Computational Commutative and Non-Commutative Algebraic Geometry

Edited by
Svetlana Cojocaru
Gerhard Pfister
Victor Ufnarovski

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Edited by

Svetlana Cojocaru

Institute of Mathematics and Computer Science, Chisinau, Moldova

Gerhard Pfister

Technische Universität Kaiserslautern, Fachbereich Mathematik, Kaiserslautern, Germany

Victor Ufnarovski

Centre for Mathematical Sciences, Mathematics, Lund Institute of Technology, Lund University, Lund, Sweden

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Preface

A NATO Advanced Research Workshop "Computational Commutative and Non-Commutative Algebraic Geometry" was held in Chishinau, Moldova from 6 to 11 June, 2004. The financial support was offered by the NATO grant PST.ARW.980392. The workshop was organized by Gerhard Pfister and Svetlana Cojocaru with assistance of members of the Organizing Committee: Jürgen Herzog, Lorenzo Robbiano, Victor Ufnarovski. There were 25 lectures given by the participants and open discussions on the subjects presented. The central theme of this workshop was the interplay between commutative and non-commutative algebraic geometry, with its theoretical and computational aspects. The Scientific Program emphasizes current trends in Commutative and Non-Commutative Algebraic Geometry and Algebra. The contributors to this volume review the state of the art and present new evolution and progress reflecting the topics discussed in the lectures. The volume addresses in the first place researchers and graduate students. The editors would like to thank the contributors to this volume. Thanks are due to the NATO Public Diplomacy Division for their generous financial support. Special thanks to Tatiana Verlan, who had the task to correct and prepare the manuscript.

December 2004,
Svetlana Cojocaru,
Gerhard Pfister,
Victor Ufnarovski

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The Structure of $Simp_{<\infty}(A)$ for Finitely Generated k-algebras A^1

Olav Laudal [a,2]
[a] *Institute of Mathematics, University of Oslo*

Abstract. In this paper we study the non-commutative scheme structure of the set of iso-classes of simple modules on a finitely generated k-algebra, k an algebraically closed field. We introduce the notion of *geometric algebra* and, referring to [12], we prove that these algebras are determined by the structure of this non-commutative scheme. We consider natural completions of these schemes, adding indecomposable modules at infinity. This leads to a notion of correspondence on plane curves, which we explore to some degree. We end the paper with a sketch of how to relate global invariants of the algebra, like cyclic homology, to corresponding invariants of the scheme of simple finite-dimensional modules.

Keywords. Geometric algebras, modules, simple modules, extensions, deformation theory, moduli spaces, non-commutative schemes, non-commutative plane curves.

Introduction

Let k be any field, most often assumed to be algebraically closed, and consider a finitely generated k-algebra A. Let

$$Simp_{<\infty}(A) = \bigcup_n Simp_n(A)$$

be the set of (iso-classes of) finite dimensional simple right A-modules. An n-dimensional simple A-module $V \in Simp_n(A)$ defines a surjective homomorphism of k-algebras, $\rho : A \to End_k(V)$, the kernel of which is a two-sided maximal ideal $\mathfrak{m}_V$ of A. Let $Max_{\leq\infty}(A)$ be the set of all such maximal ideals of A, for $n \geq 1$. To exclude some strange and for our purposes non-interesting cases, we shall assume that A has the following property:

$$Rad(A)^{\infty} := \bigcap_{\mathfrak{m} \in Max_{<\infty}(A), n \geq 0} \mathfrak{m}^n = 0$$

For want of a better name, we shall call such algebras *geometric*. This condition is actually satisfied for most finitely generated k-algebras that we shall be interested in, and in

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[2]Olav Arnfinn Laudal, Box.1053, Blindern, 0316 Oslo, Norway, Institute of Mathematics, University of Oslo. E-mail: arnfinnlmath.uio.no, http://www.math.uio.no.

particular it is satisfied for the free k-algebra on d symbols, $A = k < x_1, x_2, ..., x_d >$, see the example (4.19) of [10], [12].

We shall be concerned with the structure of the individual $Simp_n(A)$, $n \geq 1$, and we shall construct natural completions $Simp_\Gamma(A)$, of the scheme $Simp_n(A)$, adding indecomposable modules. We shall also see that the scheme of indecomposable two-dimensional representations induces interesting correspondences for hypersurfaces, and in particular for plane curves. The study of $Ind_\Gamma(A) := Simp_\Gamma(A) - Simp_n(A)$ may also throw light on the classical McKay correspondence. As a tool for studying $Simp_\Gamma(A)$ we introduce the Jordan morphism and a corresponding generalization of the Deligne-Simpson problem. Finally we shall discuss to what extent the family $\{Simp_n(A)\}_{n\geq 1}$ of schemes determines the *global* structure of A. In particular, are the K-groups (resp. the cyclic homology) of A determined by the K-groups, (resp. the de Rham cohomology) of the different $Simp_n(A)$? Conversely, what can we learn about the de Rham cohomology of $Simp_n(A)$, knowing the cyclic cohomology of A?

This paper will be followed by a more comprehensive study of non-commutative plane curves, see [5].

1. Some General Results

In [10], [11], [12], we introduced non-commutative deformations of families of modules of non-commutative k-algebras, and the notion of *swarm* of right modules (or more generally of objects in a k-linear abelian category).
Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras,

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra R* is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories

$$\underline{l} \subseteq \underline{a}_1$$

where $\underline{l}$, as usual, denotes the category of commutative local artinian k-algebras with residue field k.

Fix a not necessarily commutative k-algebra A and consider a right A-module M. The ordinary deformation functor

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined. Assuming $Ext^i_A(M, M)$ has finite k-dimension for $i = 1, 2$, it is well known, see [18], or [9], that Def_M has a Noetherian pro-representing hull H, *the formal moduli of M*. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M, M)$, and H can be computed in terms of $Ext^i_A(M, M)$, $i = 1, 2$ and their *matric* Massey products, see [9].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A-modules. Assume that,

$$dim_k Ext_A^1(V_i, V_j) < \infty.$$

Any such family of A-modules will be called a *swarm*. We shall define a deformation functor,

$$Def_{\mathcal{V}} : \underline{a}_r \to \underline{Sets}$$

generalizing the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that $End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j))$. Clearly π defines a k-linear and left R-linear map,

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras, $\eta_0 : A \to \oplus_{i=1}^r End_k(V_i)$. Let

$$Def_{\mathcal{V}}(R) \in \underline{Sets}$$

be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that,

$$\tilde{\pi}(R) \circ \eta' = \eta_0,$$

where the equivalence relation is defined by inner automorphisms in the k-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we prove the following, see [11]:

Theorem 1. *The functor $Def_{\mathcal{V}}$ has a pro-representable hull, i.e. an object H of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family,*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathcal{V}}(H/\mathfrak{m}^n),$$

where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$,

$$\kappa : Mor(H, -) \to Def_{\mathcal{V}}$$

defined for $\phi \in Mor(H,R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matric Massey products defined on subspaces,

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k),$$

with values in $Ext^2(V_i, V_k)$.

The right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)),$$

and the k-algebra $O(\mathcal{V})$ acts on the family of A-modules $\mathcal{V} = \{V_i\}$, extending the action of A. If $dim_k V_i < \infty$, for all $i = 1, ..., r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, we prove the crucial result,

A generalized Burnside theorem *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H, see [8]. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$, see [8]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [16].

Using this we consider, in [11], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i,\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$, and extension type given by a directed representation graph Γ, see §3. The main result is the following, see [11],

Proposition 2. *Let A be any k-algebra, $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of A-modules, i.e. such that,*

$$\dim_k Ext^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \ldots, r.$$

(i): Consider an iterated extension E *of* $\mathcal{V}$*, with representation graph* Γ*. Then there exists a morphism of k-algebras*

$$\phi : H(\mathcal{V}) \to k[\Gamma]$$

such that

$$E \simeq k[\Gamma]\otimes_\phi \tilde{V}$$

as right A*-algebras.*

(ii): The set of equivalence classes of iterated extensions of $\mathcal{V}$ *with representation graph* Γ*, is a quotient of the set of closed points of the affine algebraic variety*

$$\underline{A}[\Gamma] = Mor(H(\mathcal{V}), k[\Gamma])$$

(iii): There is a versal family $\tilde{V}[\Gamma]$ *of* A*-modules defined on* $\underline{A}[\Gamma]$*, containing as fibres all the isomorphism classes of iterated extensions of* $\mathcal{V}$ *with representation graph* Γ*.*

To any, not necessarily finite, swarm $\underline{c} \subset \underline{mod}(A)$ of right-A-modules, we have associated two associative k-algebras, see [10] and [12], $O(|\underline{c}|, \pi) = \varprojlim_{\mathcal{V}\subset|\underline{c}|} O(\mathcal{V})$, and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural k-algebra homomorphisms,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and,

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

with the property that the A-module structure on $\underline{c}$ is extended to an $\mathcal{O}$-module structure in an optimal way, see also §4. We then defined an *affine non-commutative scheme* of right A-modules to be a swarm $\underline{c}$ of right A-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular we considered, for finitely generated k-algebras, the swarm $Simp^*_{<\infty}(A)$ consisting of the finite dimensional simple A-modules, and the *generic* point A, together with all morphisms between them. The fact that this is a swarm, i.e. that for all objects $V_i, V_j \in Simp_{<\infty}$ we have $dim_k Ext^1_A(V_i, V_j) < \infty$, is easily proved. We have in [12] proved the following result, (see (4.1), loc.cit. and Lemma 2. above.)

Proposition 3. *Let* A *be a geometric* k*-algebra, then the natural homomorphism,*

$$\eta(Simp^*(A)) : A \longrightarrow \mathcal{O}_\pi(Simp^*_{<\infty}(A))$$

is an isomorphism, i.e. $Simp^*_{<\infty}(A)$ *is a scheme for* A*.*

In particular, $Simp^*_{<\infty}(k < x_1, x_2, ..., x_d >)$, is a scheme for $k < x_1, x_2, ..., x_d >$. To analyze the local structure of $Simp_n(A)$, we need the following, see [12],(3.23),

Lemma 4. *Let* $\mathcal{V} = \{V_i\}_{i=1,..,r}$ *be a finite subset of* $Simp_{<\infty}(A)$, *then the morphism of k-algebras,*

$$A \to O(\mathcal{V}) = (H_{i,j} \otimes_k Hom_k(V_i, V_j))$$

is topologically surjective.

Proof. Since the simple modules V_i, $i = 1,..,r$ are distinct, there is an obvious surjection, $\eta_0 : A \to \prod_{i=1,..,r} End_k(V_i)$. Put $\mathfrak{r} = ker\eta_0$, and consider for $m \geq 2$ the finite-dimensional k-algebra, $B := A/\mathfrak{r}^m$. Clearly $Simp(B) = \mathcal{V}$, so that by the generalized Burnside theorem, see [12], (2.6), we find, $B \simeq O^B(\mathcal{V}) := (H^B_{i,j} \otimes_k Hom_k(V_i, V_j))$. Consider the commutative diagram,

$$\begin{array}{ccccc}
A & \longrightarrow & (H^A_{i,j} \otimes_k Hom_k(V_i, V_j)) =: O^A(\mathcal{V}) & & \\
\downarrow & & \downarrow & \searrow & \\
B & \longrightarrow & (H^B_{i,j} \otimes_k Hom_k(V_i, V_j)) & \xrightarrow{\alpha} & O^A(\mathcal{V})/\mathfrak{m}^m
\end{array}$$

where all morphisms are natural. In particular α exists since $B = A/\mathfrak{r}^m$ maps into $O^A(\mathcal{V})/rad^m$, and therefore induces the morphism α commuting with the rest of the morphisms. Consequently α has to be surjective, and we have proved the contention. □

Localization and topology Let $s \in A$, and consider the *open subset* $D(s) = \{V \in Simp(A) | \ \rho(s)$ invertible in $End_k(V)\}$. The Jacobson topology on $Simp(A)$ is the topology with basis $\{D(s) | \ s \in A\}$. It is clear that the natural morphism,

$$\eta : A \to O(D(s), \pi)$$

maps s into an invertible element of $O(D(s), \pi)$. Therefore we may define the localization $A_{\{s\}}$ of A, as the k-algebra generated in $O(D(s), \pi)$ by $im\eta$ and the inverse of $\eta(s)$. This furnishes a general method of localization with all the properties one would wish. And in this way we also find a canonical (pre)sheaf, $\mathcal{O}$ defined on $Simp(A)$.

Definition 5. *When the k-algebra A is geometric, such that* $Simp^*(A)$ *is a scheme for A, we shall refer to the presheaf* $\mathcal{O}$, *defined above on the Jacobson topology, as the structure presheaf of the scheme* $Simp(A)$.

In the next § we shall see that the Jacobson topology on $Simp(A)$, restricted to each $Simp_n(A)$ is the Zariski topology for a classical scheme-structure on $Simp_n(A)$.

2. The Algebraic (Scheme) Structure on $Simp_n(A)$

Recall that a standard n-commutator relation in a k-algebra A is a relation of the type,

$$[a_1, a_2, ..., a_{2n}] := \sum_{\sigma \in \Sigma_{2n}} sign(\sigma) a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(2n)} = 0$$

where $\{a_1, a_2, ..., a_{2n}\}$ is a subset of A. Let $I(n)$ be the two-sided ideal of A generated by the subset,

$$\{[a_1, a_2, ..., a_{2n}] | \{a_1, a_2, ..., a_{2n}\} \subset A\}.$$

Consider the canonical homomorphism,

$$p_n : A \longrightarrow A/I(n) =: A(n).$$

It is well known that any homomorphism of k-algebras,

$$\rho : A \longrightarrow End_k(k^n) =: M_n(k)$$

factors through p_n, see e.g. [4].

Corollary 6. *(i). Let $V_i, V_j \in Simp_{\leq n}(A)$ and put $\mathfrak{r} = \mathfrak{m}_{V_i} \cap \mathfrak{m}_{V_j}$. Then we have, for $m \geq 2$,*

$$Ext^1_A(V_i, V_j) \simeq Ext^1_{A/\mathfrak{r}^m}(V_i, V_j)$$

(ii). Let $V \in Simp_n(A)$. Then,

$$Ext^1_A(V, V) \simeq Ext^1_{A(n)}(V, V)$$

Proof. (i) follows directly from Lemma 4. To see (ii), notice that $Ext^1_A(V, V) \simeq Der_k(A, End_k(V))/Triv \simeq Der_k(A(n), End_k(V))/Triv \simeq Ext^1_{A(n)}(V, V)$. The second isomorphism follows from the fact that any derivation maps a standard n-commutator relation into a sum of standard n-commutator relations. □

Example 7. *Notice that, for distinct $V_i, V_j \in Simp_{\leq n}(A)$, we may well have,*

$$Ext^1_A(V_i, V_j) \neq Ext^1_{A(n)}(V_i, V_j).$$

In fact, consider the matrix k-algebra,

$$A = \begin{pmatrix} k[x] & k[x] \\ 0 & k[x] \end{pmatrix},$$

and let $n = 1$. Then $A(1) = k[x] \oplus k[x]$. Put $V_1 = k[x]/(x) \oplus (0)$, $V_2 = (0) \oplus k[x]/(x)$, then it is easy to see that,

$$Ext^1_A(V_i, V_j) = k,\ Ext^1_{A(1)}(V_i, V_j) = 0, i \neq j,$$

but,

$$Ext^1_A(V_i, V_i) = Ext^1_{A(1)}(V_i, V_i) = k, i = 1, 2.$$