values as Γ_0 is approached within Ω_I, Ω_{II} respectively.

The three quantities that we shall be interested in evaluating are:

(i) the total outflow per unit time through the downstream surface AB,

$$\Phi_1 = -\int_{AB} k_I \frac{\partial w}{\partial x_2} \, ds \quad .$$

(ii) the outflow speed $\Phi_2 = -k_I \frac{\partial w}{\partial x_2}$ at a point on the downstream surface AB.

(iii) the piezometric head $\Phi_3 = w$ at a point Q in Ω_1 .

4.2 <u>An extraction expression for Φ_1</u>

Suppose that ϕ is a smooth function which satisfies (i) $\phi = -1$ on AB, and (ii) $\phi = 0$ on CD. Then an integration by parts shows

$$\int_\Omega k\nabla w \cdot \nabla\phi \, dA = \int_{\partial\Omega} k\nabla w \cdot \hat{n} \, \phi \, ds + \int_{\Gamma_0} [(k\nabla w \cdot \hat{N})_{II} - (k\nabla w \cdot \hat{N})_I]\phi \, ds$$

$$- \int_{\Omega_I \cup \Omega_{II}} k\nabla^2 w \, \phi \, dA$$

where $\hat{n}$ is the outward pointing unit normal on $\partial\Omega$. Using (4.1), (4.2) and the boundary conditions for w on $\partial\Omega$, we obtain the extraction expression

$$\Phi_1 = -\int_{AB} k_I \frac{\partial w}{\partial x_2}\, ds = \int_\Omega k\nabla w \cdot \nabla\phi \, dA \qquad (4.3)$$

Notice that (4.3) is different from the extraction expressions of §2 and §3 in that it entails the integration of the derivatives of w. It is not difficult to see that there are many choices for the function ϕ which lead to smooth extraction functions in (4.3).

As usual the extraction expression (4.3) suggests that we try to approximate Φ_1 by

$$\tilde{\Phi}_1 = \int_\Omega k\nabla\tilde{w} \cdot \nabla\phi \, dA$$

where $\tilde{w}$ is the finite element approximation to w.

4.3 <u>An extraction expression for Φ_2</u>

If we set

$$\phi = \frac{1}{\pi}\frac{\sin\theta}{r} \qquad (r,\theta \text{ polar coordinates centered at P}) \qquad (4.4)$$

then it can be shown that

$$\Phi_2 = -k_I \frac{\partial w}{\partial x_2}(P) = \int_{\partial\Omega} k\nabla\phi\cdot\hat{n}\, w\, ds - (k_I - k_{II})\int_{\Gamma_0} \nabla\phi\cdot\hat{N} w\, ds , \qquad (4.5)$$

where $\hat{n}$ is the outward pointing unit normal on $\partial\Omega$. Notice that the integral over $\partial\Omega$ includes no contribution from AB, since $w = 0$ there. The extraction function in (4.5) is smooth provided P is "reasonably" distant from A or B.

Based on (4.5) we have the approximation

$$\tilde{\Phi}_2 = \int_{\partial\Omega} k\nabla\phi\cdot\hat{n}\tilde{w}\, ds - (k_I - k_{II})\int_{\Gamma_0} \nabla\phi\cdot\hat{N}\tilde{w}\, ds$$

to Φ_2.

4.4 <u>An extraction expression for Φ_3</u>

Set $\phi = \frac{1}{2\pi k_I}(\log r - \phi_0)$ (r,θ polar coordinates centered on Q) ,

where ϕ_0 is some smooth blending function which ensures that ϕ vanishes on AB and CD. Proceeding much as in §2.1 we find

$$\Phi_3 = w(Q) = \int_{\partial\Omega} k\nabla\phi\cdot\hat{n}\, w \, ds - (k_I - k_{II})\int_{\Gamma_0} \nabla\phi\cdot\hat{N} w\, ds - \int_\Omega k\nabla^2\phi_0 w\, dA . \qquad (4.6)$$

We could, for instance, use $\phi_0(x) = \log|Q-(x_1,d)|$ where d is the perpendicular distance of x from the line CB. Then provided Q is "reasonably" distant from $\partial\Omega$ and Γ_0, the extraction functions appearing in (4.6) are smooth.

As usual we may use (4.6) to obtain the approximation

$$\tilde{\Phi}_3 = \int_{\partial\Omega} k\nabla\phi\cdot\hat{n}\tilde{w}\, ds - (k_I - k_{II})\int_{\Gamma} \nabla\phi\cdot\hat{N}\tilde{w}\, ds - \int_\Omega k\nabla^2\phi_0\tilde{w}\, dA .$$

5. MODEL PROBLEM IV (EDGE CRACKED SQUARE CYLINDER)

Consider the edge cracked square cylinder whose cross section Ω is shown in Figure 3. We suppose that a plane strain assumption is valid. The crack faces $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4 are traction free and the non-crack boundary of the region Ω has a traction loading of $g = (g_1,g_2)$. We assume that no body forces act. Let w denote the displacement vector and suppose that (r,θ) are polar coordinates centered at A.

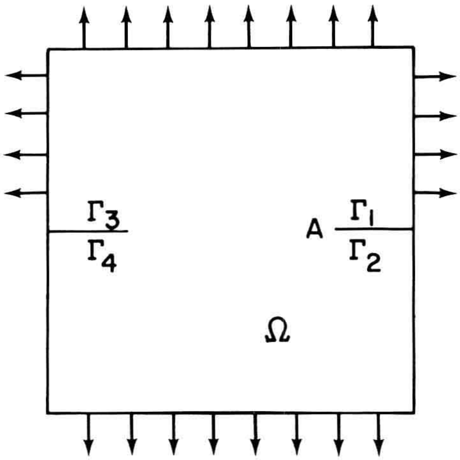

Figure 3. Cross section of edge cracked square cylinder

In the neighborhood of A the displacement vector w can be written in the form

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{K_I r^{\frac{1}{2}}}{2\mu} \begin{pmatrix} -(\kappa-\frac{1}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2} \\ (\kappa+\frac{1}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2} \end{pmatrix} + \frac{K_{II} r^{\frac{1}{2}}}{2\mu} \begin{pmatrix} -(\kappa+\frac{3}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2} \\ -(\kappa-\frac{3}{2})\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2} \end{pmatrix}$$

$$+ O(r^{3/2})$$

where μ is the shear modulus, $\kappa=3-4\nu$ and ν is Poisson's ratio. The constants K_I and K_{II} are (up to a normalizing constant) the mode I and II stress intensity factors of linear fracture mechanics. There are a number of extraction expressions that yield the stress intensity factors K_I and K_{II}. We shall briefly describe two kinds of such expressions.

Define the vector functions v^I and v^{II} by

$$v^I = \begin{pmatrix} v_1^I \\ v_2^I \end{pmatrix} = \frac{r^{-\frac{1}{2}}}{2\mu} \begin{pmatrix} (-\kappa+\frac{3}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{5\theta}{2} \\ -(\kappa+\frac{3}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{5\theta}{2} \end{pmatrix}, \text{ and}$$

$$v^{II} = \begin{pmatrix} v_1^{II} \\ v_2^{II} \end{pmatrix} \frac{r^{-\frac{1}{2}}}{2\mu} \begin{pmatrix} (\kappa+\frac{1}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{5\theta}{2} \\ (-\kappa+\frac{1}{2})\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{5\theta}{2} \end{pmatrix} .$$

Then for α = I, II it may be shown that

$$K_o = \frac{\mu}{4\pi(1-\nu)} \int_{\partial\Omega} (g_k v_k^\alpha - t_{k\ell}(v^\alpha) n_\ell w_k)\, ds \quad , \qquad (5.1)$$

where, for any vector function u , $t_{k\ell}(u)$ denotes the corresponding stress tensor

$$t_{k\ell}(u) = \frac{2\mu\nu}{1-2\nu} \delta_{k\ell} \frac{\partial u_s}{\partial x_s} + \mu \left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k} \right) \qquad (k,\ell,s = 1,2);$$

$n = (n_1,n_2)$ is the outward pointing unit normal on $\partial\Omega$; $\delta_{k\ell}$ is the Kronecker delta; and a repeated index indicates summation. Notice that the integration in (5.1) is in fact only over the non-crack portion of $\partial\Omega$.

The second kind of extraction expression that we shall describe is related to the Energy Release principle. Suppose that ϕ is any sufficiently smooth function on Ω which is unity in the vicinity of A and vanishes on $\partial\Omega-(\Gamma_3 \cup \Gamma_4)$. Let

$$u^I = \begin{pmatrix} u_1^I \\ u_2^I \end{pmatrix} = \frac{r^{\frac{1}{2}}}{2\mu} \begin{pmatrix} -(\kappa-\frac{1}{2})\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2} \\ (\kappa+\frac{1}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2} \end{pmatrix} \quad \text{and}$$

$$u_1^{II} = \begin{pmatrix} u_1^{II} \\ u_2^{II} \end{pmatrix} = \frac{r^{\frac{1}{2}}}{2\mu} \begin{pmatrix} -(\kappa+\frac{3}{2})\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2} \\ -(\kappa-\frac{3}{2})\cos\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2} \end{pmatrix} .$$

Then for α = I, II

$$K^\alpha = \frac{-\mu}{(1-\nu)\pi} \int_\Omega \left\{ \left[\frac{\partial w_i}{\partial x_k} t_{ik}(u^\alpha) - \frac{\partial w_i}{\partial x_1} t_{i1}(u^\alpha) - \frac{\partial u_i^\alpha}{\partial x_1} t_{i1}(w) \right] \frac{\partial \phi}{\partial x_1} - \left[\frac{\partial w_i}{\partial x_1} t_{i2}(u^\alpha) + \frac{\partial u_i^\alpha}{\partial x_1} t_{i2}(w) \right] \frac{\partial \phi}{\partial x_2} \right\} dA \tag{5.2}$$

Notice that the integration in (5.2) excludes the region around A in which $\phi = 1$. Therefore, the singularities in the derivatives of u^α at A should present no computational difficulties in any post-processing based on (5.2).

6. SOME NUMERICAL EXAMPLES

6.1 The FEARS Program

The calculations associated with the examples of this section were performed using the FEARS program. FEARS is a research oriented, adaptive finite element package developed at the University of Maryland. A detailed description of the operation and mathematical background of the program can be found in [8]. For the purposes of this paper, the following few remarks will suffice. FEARS assumes that the region under consideration has firstly been partitioned into a number of subregions, each of which is a curvilinear quadrilateral. Within the program, each of these subregions is transformed by a change of coordinates into a unit square. The actual finite element modelling is then carried out on these transformed squares. Square bilinear elements are used. FEARS has an adaptive character: starting from an initial coarse mesh (usually, uniform on each of the transformed squares), the program automatically selects, in a recursive manner, a sequence of "optimal" mesh refinements. FEARS also calculates an a posteriori estimate of the error in the finite element solution.

6.2 Example 1 (The Membrane Problem)

We consider the problem (2.1) with the load $f = -1$. By the method of separation of variables, an infinite series representation of w can be found. Using this series the following exact values (accurate to 5 significant figures) can be found:

$$E(w) = \int_\Omega |\nabla w|^2 dA = .56231 \; ; \; W = w(0,0) = -.29469 \; ; \; V = \frac{\partial w}{\partial x_1}(1,0) = .67528 .$$

Square bilinear elements were used on a uniform mesh of mesh size h. (For this problem the transformation of Ω carried out by FEARS is just a reduction in size.) Considerations of symmetry show that we need only calculate using a quarter segment of Ω . On the basis of theoretical considerations, we expect to see the following rates of convergence: $E(w-\tilde{w}) \sim O(h^2)$, and $E(w-\tilde{w})^{\frac{1}{2}} \sim O(h)$; and for the pointwise values $|W - \tilde{w}(0,0)| \sim O(h^2)$ and $|V - \frac{\partial w}{\partial x_1}(1,0)| \sim O(h)$. Post-processed values $\tilde{W}$ and $\tilde{V}$ were computed using (2.5) and (2.4) respectively.

The results of the calculations are listed in Table 1. The post-processing calculations for $\tilde{W}$ (see (2.4)) were performed using

$$\phi_0 = \frac{1}{4\pi} \log\left(\frac{(1+x_1^2)(1+x_2^2)}{2}\right) ,$$

while for the computation of $\tilde{V}$ (see (2.14) we selected

$$\phi_0 = \frac{(x_1-1)}{\pi}\left(\frac{1}{(x_1-1)^2+x_2^2} - \frac{1}{(x_1-1)^2+1}\right)X(x)$$

where
$$X(x) = \begin{cases} |x_1|^3 & -1 \le x_1 < 0 \\ 0 & 0 \le x_1 \le 1 \end{cases} .$$

The various θ ratios mentioned in the table are the ratios of the appropriate FEARS a posteriori error estimates to the corresponding true errors. For instance, for the calculation of $\tilde{W}$ with the coarsest mesh, the FEARS estimate of the error in $\tilde{W}$ was 1.07 times the true error in $\tilde{W}$.

Let us make a few comments about the results in Table 1. From the first section of the table it is seen that the energy norm of the error appears to converge at a rate of $O(h)$, which means that energy of the error is converging as $O(h^2)$. Notice, both the pointwise value $\tilde{w}(0,0)$ and the post-processed value $\tilde{W}$ for the displacement W appear to converge as $O(h^2)$. This is as theory would predict since we use bilinear elements. (See the discussion in §2.1). Nonetheless, $\tilde{W}$ is about twice as accurate as $\tilde{w}(0,0)$. The case of the stress component V is however markedly different. The post-processed value $\tilde{V}$ is considerably more accurate than the pointwise value $\frac{\partial \tilde{w}}{\partial x_1}(1,0)$. The rate of convergence of $\tilde{V}$ is $O(h^2)$, while that of the pointwise value is around $O(h)$. The table confirms that indeed the post-processed values for both displacement and stress are converging at the same rate as the strain energy of the error.

6.3 Example 2 (The Split Membrane Problem)

For our second numerical example we consider the problem dealt with in §3 with loading $g = x_2$. For this problem also we can obtain a series solution which can be manipulated to give the following exact values (accurate to 5 significant figures): $E(w) = 4.5271$; $k_1 = -1.3581$; $k_2 = .97009$; and $k_3 = .45271$.

Since the solution of this problem has a severe singularity at (0,0) we do not expect uniform meshes to yield accurate finite element solutions. When FEARS computed whis problem it was directed to produce mesh refinements that were "optimal" in a strain energy error sense. As expected, the sequence of meshes produced exhibits a quite severe refinement around the tip of the slit. Two sets of post-processing calculations were carried out for the km's. Firstly using the extraction expression (3.1), and secondly employing (3.2) with

$$\phi = \begin{cases} 1 & 0 \le r < \tfrac{1}{2} \\ 1-4(r-\tfrac{1}{2})^2 & \tfrac{1}{2} \le r \le 1 \end{cases} .$$

The results of the calculations are shown in Tables 2 and 3. The following conclusions can be drawn from these results:

(a) Despite the presence of singularities of the exact solution w, the sequence of adaptively created meshes gives an apparent rate of convergence for the energy norm of the error in $\tilde{w}$ that is close to the theoretically "optimal"

No. of elements in quarter segment (uniform mesh)	4	16	64
<u>Energy Calculations</u> ($E(w) = .56231$)			
$E(\tilde{w})$	.511608	.549340	.559040
Relative energy norm of error = $\left(\frac{E(w-\tilde{w})}{E(w)}\right)^{\frac{1}{2}}$	30.1%	15.2%	7.62%
θ ratio for energy norm of error	.94	.96	.98
<u>Calculation of displacement</u> ($W = w(0,0) = -.29469$)			
$\tilde{w}(0,0)$	-.310714 (5.4%)	-.298393 (1.3%)	-.295596 (.31%)
$\tilde{W}$	-.287306 (2.5%)	-.292829 (.63%)	-.294220 (.16%)
θ ratio for error in $\tilde{W}$	1.07	.98	.99
<u>Calculation of stress component</u>			
($V = \frac{\partial w}{\partial x_1}(1,0) = .67528$)			
$\frac{\partial \tilde{w}}{\partial x_1}(1,0)$	.482142 (29%)	.565480 (16%)	.616687 (8.7%)
$\tilde{V}$	.66623 (1.3%)	.67313 (.32%)	.67477 (.076%)
θ ratio for error in $\tilde{V}$	.93	1.02	1.06

Table 1: Numerical results for Example 1. (Percentages in parentheses are relative errors with respect to exact values.)