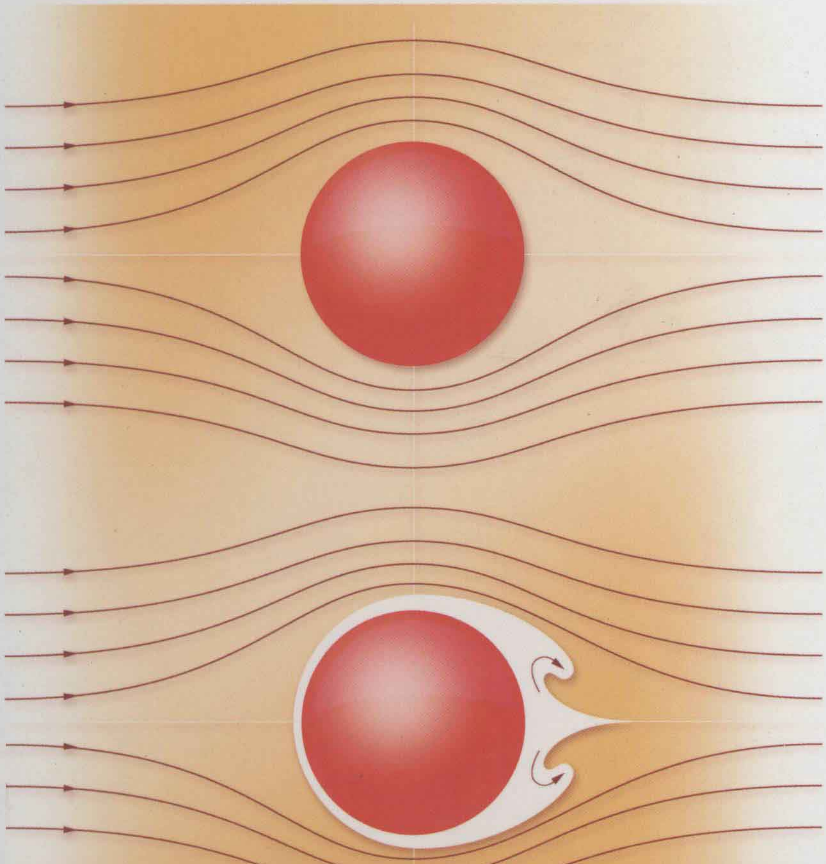
Rainer Ansorge and Thomas Sonar

WILEY-VCH

Mathematical Models of Fluid Dynamics

Modelling, Theory, Basic Numerical Facts
An Introduction

Second, Updated Edition



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Mathematical Models of Fluid Dynamics

For additional information reagarding this topic, please refer also to the following publications

Durbin, P. A., Reif, B. A. P.

Statistical Theory and Modeling for Turbulent Flows

2001

ISBN 978-0-471-49744-8

Shivamoggi, B. K.

Theoretical Fluid Dynamics

1998

ISBN 978-0-471-05659-1

Middleman, S.

Introduction to Fluid Mechanics
Principles of Analysis and Design

1998

ISBN 978-0-471-18209-2

Dedicated to our family members and to our students

Preface to the Second Edition

Because the first edition of this introduction to a mathematical description of fluid dynamics was so enthusiastically adopted by readers, the publishers encouraged the first author to revise the first edition and add any new results and considerations that are relevant to an introductory course. Because of this, the first author introduced some new sections and remarks (e.g., Sections 4.3 , 5.6 , 5.7 , 7.9, etc.), and in turn encouraged the second author (who had already written parts of Chapter 7 for the first edition) to widen his participation in the book and to contribute additional chapters touching upon some modern areas of the numerical treatment of fluid flows to this new edition. However, even in these new chapters, the introductory character of the book has been maintained.

For more extensive representations of special areas of mathematical fluid dynamics, like modeling, theory or numerical methods, we refer the reader to some monographs listed at the end of this book.

Again, we thank the publishers for their support, our colleagues for fruitful discussions, and Mrs. Monika Jampert for technical help with respect to some LaTeX problems. We also hope that readers of the new edition will again consider it useful for enriching their scientific interests and for executing their work.

Hamburg and Brunswick, September 2008

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Thomas Sonar

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Preface to the First Edition

Mathematical modeling is the process of representing problems from fields beyond mathematics itself using mathematics. The subsequent mathematical treatment of this model using theoretical and/or numerical procedures proceeds as follows:

1. Transition from the nonmathematical phenomenon to a mathematical description, which at the same time leads to the translation of problems formulated in terms of the original problem into mathematical problems.

   This task forces the scientist or engineer who intends to use mathematical tools to:
   - Cooperate with experts working in the field that the original problem comes from. Thus, he/she has to learn the language of these experts and to understand their way of thinking (teamwork).
   - Create or accept an idealized description of the original phenomena, i.e., to ignore the properties of the original problem that are expected to be of no great relevance to the questions under consideration. These simplifications are useful since they reduce the complexity of both the model and its mathematical treatment.
   - Identify structures within the idealized problem and replace these structures with suitable mathematical structures.

2. Treatment of the mathematical substitute.

   This task normally requires:
   - Independent activity from the person working on the problem, who must work theoretically.
   - Treatment of the problem using tools from mathematical theory.
   - The solution of the particular mathematical problems that occur using these theoretical tools; in other words, differential equations or integral equations, optimal control problems, or systems of algebraic equations, etc., must be solved. Numerical procedures are often the only way to do this and to answer the particular questions of interest, at least approximately. The error in the approximate solution compared with the unknown

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so-called exact solution does not normally affect the answer to the original problem a great deal, provided that the numerical method and tools applied are of sufficiently high accuracy. In this context, it should be realized that the quest for an exact solution does not make sense because of the idealizations mentioned above, and because the initial data presented with the original problem normally originate from statistics or from experimental measurements.

3. Retranslation of the results.

   The qualitative and quantitative statements obtained from the mathematical model then need to be retranslated into the language in which the original problem was formulated. In other words, the results must be interpreted with respect to their real-world meaning. This process again requires teamwork with the experts from the field in which the problem originated.

4. Model checking.

   After retranslation, the results must be checked for relevance and accuracy, e.g., by performing experimental measurements. This work must be done by the experts from the field in which the problem originated. If the mathematical results coincide sufficiently well with the results from experiments stimulated by the theoretical forecasts, the mathematical part is then completed, and a new tool that can be applied by the physicists, engineers, etc., to similar situations has been created.

   On the other hand, if the logical or computational errors are nontrivial, the model must be revised. In this situation, the gap between the results from the mathematical model and the real results can only have originated from using too much idealization during the modeling process.

The development of mathematical models not only stimulates new experiments and leads to constructive prognostic (and hence technical) tools for physicists or engineers, but it is also important from the point of view of the theory of cognition: it allows us to understand the connections between different elements from an unstructured set of observations; in other words, to create theories.

Mathematical descriptions have been used for centuries in various fields, such as physics, engineering, music, etc. Mathematical models are also used in modern biology, medicine, philology and economics, as well as in certain fields of art, like architecture or oriental ornaments.

This book presents an introduction to models used in fluid mechanics. Important properties of fluid flows can be derived theoretically from such models. We discuss some basic ideas for the construction of effective numerical procedures. Hence, all aspects of theoretical fluid dynamics are addressed: modeling, mathematical theory and numerical methods.

We do not expect the reader to be familiar with a lot of experimental work. A knowledge of some fundamental principles of physics, like conservation of mass, conservation of energy, etc., is sufficient. The most important idealization is (in contrast to the molecular structure of materials) the assumption of fluid continua.

The reader will find the mathematics in the text easier to understand if he/she is acquainted with some basic elements of:

- Linear algebra
- Calculus
- Partial differential equations
- Numerical analysis
- The theory of complex functions
- Functional analysis

Functional analysis only plays a role in the somewhat general theory of discretization algorithms described in Chapter 6. In this chapter, the question of the existence of weak entropy solutions of the problems under consideration is discussed. Physicists and engineers are normally not very interested in the treatment of this problem. Nevertheless, we felt that it should be included in this work so that this question not left unanswered. The lack of a solution immediately shows that a model does not fit reality if there is a measurable course of physical events. Existence theorems are therefore important beyond the field of mathematics alone. However, readers who are unacquainted with functional analytic terminology can of course skip this chapter.

With respect to models and their theoretical treatment, as well as to the numerical procedures that occur in Sections 4.1, 5.3, 6.3 and Chapter 7, a brief introduction to mathematical fluid mechanics as provided by this book can only present the most basic facts. However, the author hopes that this overview will generate interest in this field among young scientists, and that it will familiarize people working in institutes and industry with some fundamental mathematical aspects.

Finally, I wish to thank several colleagues for suggestions, particularly Thomas Sonar, who contributed to Chapter 7 when we organized a joint course for graduate students,[1] and Dr. Michael Breuss, who read the manuscript carefully. Last but not least, I thank the publishers, especially Dr. Alexander Grossmann, for their encouragement.

1) Parts of Chapters 1 and 5 are translations from parts of Sections 25.1, 25.2, 29.9 of: R. Ansorge, H. Oberle: *Mathematik für Ingenieure, vol. 2, 2nd ed.* Berlin: Wiley-VCH 2000.

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1 Ideal Fluids

1.1 Modeling by Euler's Equations

Physical laws are mainly derived from conservation principles, such as conservation of mass, conservation of momentum, and conservation of energy.

Let us consider a fluid (gas or liquid) in motion, i.e., the flow of a fluid.[1] Let

$$u(x, y, z, t) = \begin{pmatrix} u_1(x, y, z, t) \\ u_2(x, y, z, t) \\ u_3(x, y, z, t) \end{pmatrix}$$

be the velocity,[2] and denote by $\varrho = \varrho(x, y, z, t)$ the density of this fluid at point $\boldsymbol{x} = (x, y, z)$ and at time instant t.

Let us take out of the fluid at a particular instant t an arbitrary portion of volume $W(t)$ with surface $\partial W(t)$. The particles of the fluid now move, and assume that $W(t + h)$ is the volume formed at the instant $t + h$ by the same particles that formed $W(t)$ at time t.

Moreover, let $\varphi = \varphi(x, y, z, t)$ be one of the functions describing a particular state of the fluid at time t at point $\boldsymbol{x}$, such as mass per unit volume (= density), interior energy per volume, momentum per volume, etc. Hence, $\int_{W(t)} \varphi \, d(x, y, z)$ gives the full amount of mass or interior energy, momentum, etc., of the volume $W(t)$ under consideration.

We would like to find the change in $\int_{W(t)} \varphi \, d(x, y, z)$ with respect to time, i.e.,

$$\frac{d}{dt} \int_{W(t)} (x, y, z, t) \, d(x, y, z) \, . \tag{1.1}$$

1) Flows of other materials can be included too, e.g., the flow of cars on highways, provided that the density of cars or particles is sufficiently high.

2) Note that bold letters in equations normally indicate vectors or matrices.

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We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\,\mathrm{d}(x,y,z) = \lim_{h\to 0}\frac{1}{h}\left\{\int\limits_{W(t+h)} \varphi(\tilde{\boldsymbol{y}},t+h)\,\mathrm{d}(y_1,y_2,y_3) - \int\limits_{W(t)} \varphi(x,y,z,t)\,\mathrm{d}(x,y,z)\right\},$$

where the change from $W(t)$ to $W(t+h)$ is obviously given by the mapping

$$\tilde{\boldsymbol{y}} = \boldsymbol{x} + h\cdot \boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{o}(h)$$
$$\left(\tilde{\boldsymbol{y}} = (y_1,y_2,y_3)^T\right).$$

The error term $\boldsymbol{o}(h)$ also depends on $\boldsymbol{x}$ but the property $\lim_{h\to 0}\frac{1}{h}\boldsymbol{o}(h) = 0$ if differentiated with respect to space, provided that these spatial derivatives are bounded.

The transformation of the integral taken over the volume $W(t+h)$ to an integral over $W(t)$ by substitution requires the integrand to be multiplied by the determinant of this mapping, i.e., by

$$\begin{vmatrix} (1+h\,\partial_x u_1) & h\,\partial_y u_1 & h\,\partial_z u_1 \\ h\,\partial_x u_2 & (1+h\,\partial_y u_2) & h\,\partial_z u_2 \\ h\,\partial_x u_3 & h\,\partial_y u_3 & (1+h\,\partial_z u_3) \end{vmatrix} + o(h)$$
$$= 1 + h\cdot\left(\partial_x u_1 + \partial_y u_2 + \partial_z u_3\right) + o(h)$$
$$= 1 + h\cdot \operatorname{div}\boldsymbol{u}(\boldsymbol{x},t) + o(h)\,.$$

Taylor expansion of $V\varphi(\tilde{\boldsymbol{y}},t+h)$ around $(\boldsymbol{x},t)$ therefore leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\;\mathrm{d}(x,y,z) = \int\limits_{W(t)} \{\partial_t\varphi + \varphi\;\operatorname{div}\boldsymbol{u} + \langle \boldsymbol{u},\nabla\varphi\rangle\}\;\mathrm{d}(x,y,z)\,. \tag{1.2}$$

Here, ∇v denotes the gradient of a scalar function v, and $\langle\cdot\,,\,\cdot\rangle$ means the standard scalar product of two vectors out of $\mathbb{R}^3$.

The product rule from differentiation gives:

$$\varphi\operatorname{div}\boldsymbol{u} + \langle\boldsymbol{u},\nabla\varphi\rangle = \operatorname{div}(\varphi\cdot\boldsymbol{u})\,,$$

so that (1.2) leads to the so-called *Reynolds' transport theorem*[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{W(t)} \varphi(x,y,z,t)\;\mathrm{d}(x,y,z) = \int\limits_{W(t)} \{\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{u})\}\;\mathrm{d}(x,y,z)\,. \tag{1.3}$$

As already mentioned, the dynamics of fluids can be described directly by conservation principles and – as far as gases are concerned – by an additional equation of state.

3) Osborne Reynolds (1842–1912); Manchester

1. Conservation of mass: If there are no sources or losses of fluid within the subdomain of the flow under consideration, the mass remains constant.

Because $W(t)$ and $W(t+h)$ consist of the same particles, they have the same mass. The mass of $W(t)$ is given by $\int_{W(t)} \varrho(x,y,z,t)\, \mathrm{d}(x,y,z)$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{W(t)} \varrho(x,y,z,t)\, \mathrm{d}(x,y,z) = 0$$

must hold. Taking (1.3) into account (particularly for $\varphi = \varrho$), this leads to the requirement

$$\int\limits_{W(t)} \{\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u})\}\, \mathrm{d}(x,y,z) = 0\,.$$

Since this has to hold for arbitrary $W(t)$, the integrand has to vanish:

$$\boxed{\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0\,.} \tag{1.4}$$

This equation is called the *continuity equation.*

2. Conservation of momentum: Another conservation principle concerns the momentum of a mass, which is defined as

mass × velocity .

Thus,

$$\int\limits_{W(t)} \varrho \boldsymbol{u}\, \mathrm{d}(x,y,z)$$

gives the momentum of the mass at time t of the volume $W(t)$ and

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \varrho \boldsymbol{u}$$

describes the *density of momentum.*

The principle of the conservation of momentum, i.e., Newton's second law

force = mass × acceleration,

then states that the change of momentum with respect to time equals the sum of all of the exterior forces acting on the mass of $W(t)$.

In order to describe these exterior forces, we take into account that there is a certain pressure $p(\boldsymbol{x}, t)$ at each point $\boldsymbol{x}$ in the fluid at each instant t. If $\boldsymbol{n}$ is considered to be the unit vector normal on the surface $\partial W(t)$ of $W(t)$, and it is directed outwards, the fluid outside of $W(t)$ acts on $W(t)$ with a force given by

$$-\int_{\partial W(t)} p\,\boldsymbol{n}\,\mathrm{do} \quad (\mathrm{do} = \text{area element of } \partial W(t))\ .$$

Besides the normal forces per unit surface area generated by the pressure, there are also tangential forces which act on the surface due to the friction generated by exterior particles along the surface.

Though this so-called fluid *viscosity* leads to a lot of remarkable phenomena, we are going to neglect this property at the first step. Instead of *real fluids* or *viscous fluids*, we restrict ourselves in this chapter to so-called *ideal fluids* or *inviscid fluids*. This restriction to ideal fluids, particularly to *ideal gases*, is one of the idealizations mentioned in the Preface.

However, as well as exterior forces per unit surface area, there are also exterior forces per unit volume – e.g., the weight.

Let us denote these forces per unit volume by $\boldsymbol{k}$, such that Newton's second law leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{W(t)} \boldsymbol{q}\,\mathrm{d}(x,y,z) = \int_{W(t)} \boldsymbol{k}(x,y,z,t)\,\mathrm{d}(x,y,z) - \int_{\partial W(t)} p\cdot\boldsymbol{n}\,\mathrm{do}\ .$$

Thus, by Gauss' divergence theorem, we find

$$\int_{\partial W(t)} p\,\boldsymbol{n}\,\mathrm{do} = \begin{pmatrix} \int_{\partial W(t)} p\,n_1\,\mathrm{do} \\ \int_{\partial W(t)} p\,n_2\,\mathrm{do} \\ \int_{\partial W(t)} p\,n_3\,\mathrm{do} \end{pmatrix} = \begin{pmatrix} \int_{W(t)} \partial_x p\,\mathrm{d}(x,y,z) \\ \int_{W(t)} \partial_y p\,\mathrm{d}(x,y,z) \\ \int_{W(t)} \partial_z p\,\mathrm{d}(x,y,z) \end{pmatrix}$$
$$= \int_{W(t)} \nabla p\ \mathrm{d}(x,y,z)\ .$$

Together with (1.3),

$$\int_{W(t)} \left\{ \partial_t \boldsymbol{q} + \begin{pmatrix} \mathrm{div}(q_1\boldsymbol{u}) \\ \mathrm{div}(q_2\boldsymbol{u}) \\ \mathrm{div}(q_3\boldsymbol{u}) \end{pmatrix} - \boldsymbol{k} + \nabla p \right\} \mathrm{d}(x,y,z) = 0$$

follows.

Again, this has to be valid for any arbitrarily chosen volume $W(t)$. If, moreover,

$$\mathrm{div}(q_i\,\boldsymbol{u}) = \langle \boldsymbol{u}, \nabla q_i\rangle + \mathrm{div}\,\boldsymbol{u}\cdot q_i$$

is taken into account,

$$\partial_t \boldsymbol{q} + \langle \boldsymbol{u}, \nabla\rangle\ \boldsymbol{q} + \mathrm{div}\,\boldsymbol{u}\cdot\boldsymbol{q} + \nabla p = \boldsymbol{k}\ ,$$